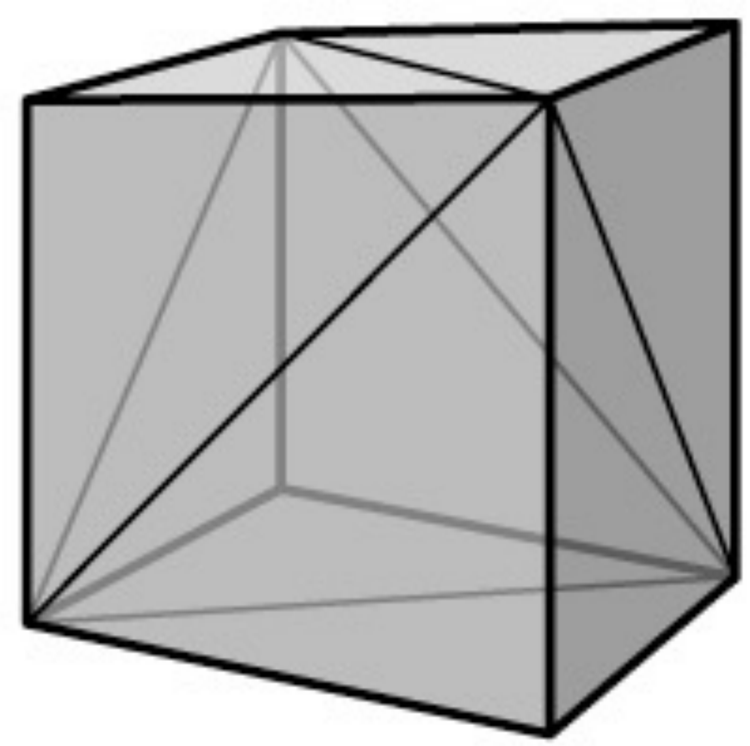
Lecture 7:

Digital Geometry Processing

Interactive Computer Graphics
Stanford CS248, Winter 2021

A small triangle mesh



8 vertices, 12 triangles

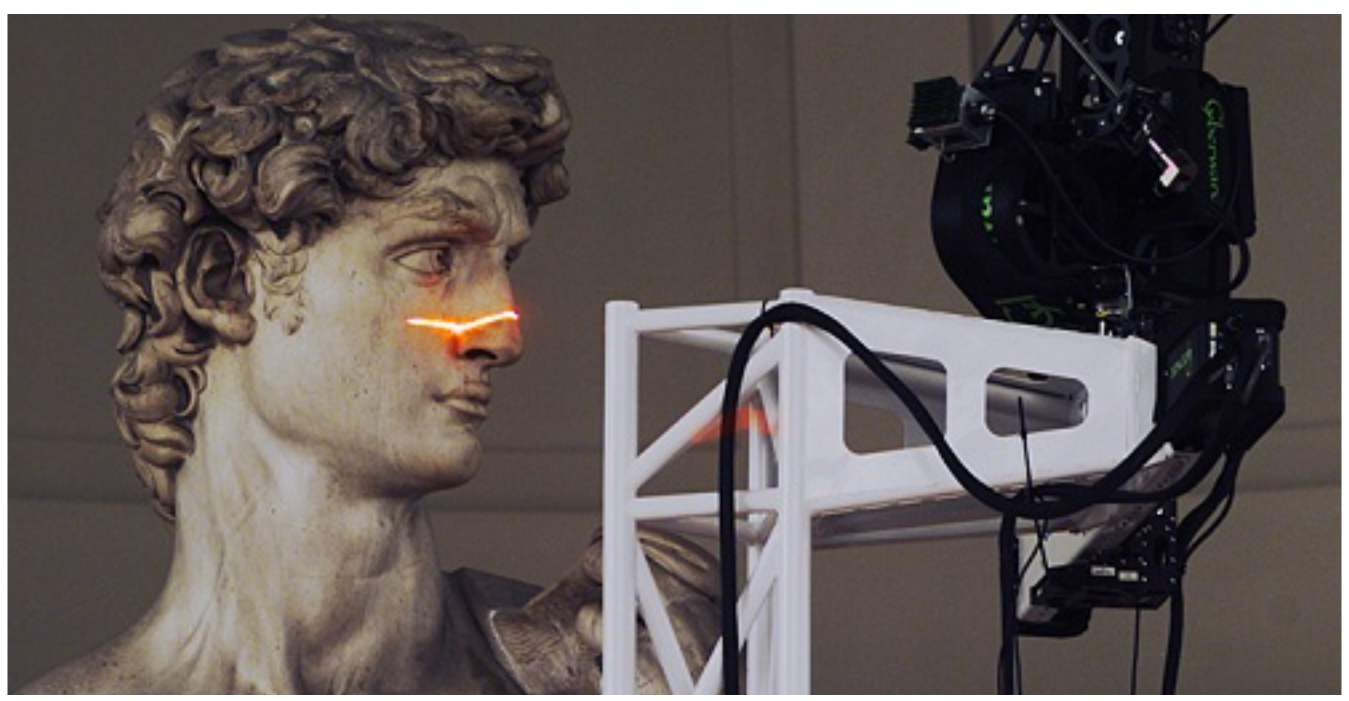
A large triangle mesh

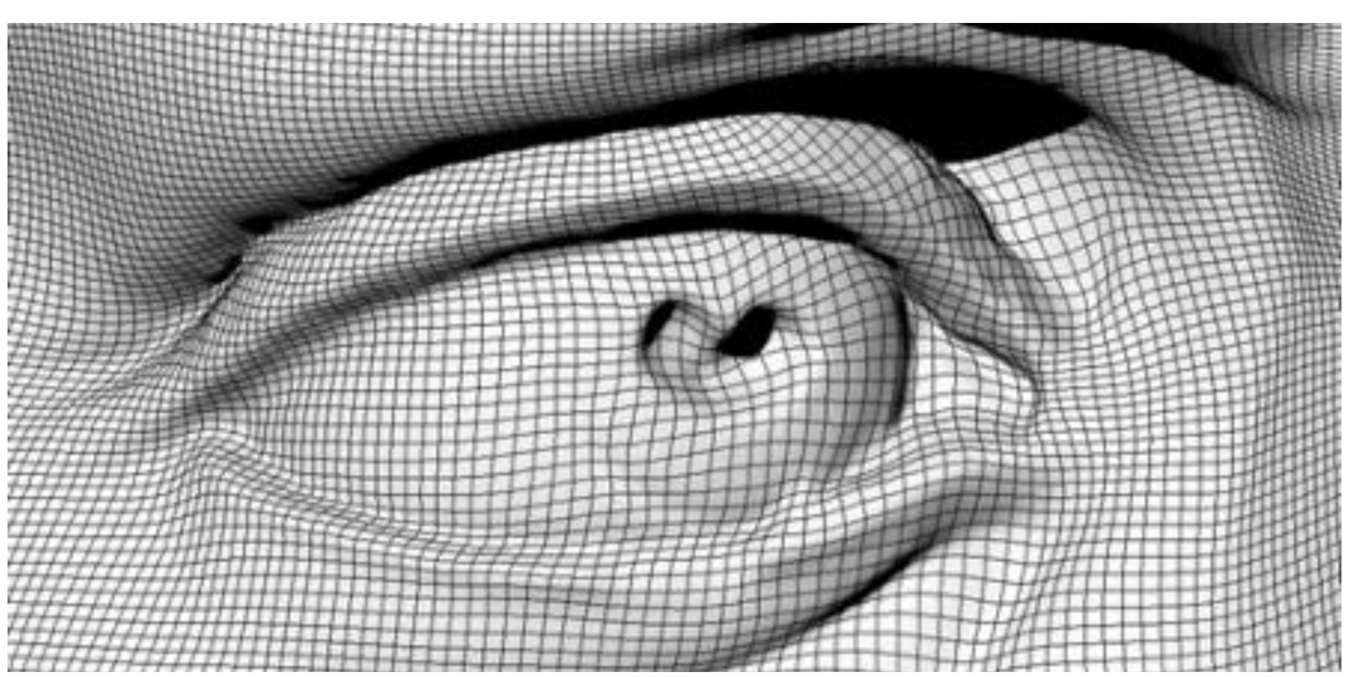
David

Digital Michelangelo Project

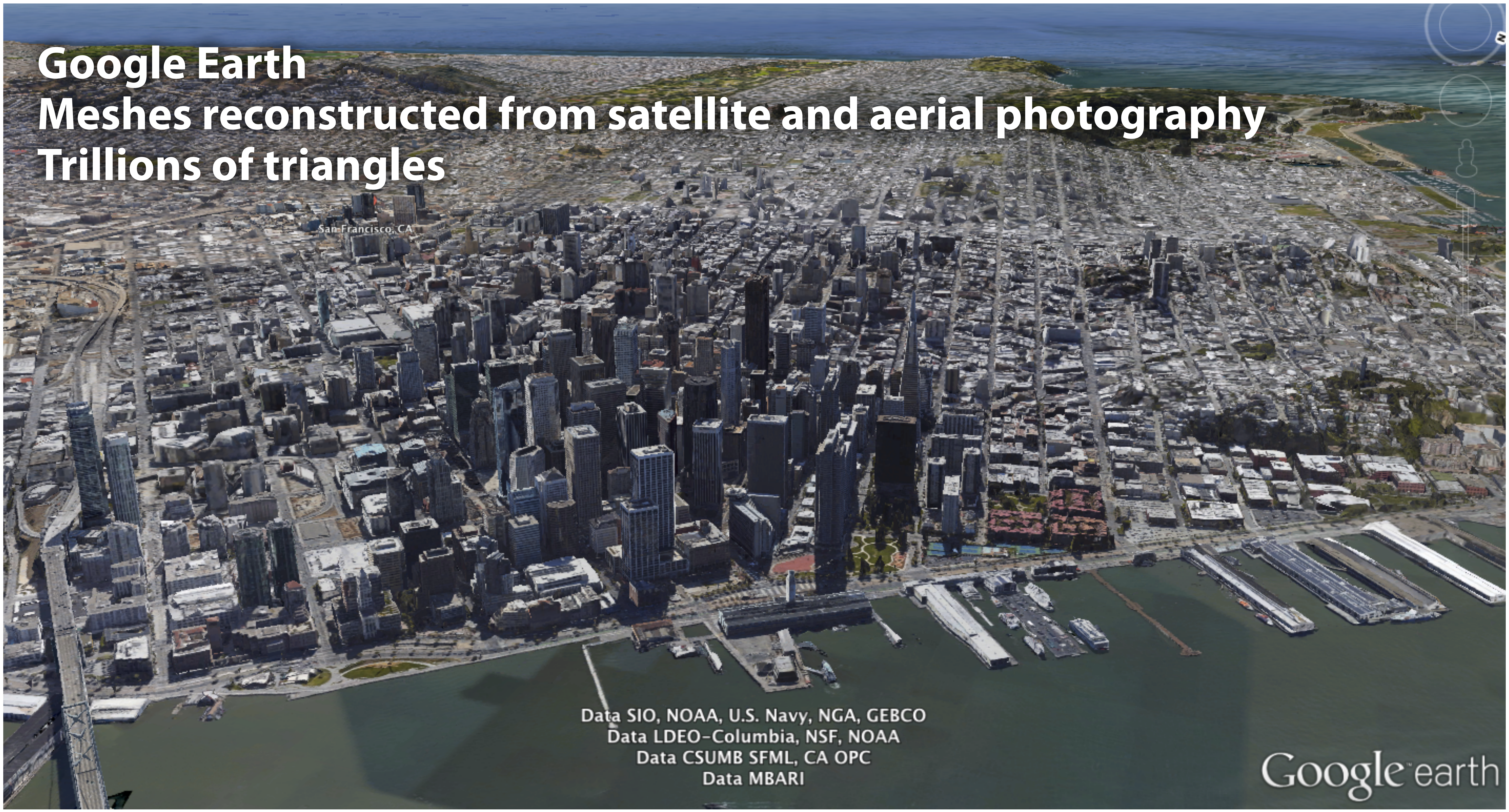
28,184,526 vertices

56,230,343 triangles

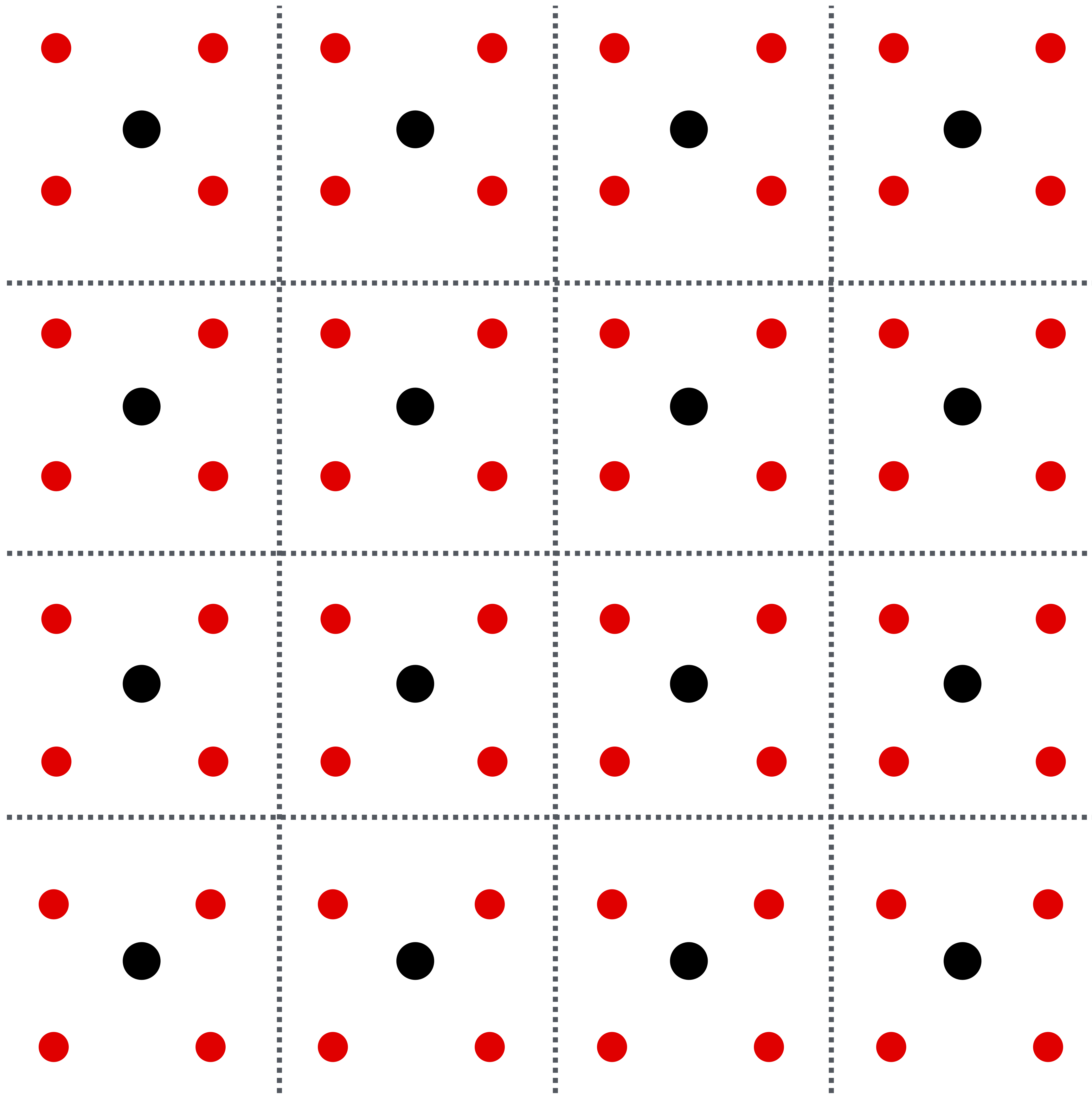




Even larger meshes



Recall: image upsampling



Convert representation of signal given by samples taken at black dots (sparse) into a representation given at new set of denser samples (red dots)

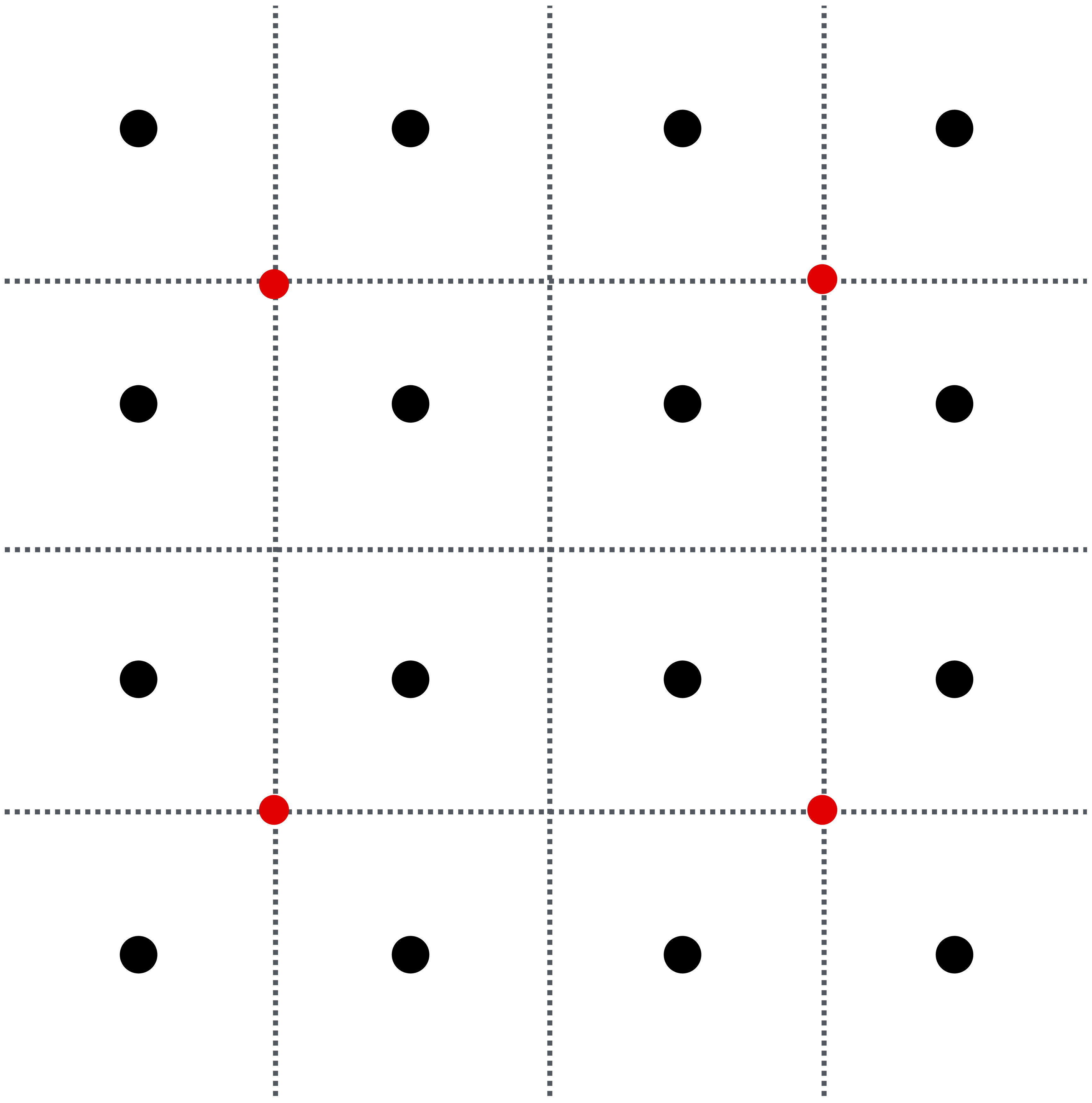
Recall: image upsampling

Upsampling via Nearest neighbor interpolation

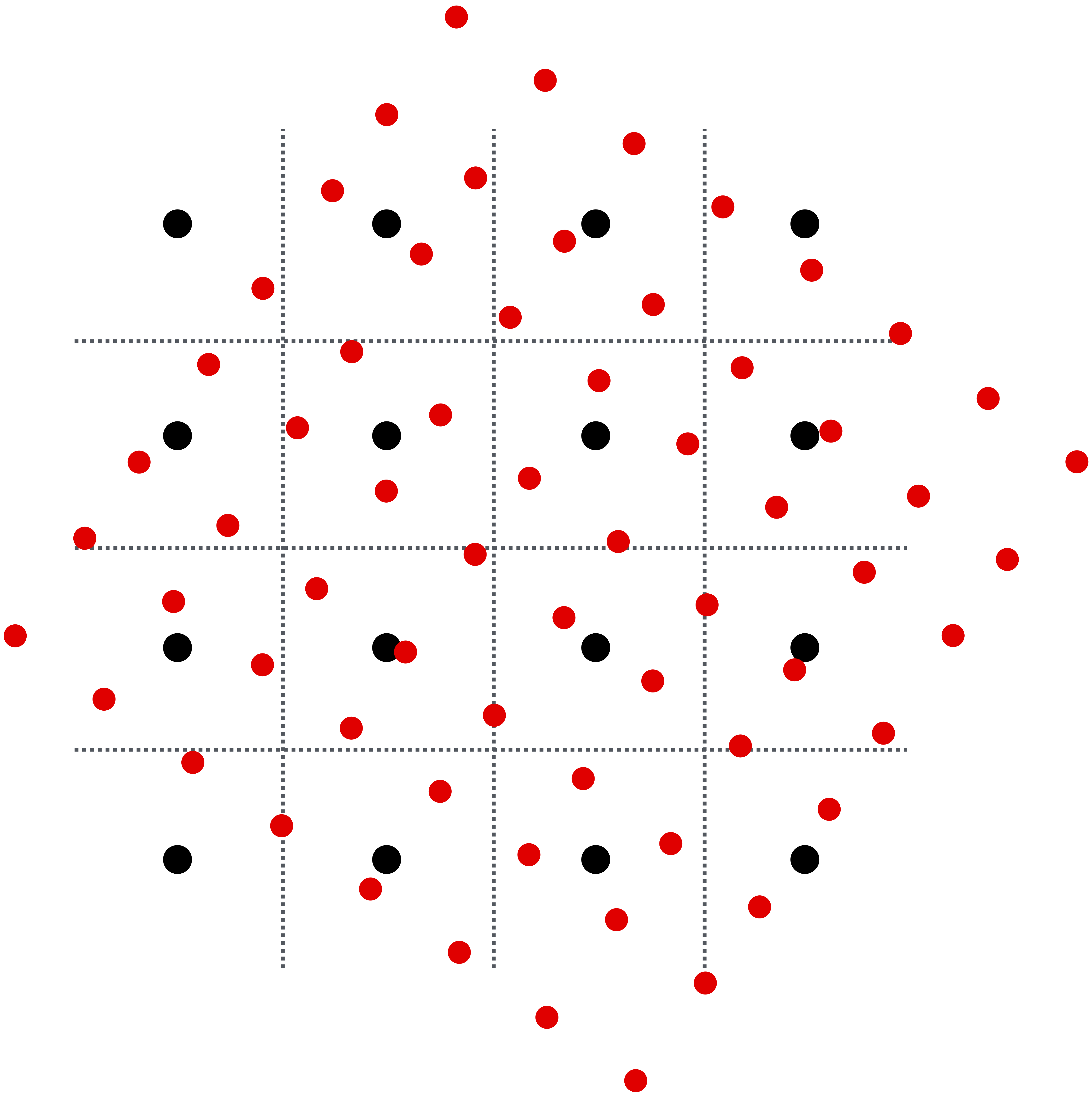
Recall: image upsampling

Upsampling via bilinear interpolation

Recall: image downsampling

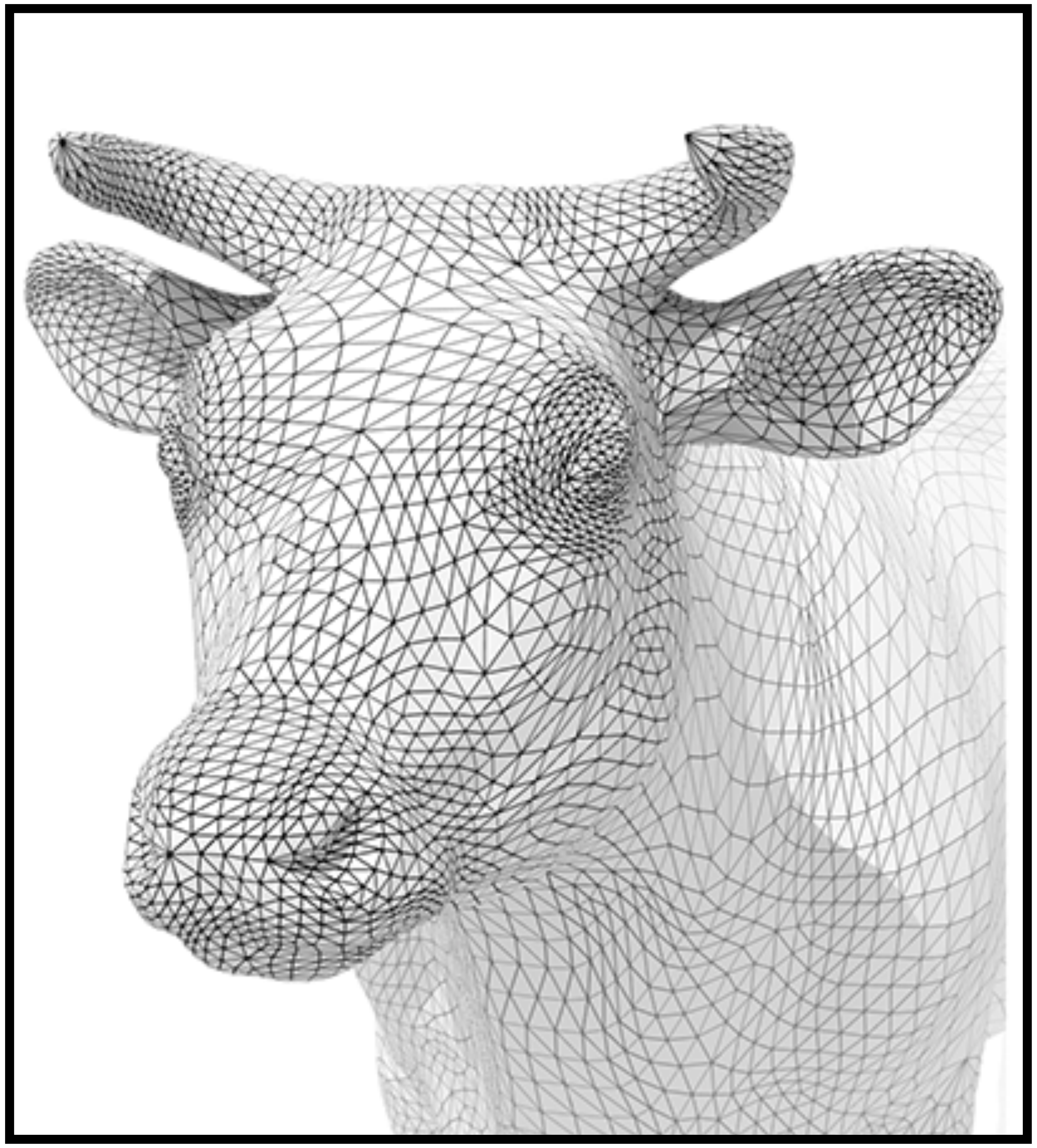


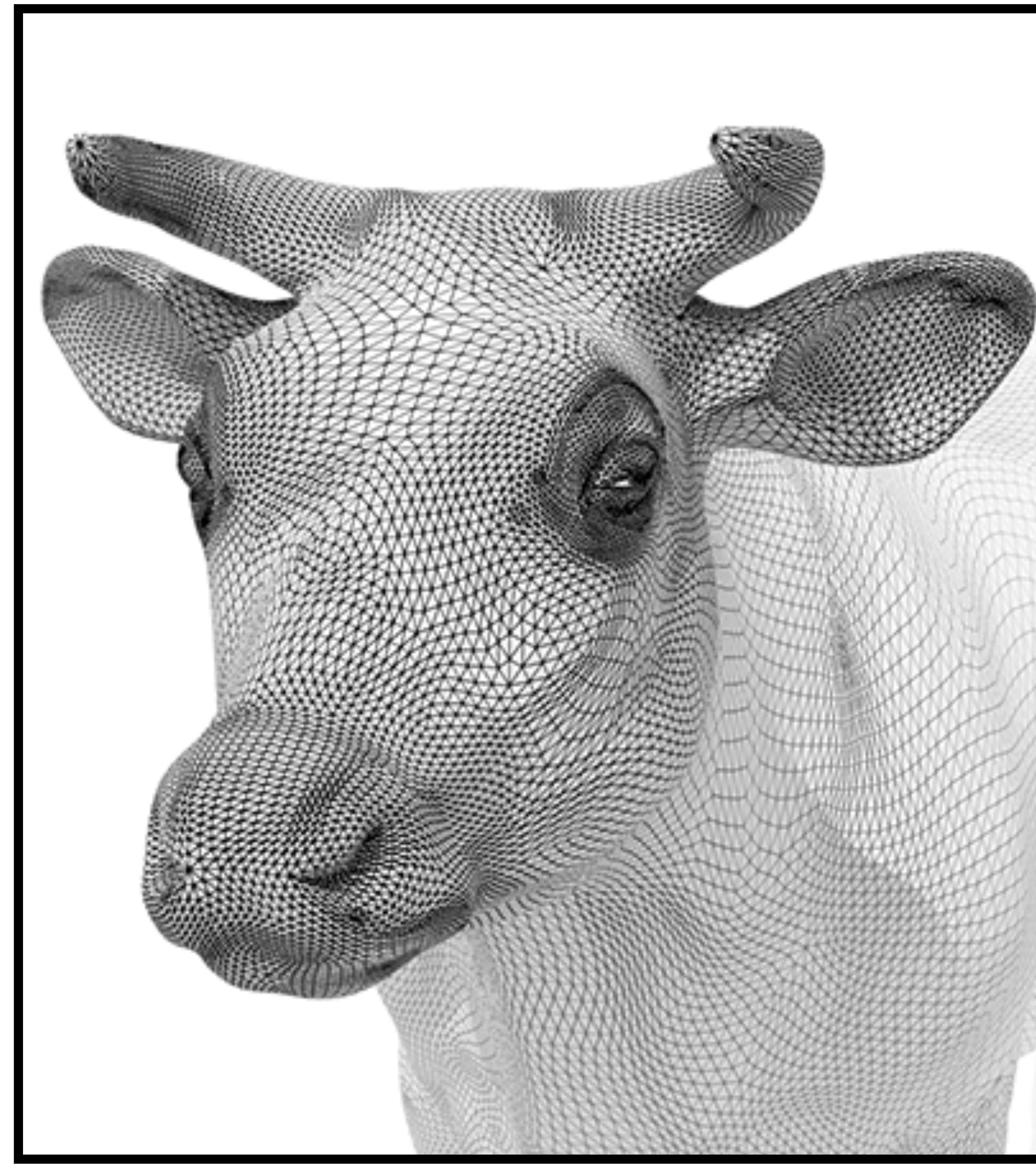
Recall: image resampling



Examples of geometry processing

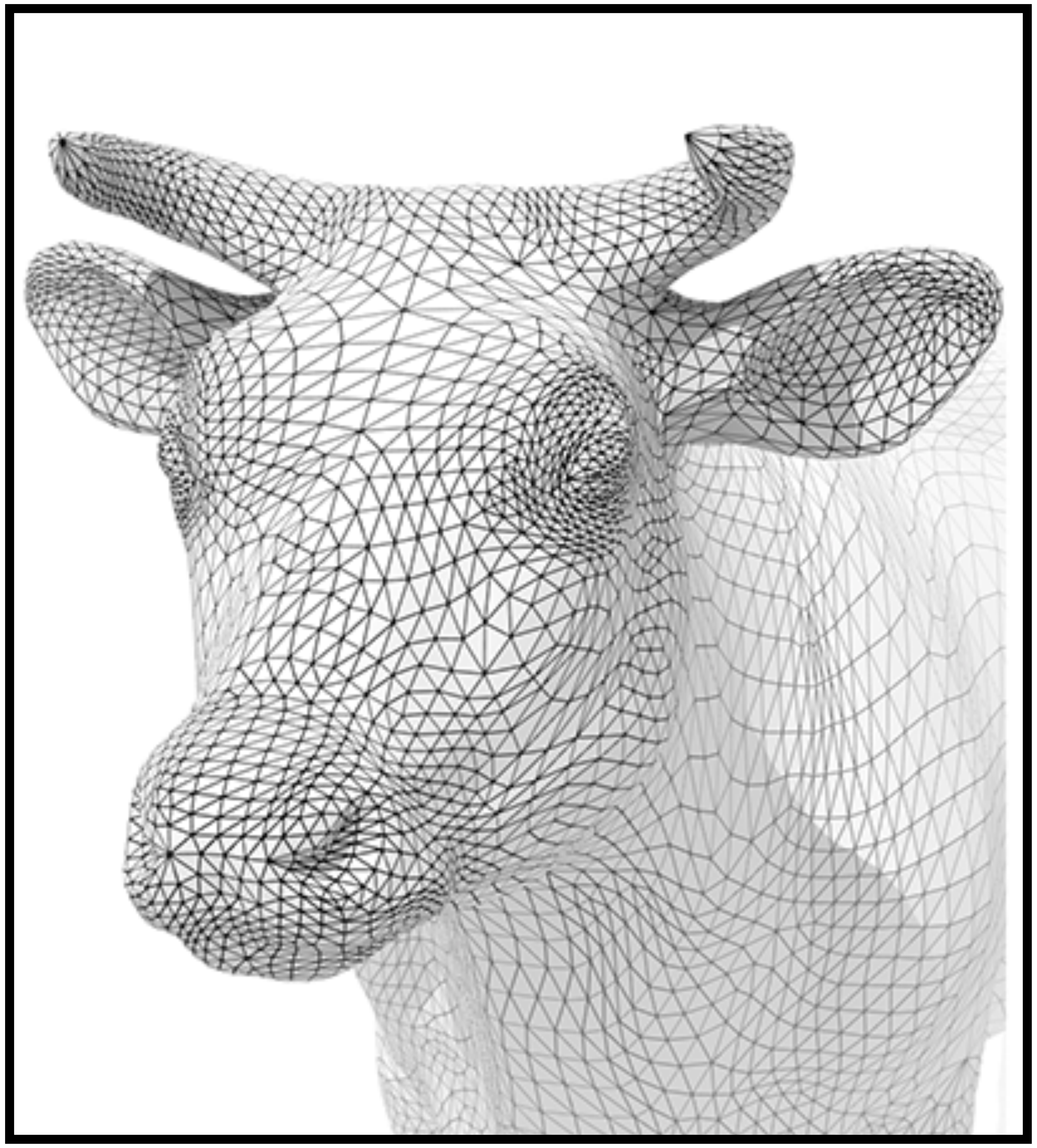
Mesh upsampling — subdivision

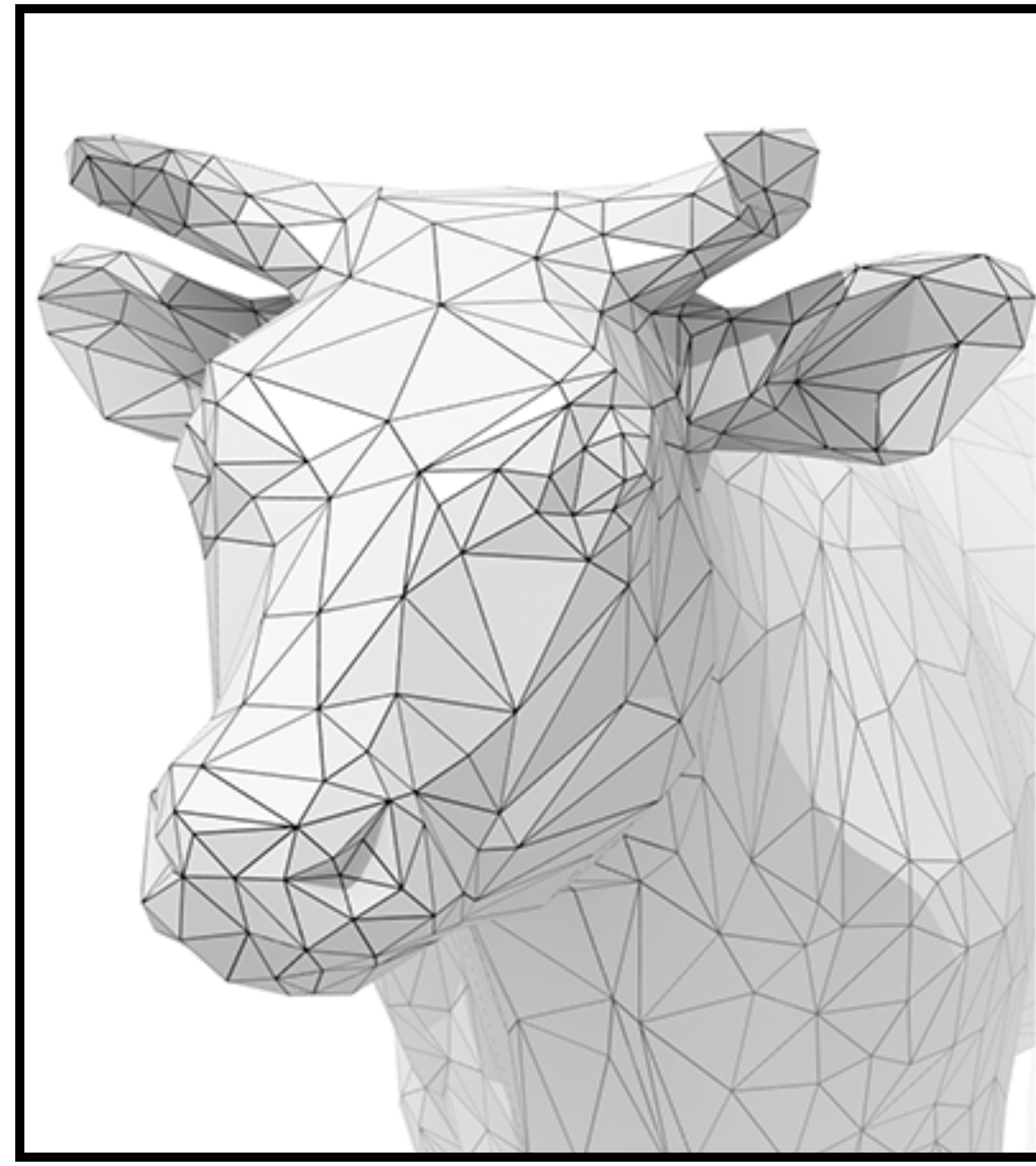




Increase resolution via interpolation

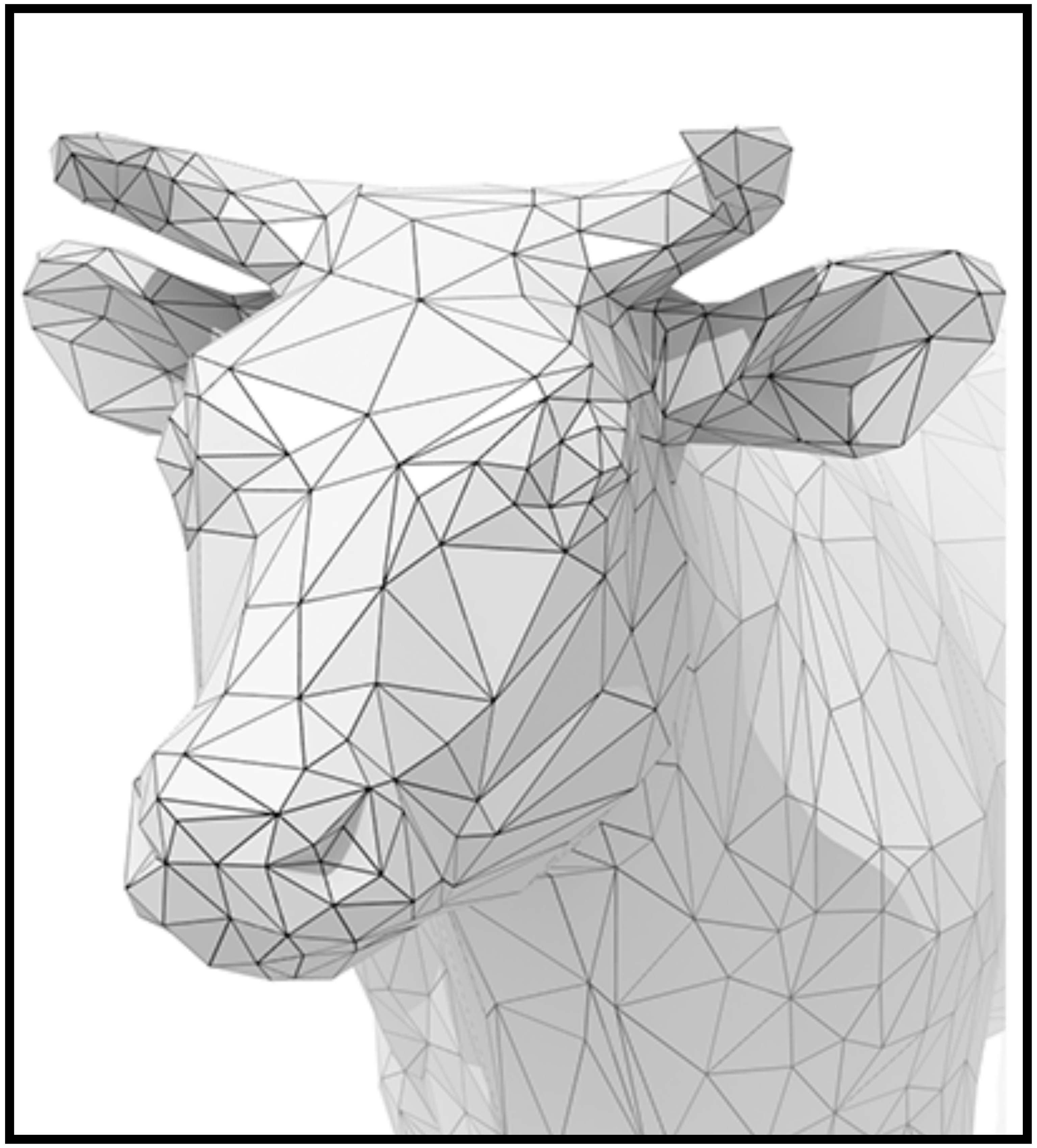
Mesh downsampling — simplification

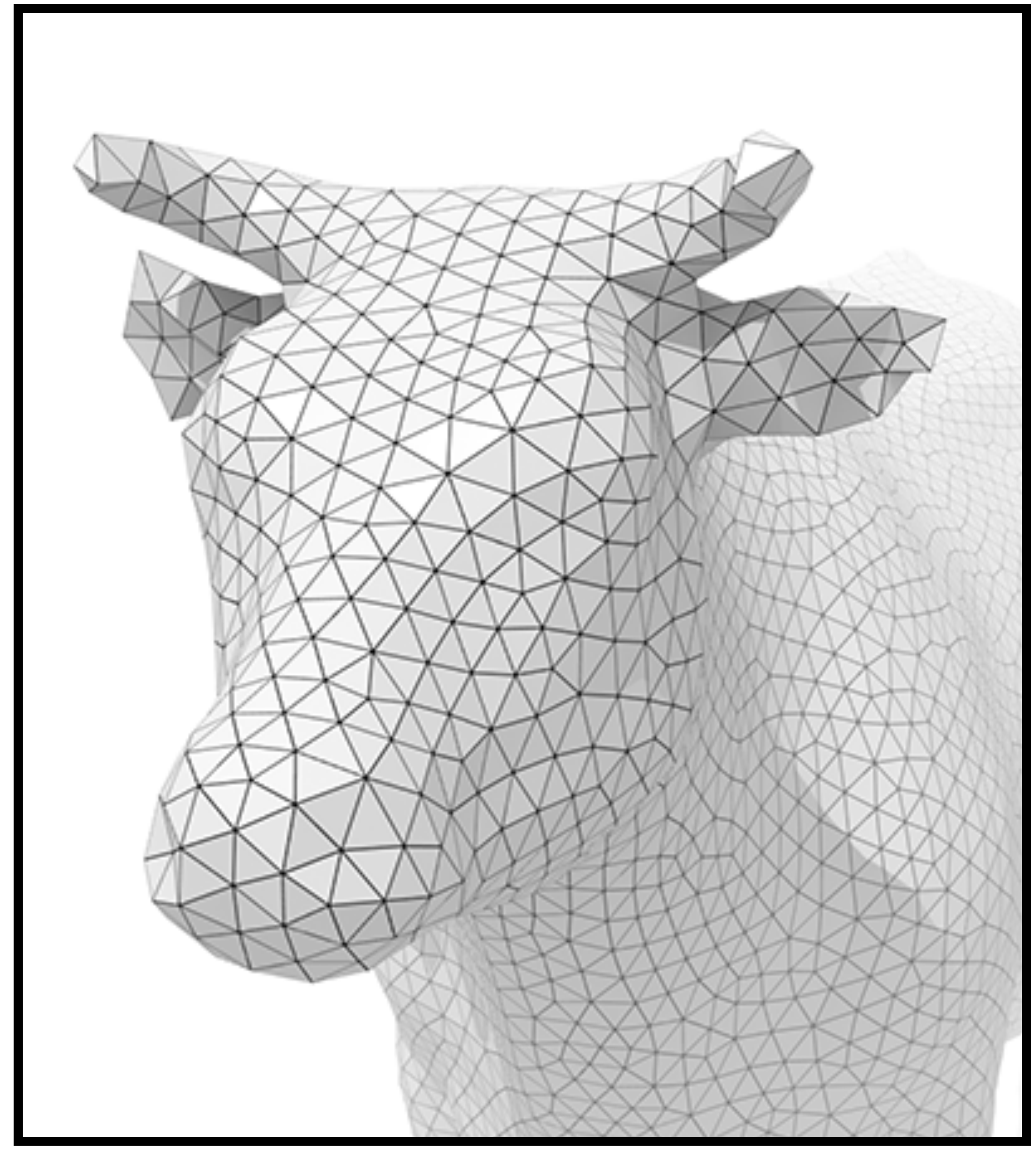




Decrease resolution; try to preserve shape/appearance

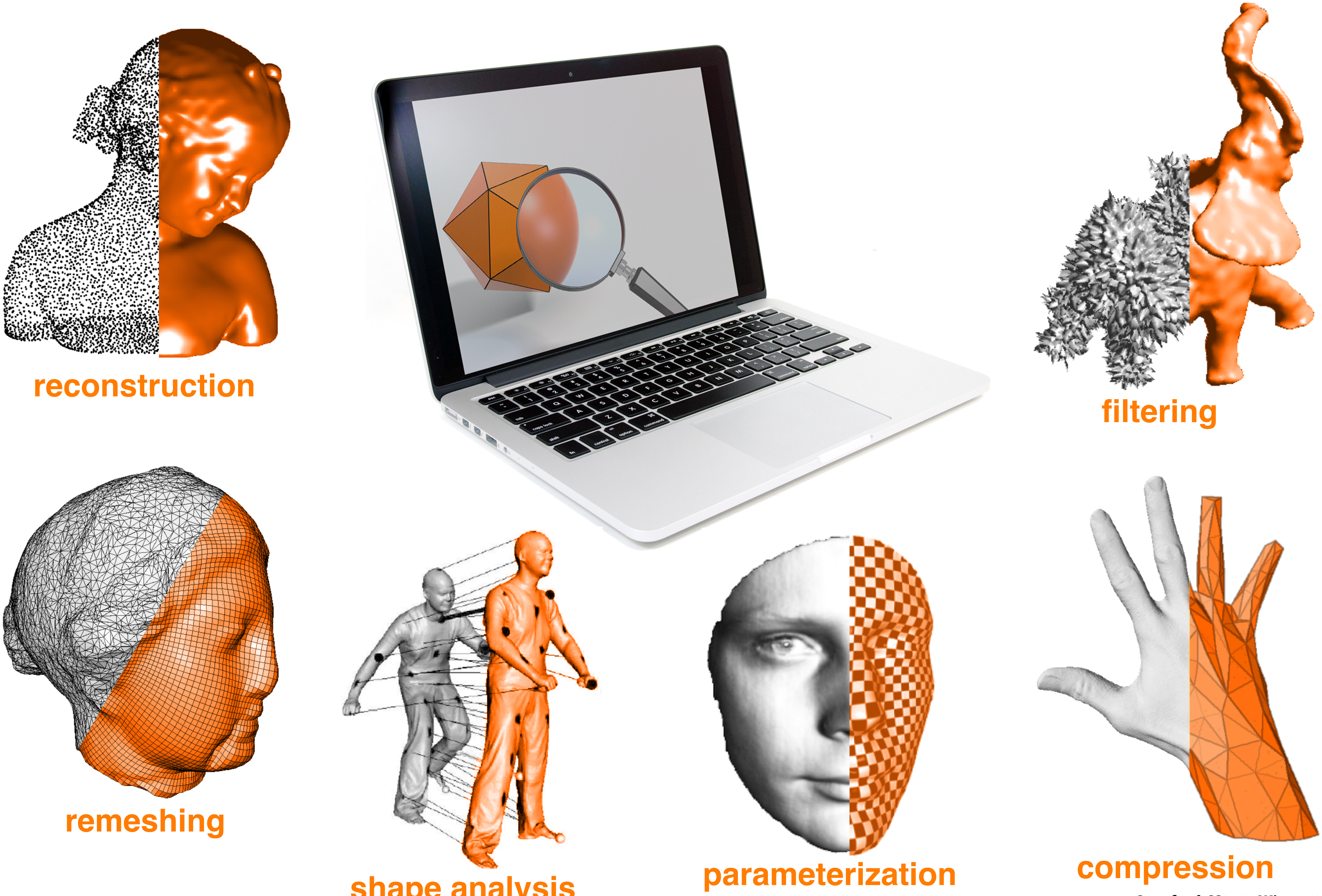
Mesh resampling — regularization





Modify sample distribution to improve quality

More geometry processing tasks



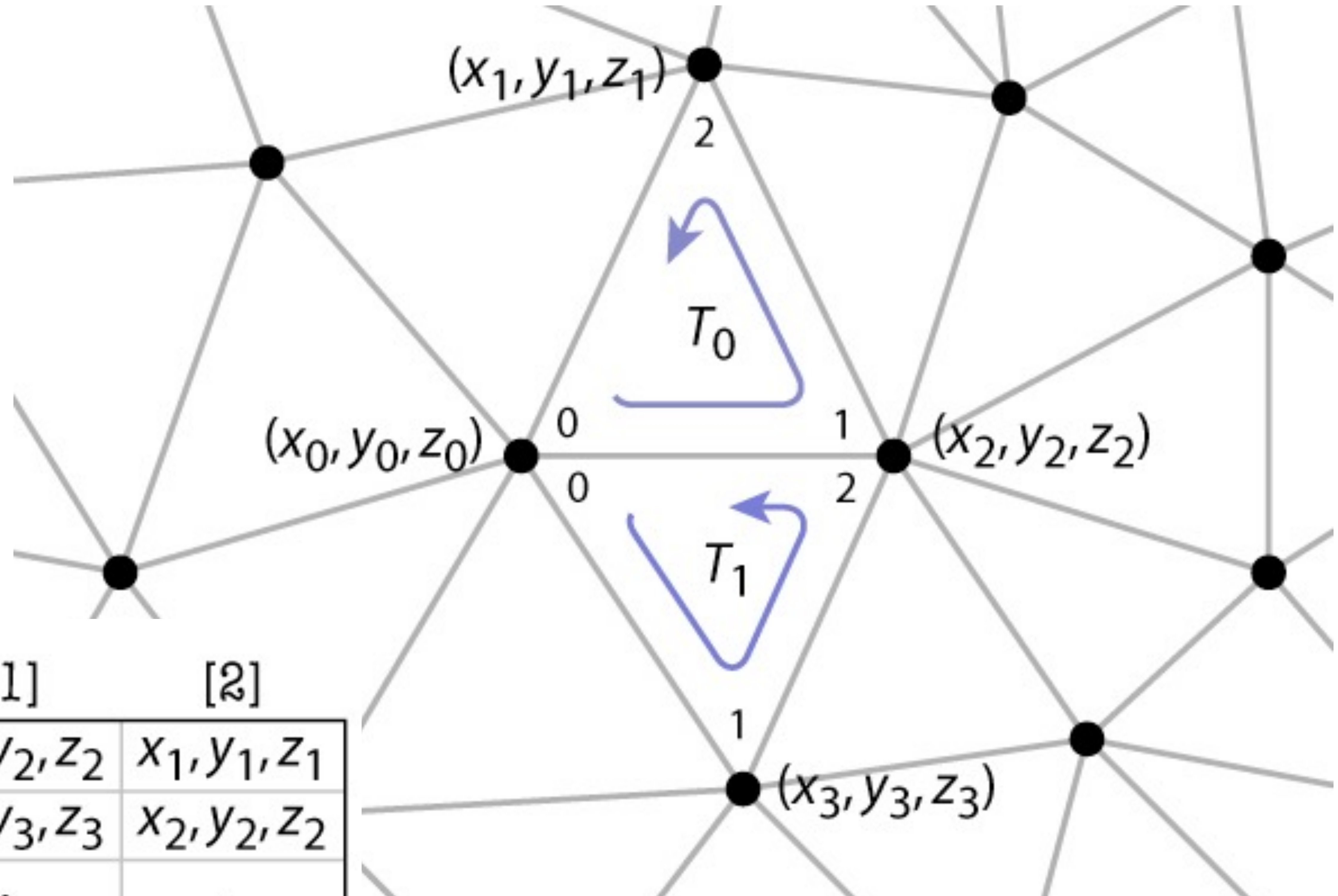
Today

- How to represent meshes (data structures)

- How to perform a number of basic mesh processing operations
  - Subdivision (upsampling)
  - Mesh simplification (downsampling)
  - Mesh resampling

Mesh representations

Basic mesh representation: list of triangles

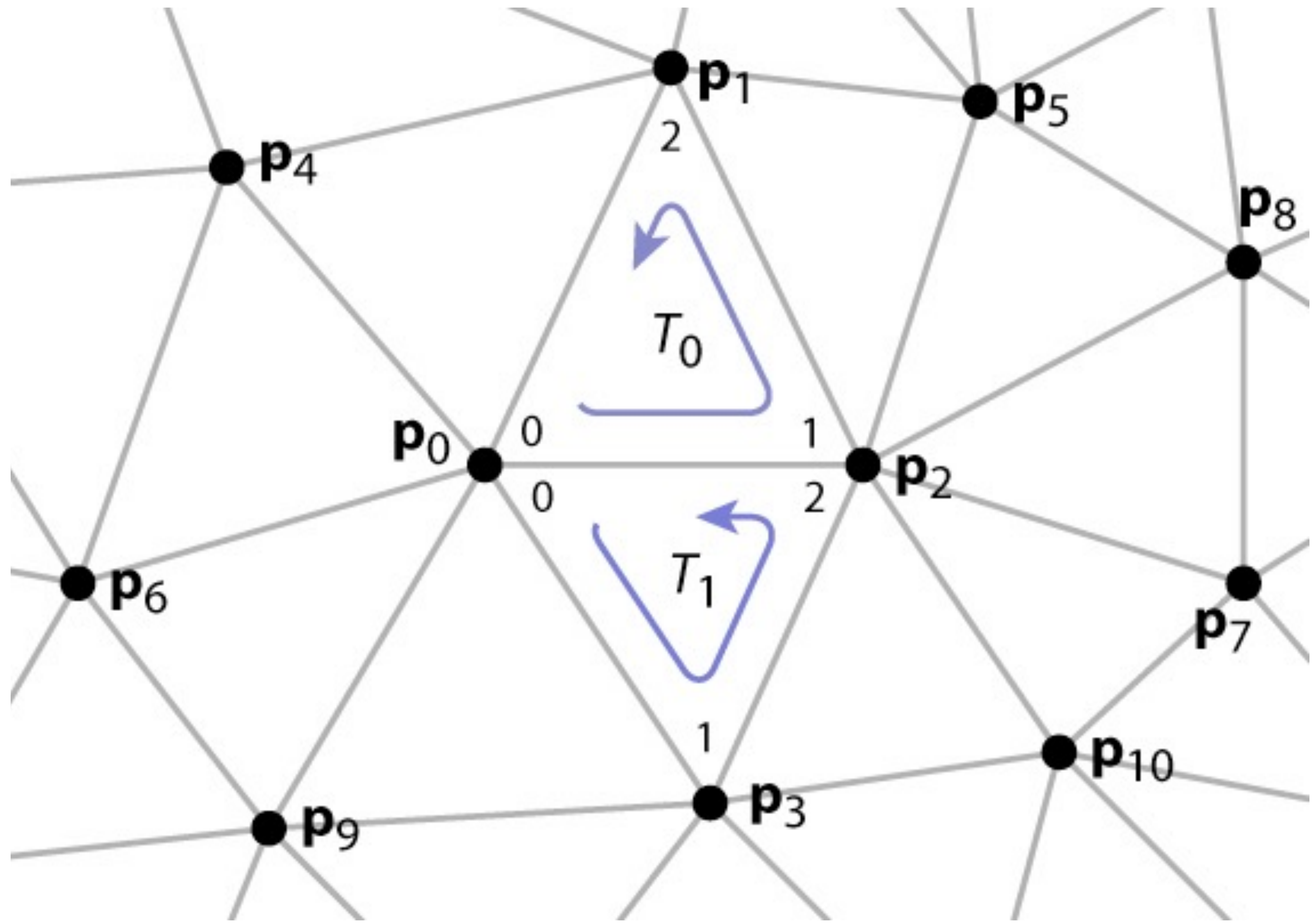


	[0]	[1]	[2]
tris[0]	x_0, y_0, z_0	x_2, y_2, z_2	x_1, y_1, z_1
tris[1]	x_0, y_0, z_0	x_3, y_3, z_3	x_2, y_2, z_2
	⋮	⋮	⋮

Another representation: Lists of vertexes / indexed triangle

verts[0]	x_0, y_0, z_0
verts[1]	x_1, y_1, z_1
	x_2, y_2, z_2
	x_3, y_3, z_3
	⋮

tInd[0]	0, 2, 1
tInd[1]	0, 3, 2
	⋮



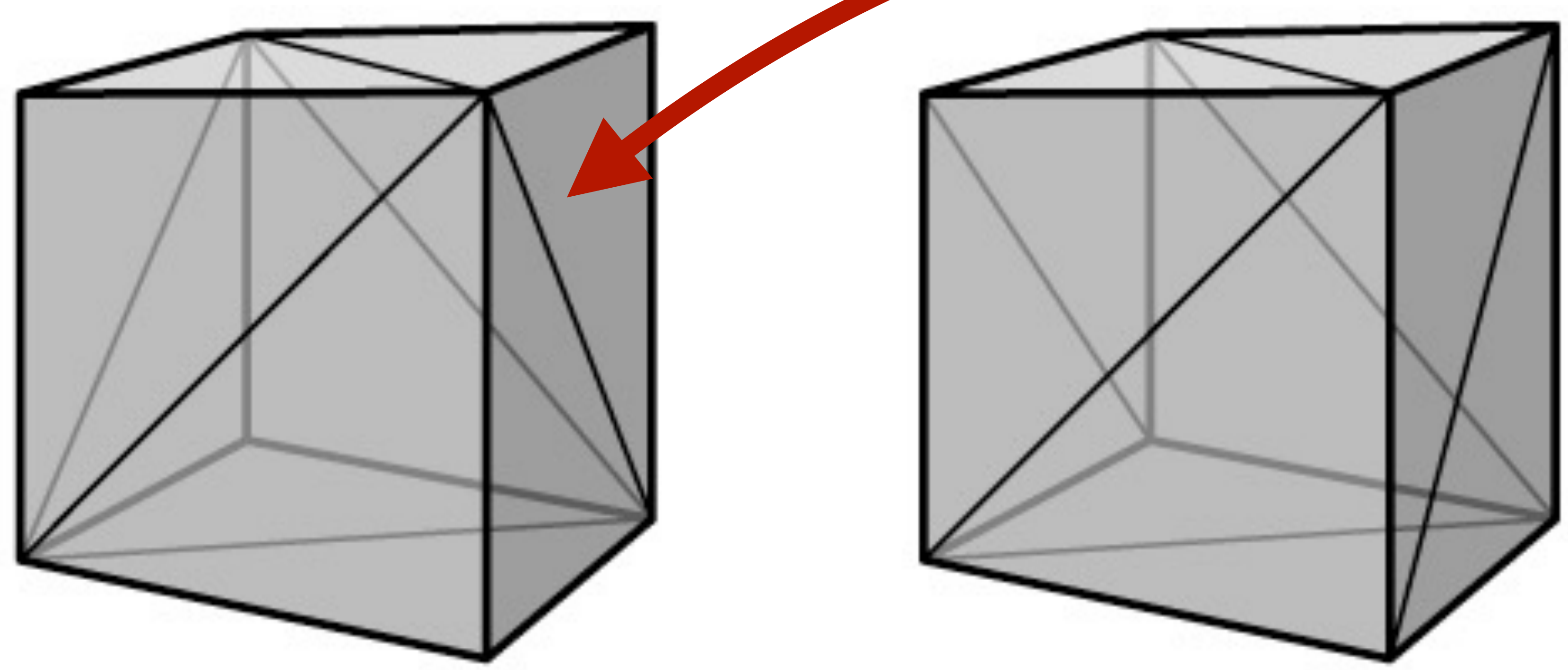
Comparison

- List of triangles
  - GOOD: simple
  - BAD: contains redundant per-vertex information

- List of vertexes + list of indexed triangles
  - GOOD: sharing vertex position information reduces memory usage
  - GOOD: ensures integrity of the mesh (changing a vertex's position in 3D space causes that vertex in all the polygons to move)

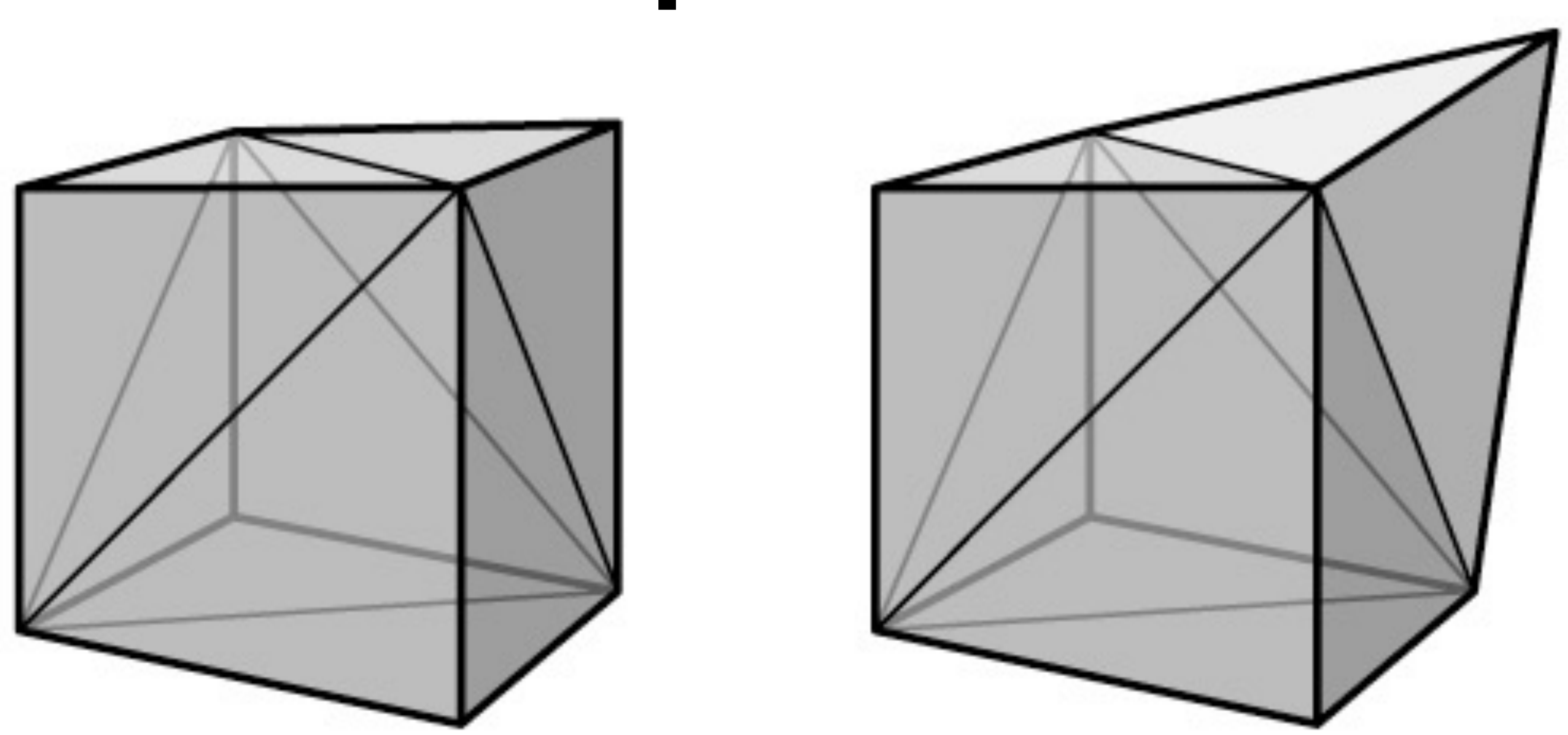
Mesh topology vs surface geometry

Same vertex positions, different mesh topology



Notice different connectivity

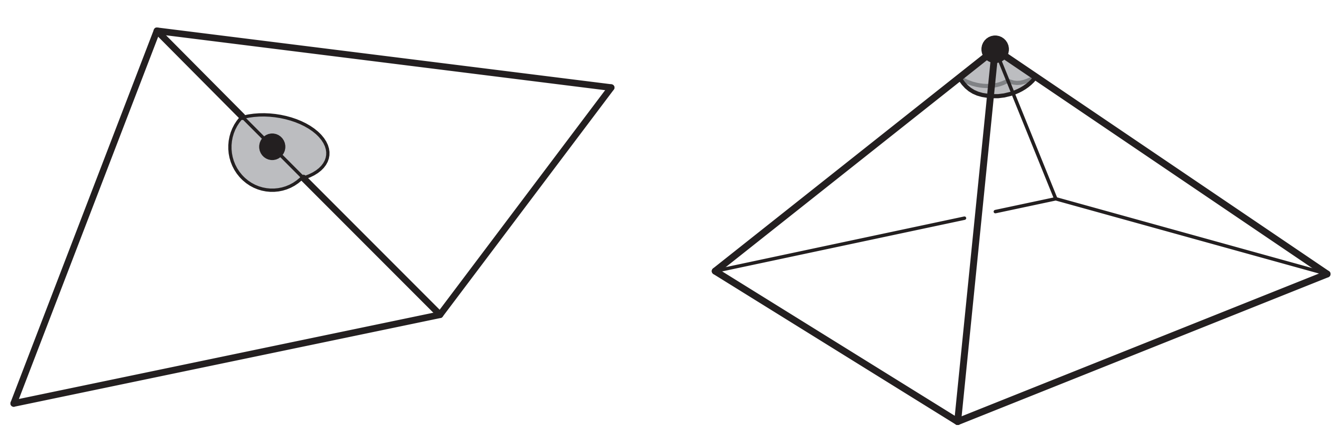
Same topology, different vertex positions



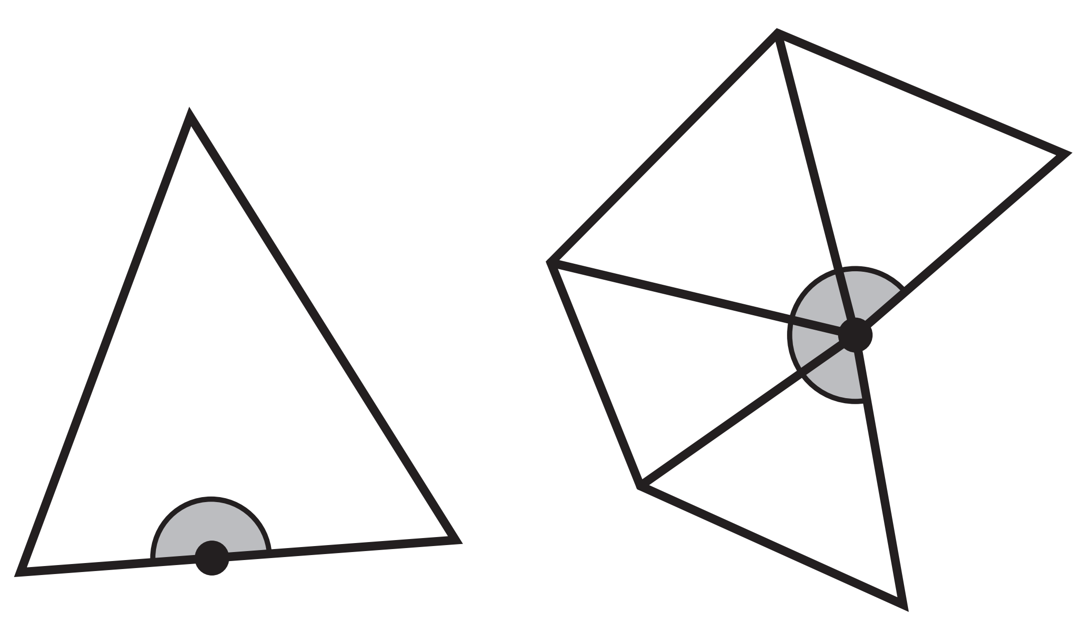
Topological validity: manifold

- **Recall, a 2D manifold is a surface that when cut with a small sphere always yields a disk (or a half disk on the boundary)**

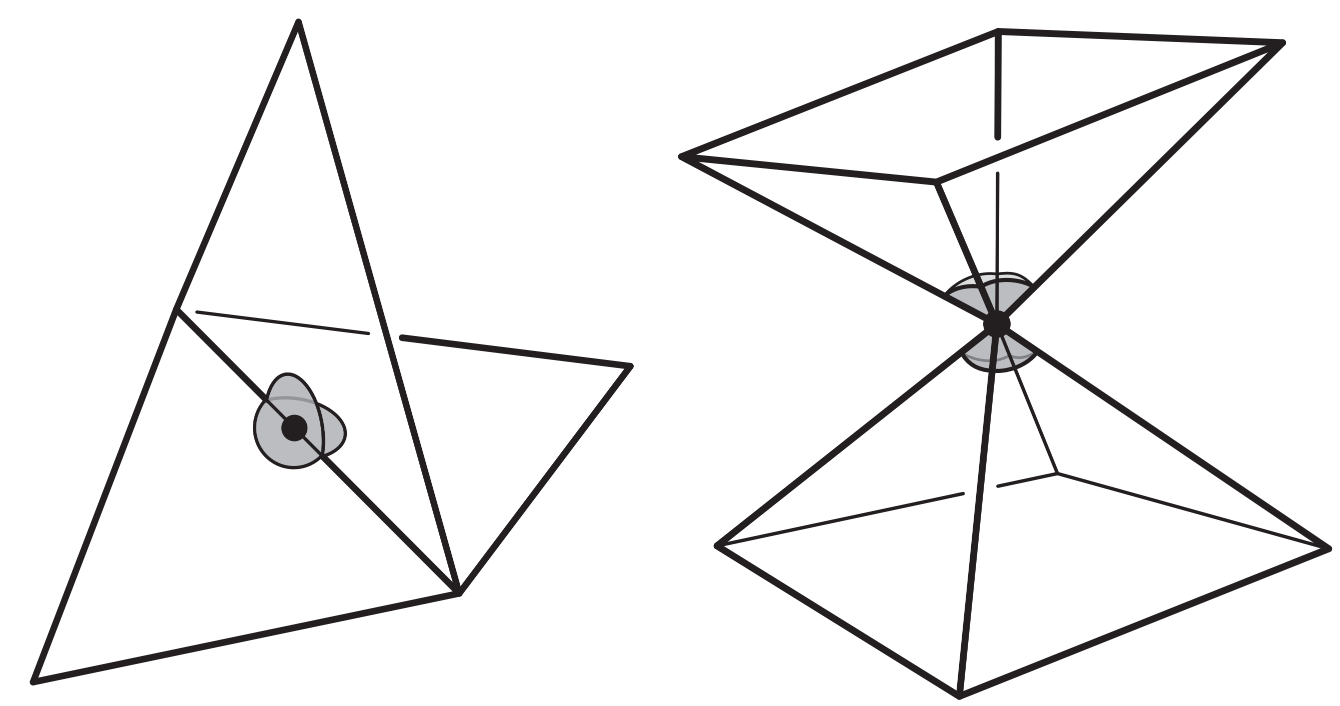
Manifold



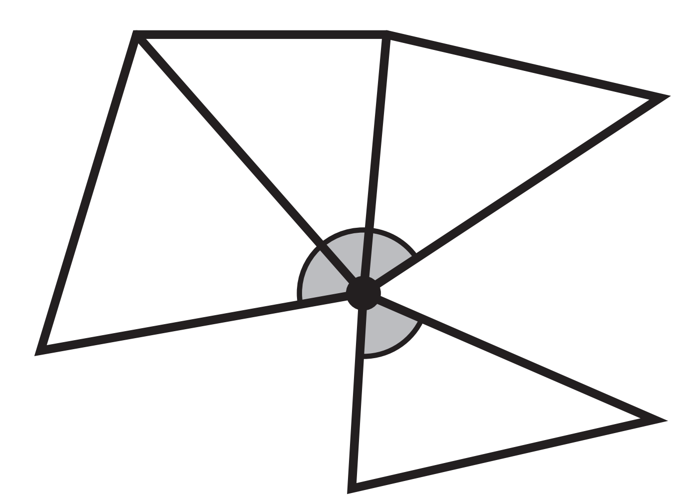
With border



Not manifold



With border



Manifolds have useful properties

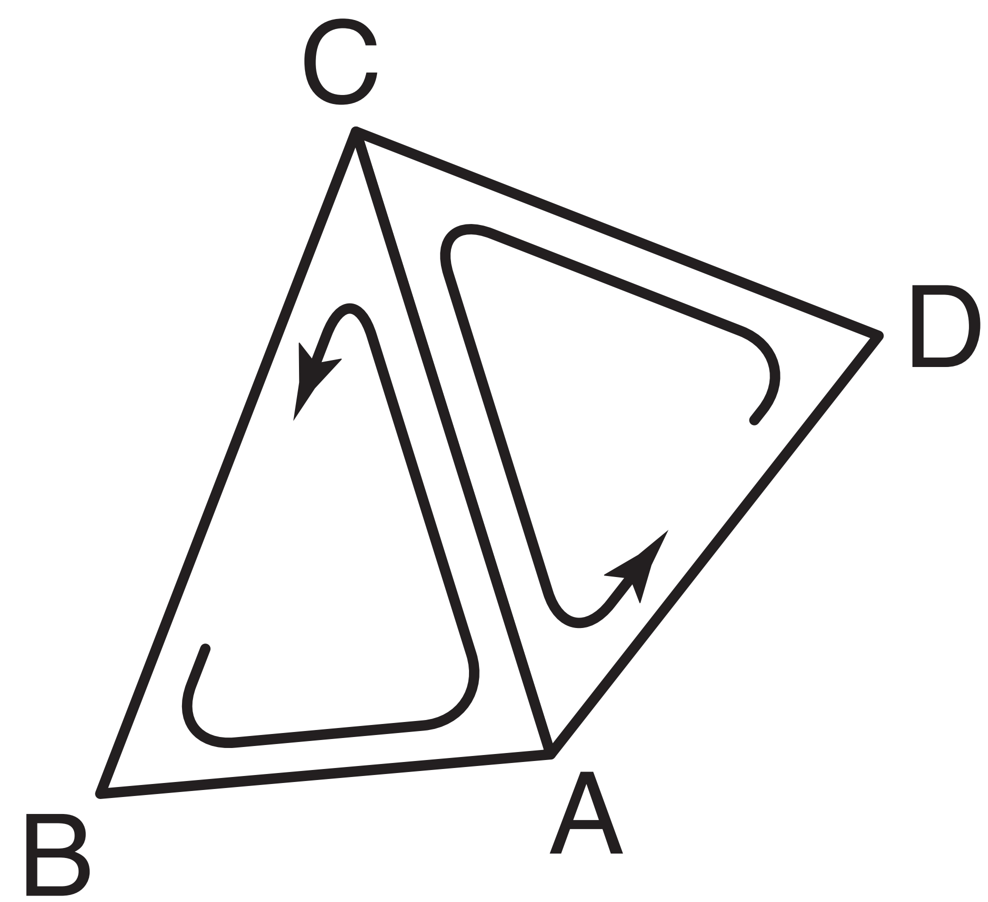
- A 2D manifold is a surface that when cut with a small sphere always yields a disk

- If a mesh is manifold, we can rely on these useful properties: *
  - An edge connects exactly two faces
  - An edge connects exactly two vertices
  - A face consists of a ring of edges and vertices
  - A vertex consists of a ring of edges and faces
  - Euler's polyhedron formula holds: #f – #e + #v = 2
    (for a surface topologically equivalent to a sphere)
    (Check for a cube: 6 – 12 + 8 = 2)

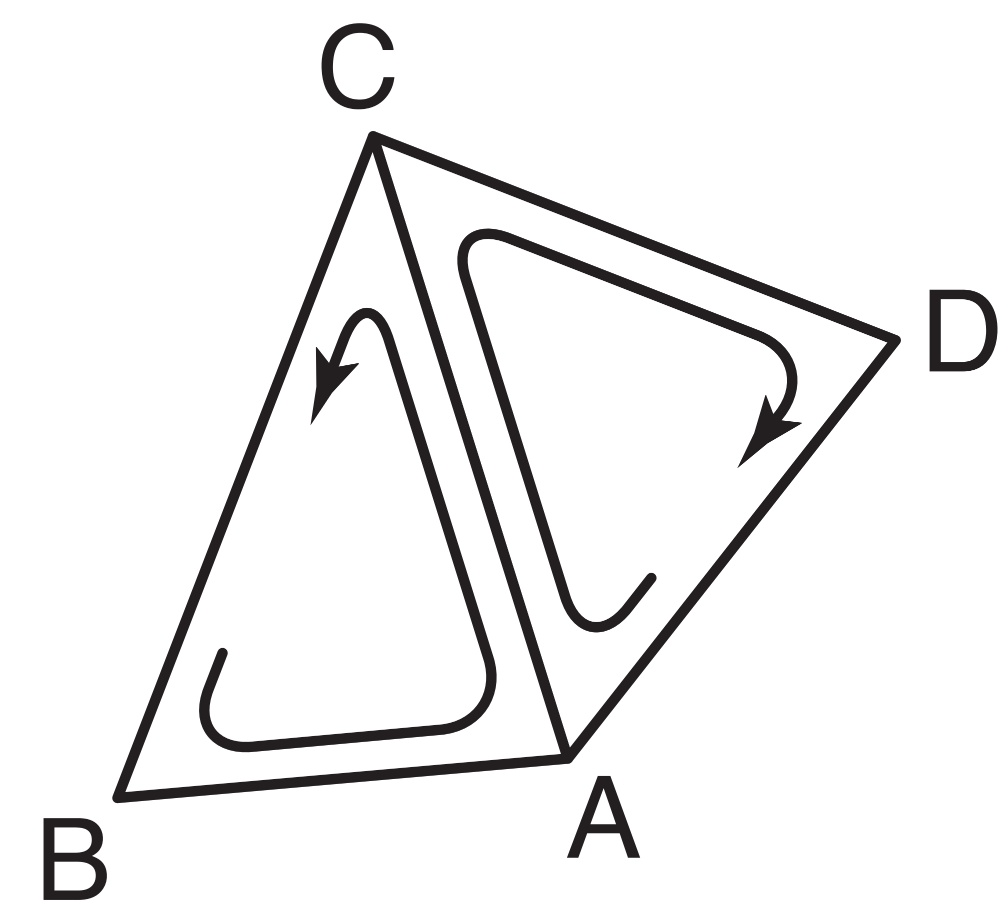
* Some of these properties only apply to non-border mesh regions

Topological validity: orientation consistency

Both facing front



Inconsistent orientations



Non-orientable (e.g., Moebius strip)

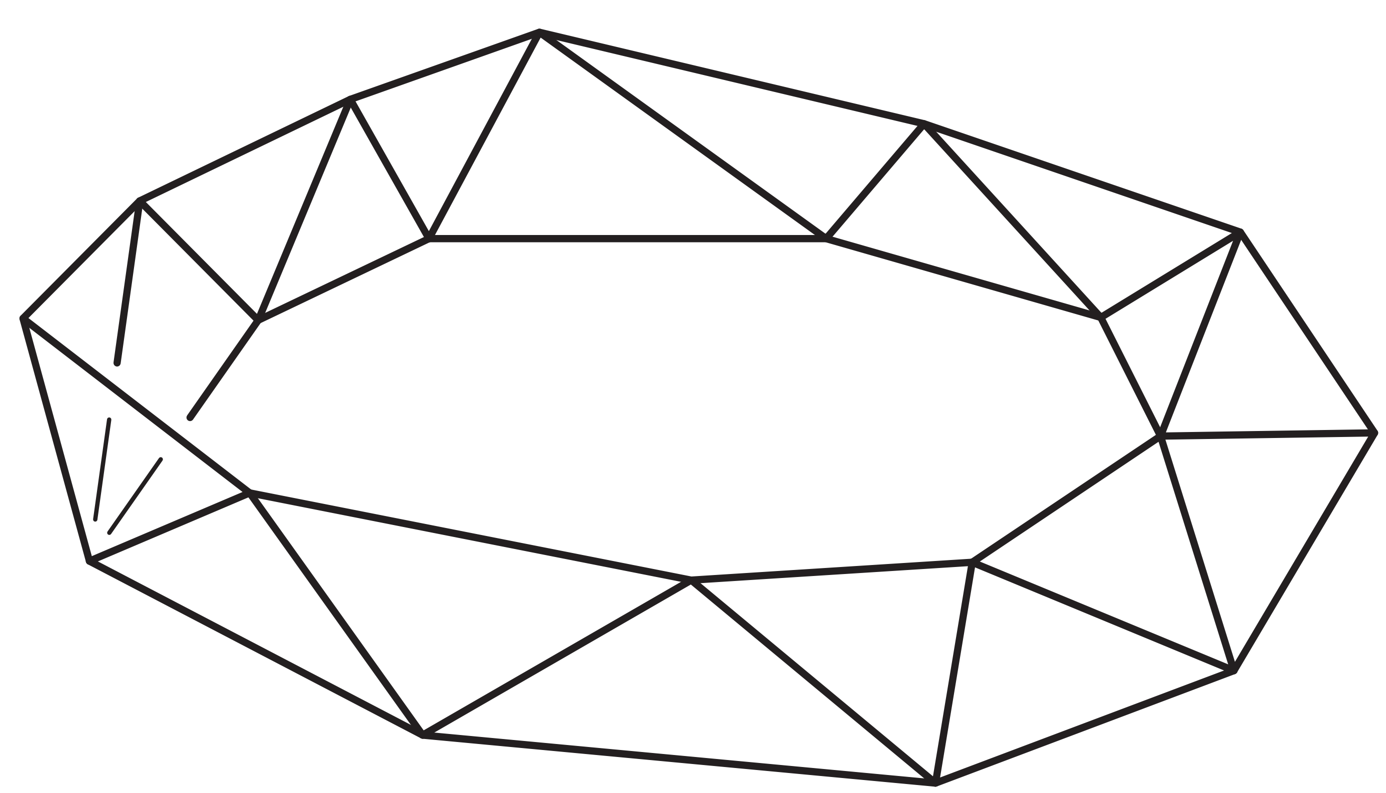
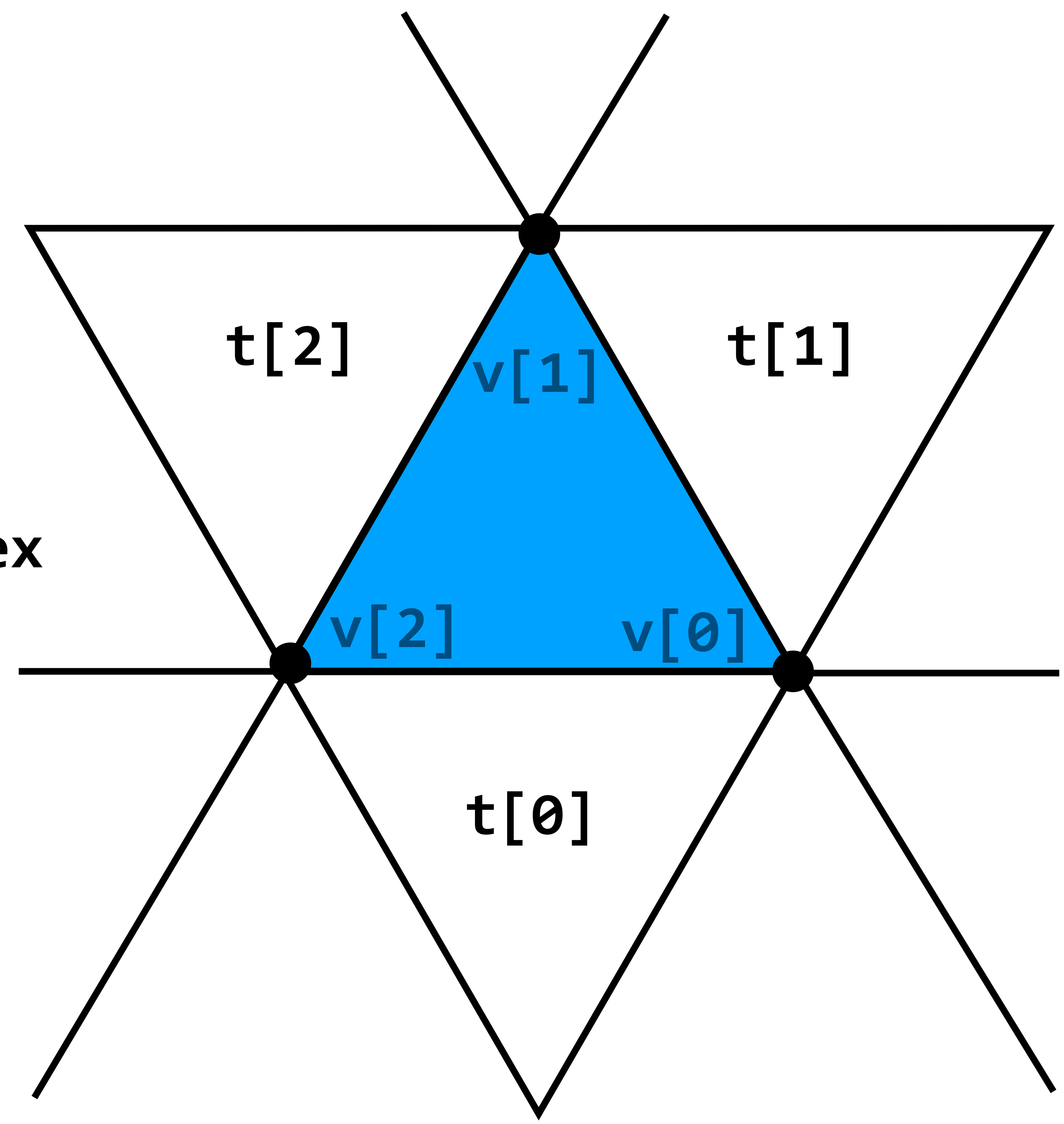


Image credit: Wikipedia

Simple example: triangle-neighbor data structure

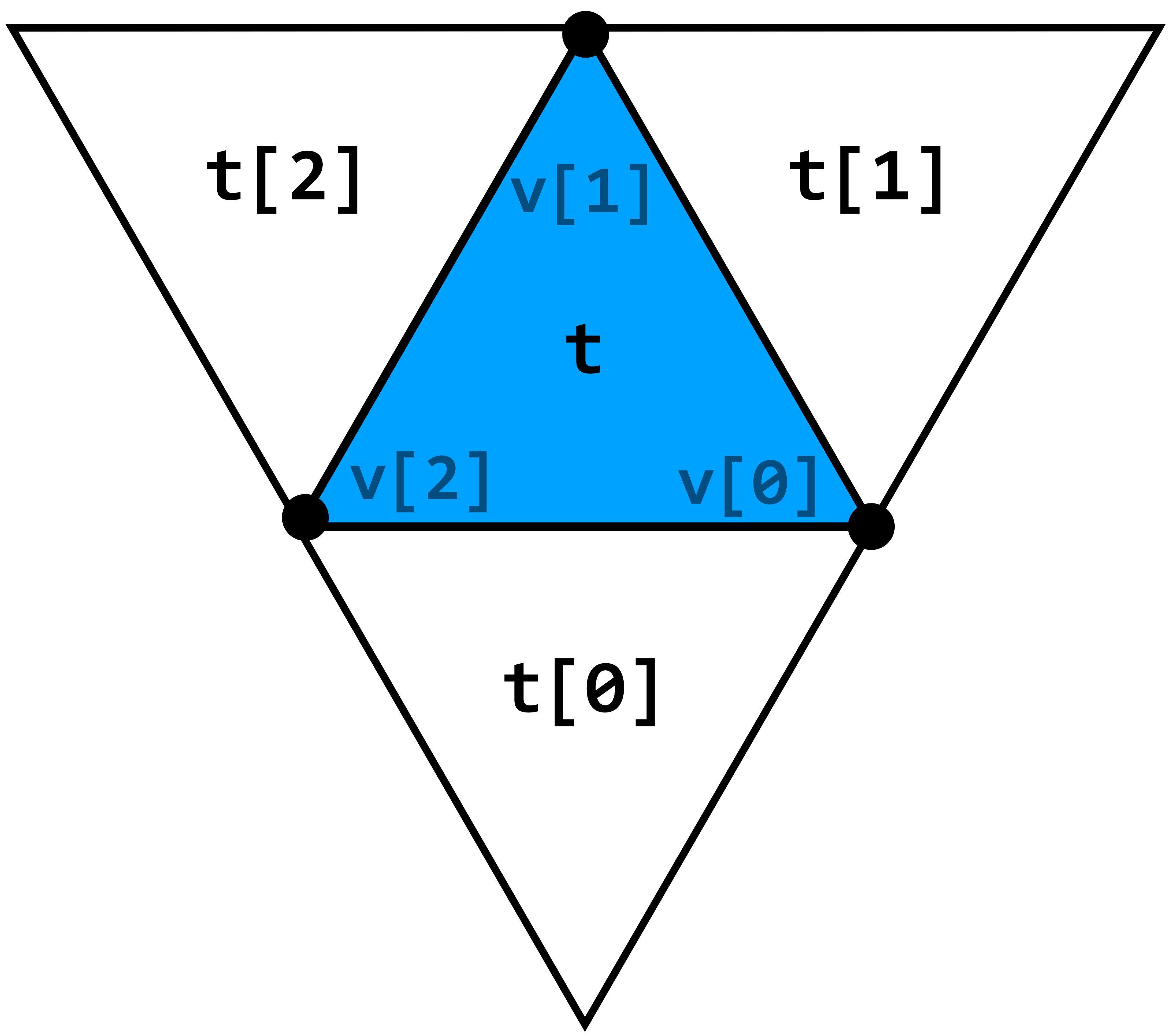
```
// definition of a triangle
struct Tri {
    Vert*  v[3];
    Tri*   t[3];
}

// definition of a triangle vertex
struct Vert {
    Vec3   pos;
    Tri*   t;
}
```



Triangle-neighbor – mesh traversal

Find next triangle counter-clockwise around vertex v from triangle t

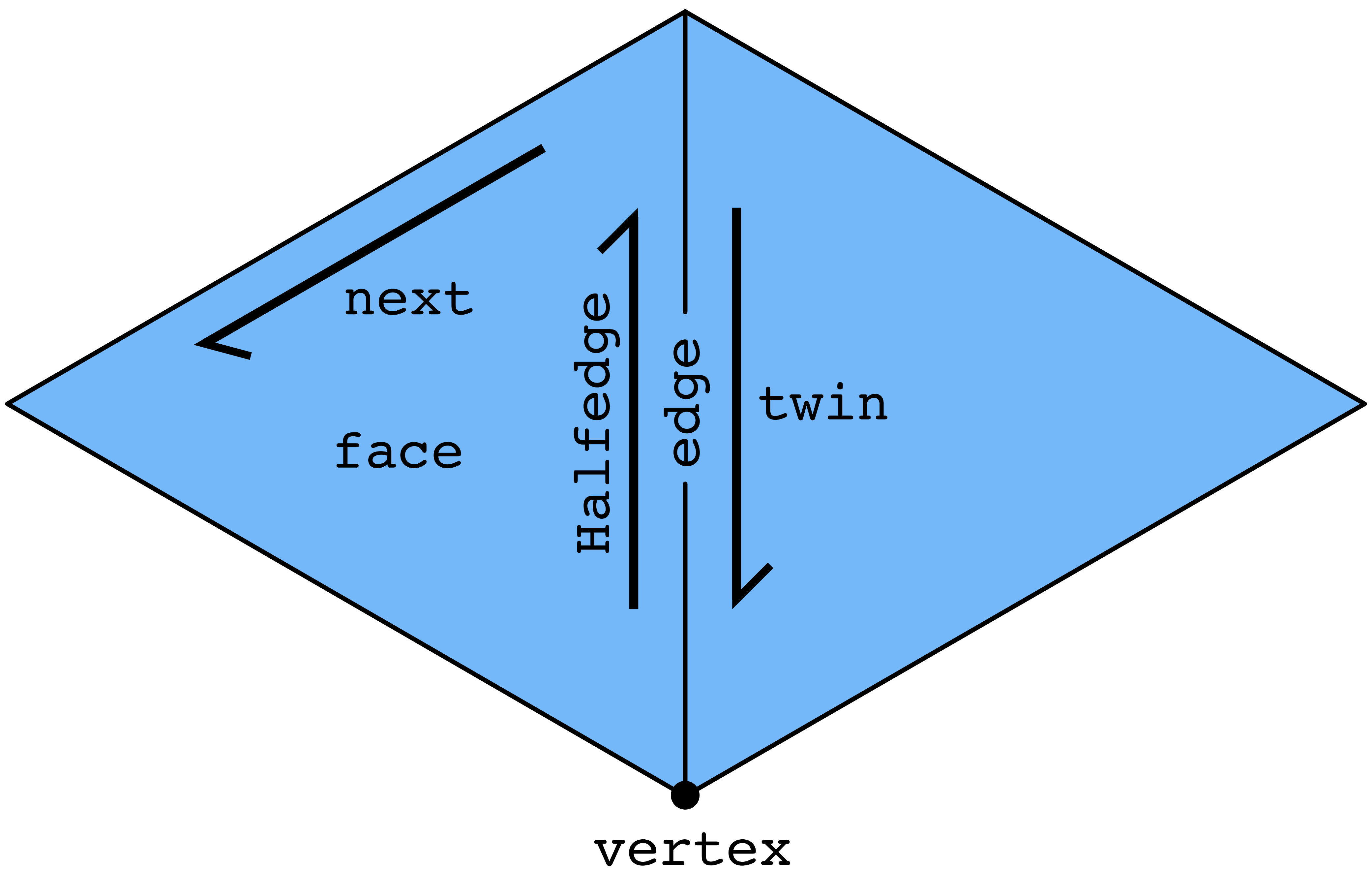
```
Tri* ccw_tri(Vert *v, Tri *t)
{
   if (v == t->v[0])
      return t[0];
   if (v == t->v[1])
      return t[1];
   if (v == t->v[2])
      return t[2];
}
```



Half-edge data structure

```
struct Halfedge {
   Halfedge *twin,
   Halfedge *next;
   Vertex *vertex;
   Edge *edge;
   Face *face;
}
struct Vertex {
   Vec3 pos;
   Halfedge *halfedge;
}
struct Edge {
   Halfedge *halfedge;
}
struct Face {
   Halfedge *halfedge;
}
```

Key idea: two half-edges act as "glue" between mesh elements



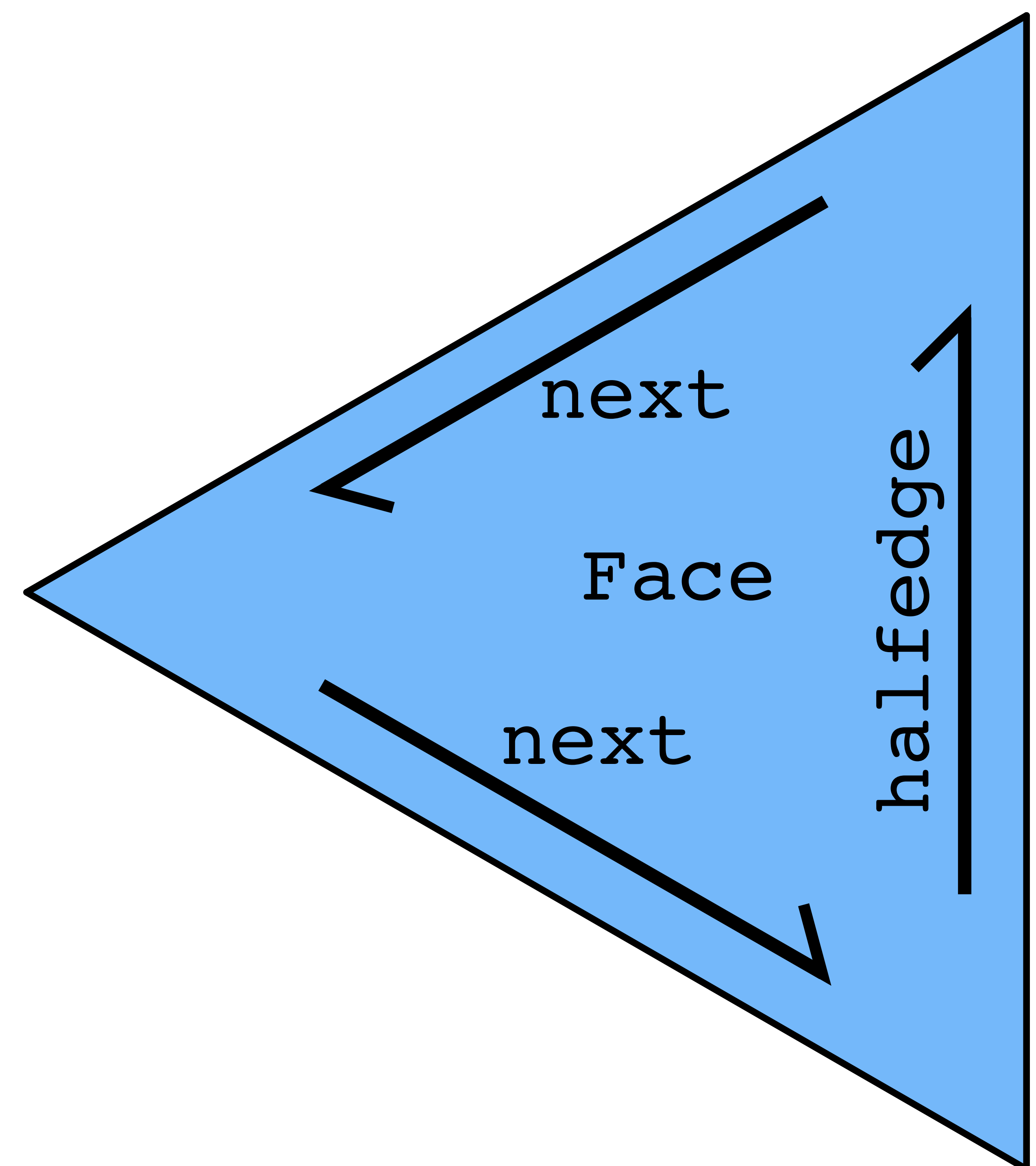
Each vertex, edge and face points to one of its half edges

Half-edge structure facilitates mesh traversal

- Use twin and next pointers to move around mesh
- Process vertex, edge, and/or face pointers

Example 1: process all vertices of a face

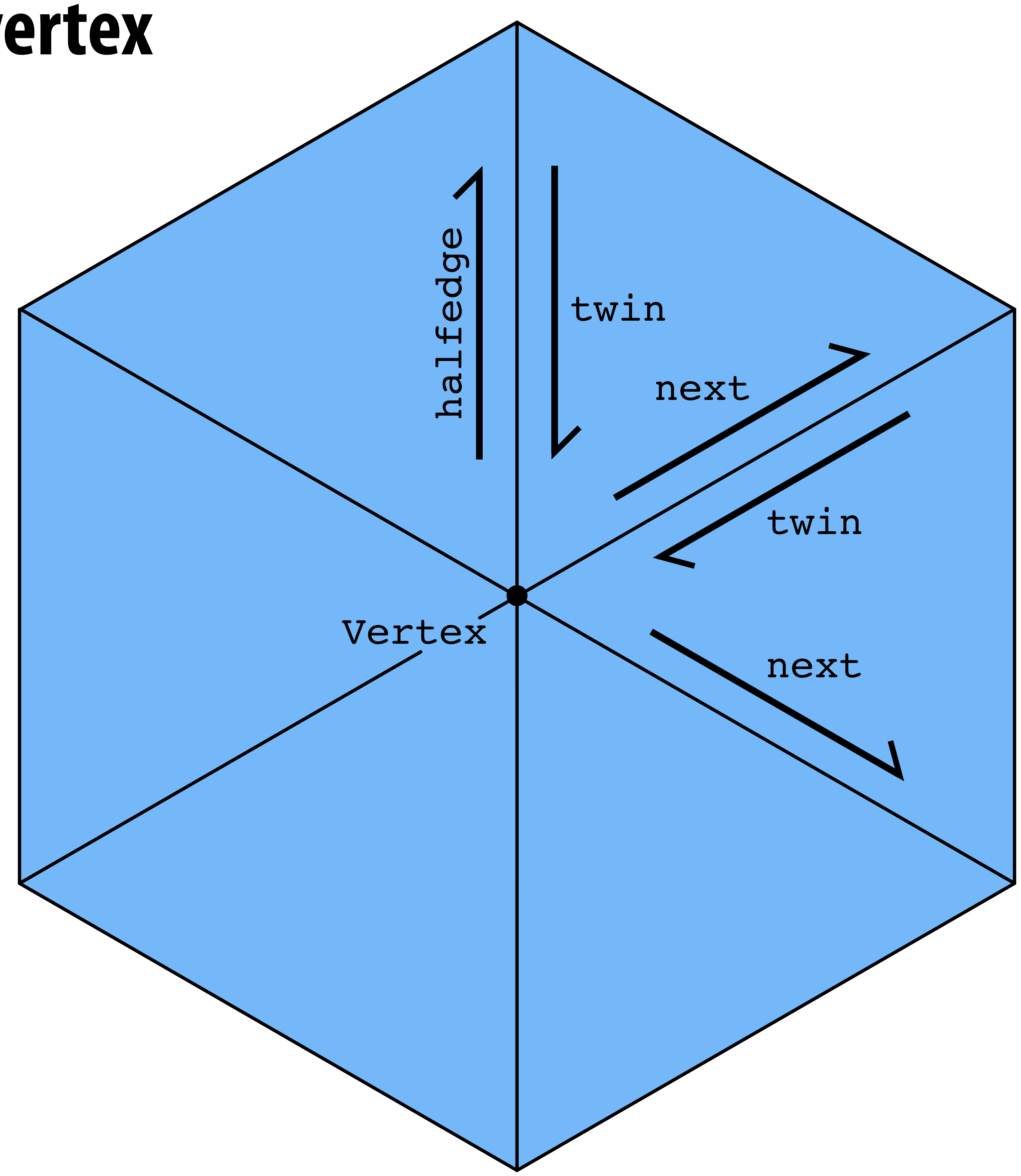
```
Halfedge* h = f->halfedge;
do {
   do_work(h->vertex);
   h = h->next;
}
while( h != f->halfedge );
```



Half-edge structure facilitates mesh traversal

Example 2: process all edges around a vertex

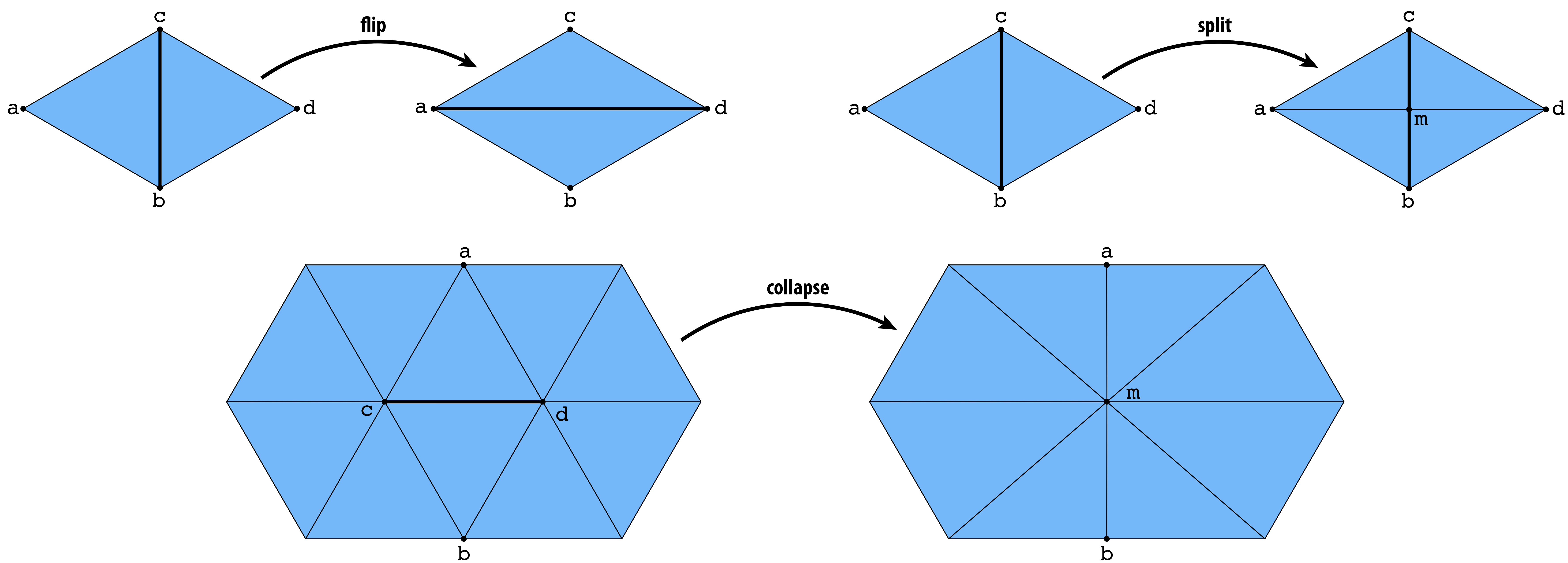
```
Halfedge* h = v->halfedge;
do {
   do_work(h->edge);
   h = h->twin->next;
}
while( h != v->halfedge );
```



Local mesh operations

Half-Edge – local mesh editing

- **Consider basic operations for linked list: insert, delete**
- **Basic ops for half-edge mesh: flip, split, collapse edges**

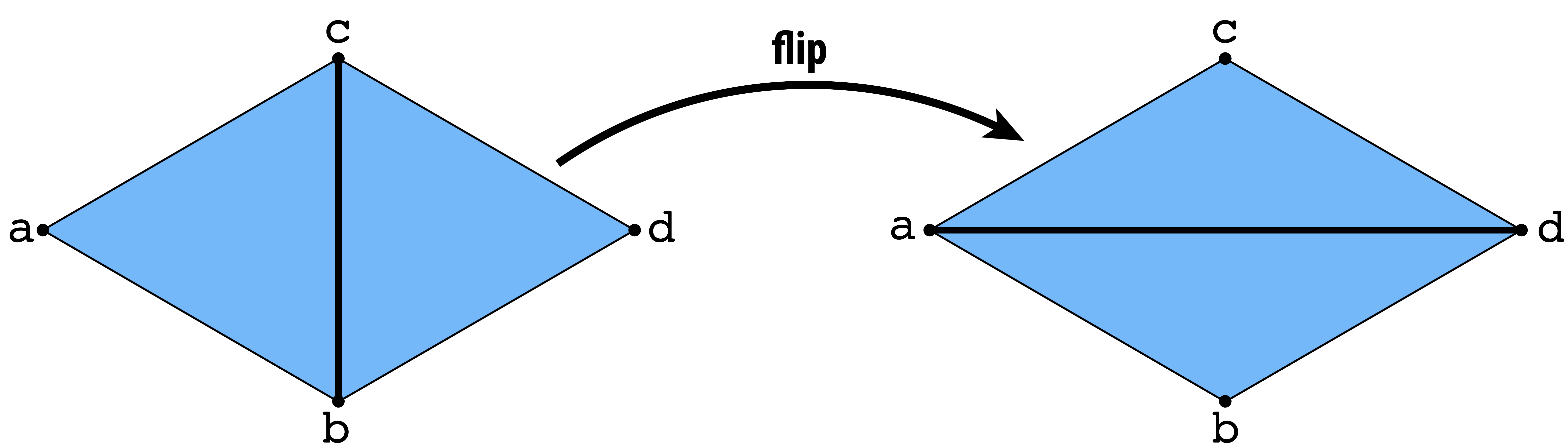


Allocate / delete elements; reassign pointers

(Care is needed to preserve mesh manifold property)

Half-edge – edge flip

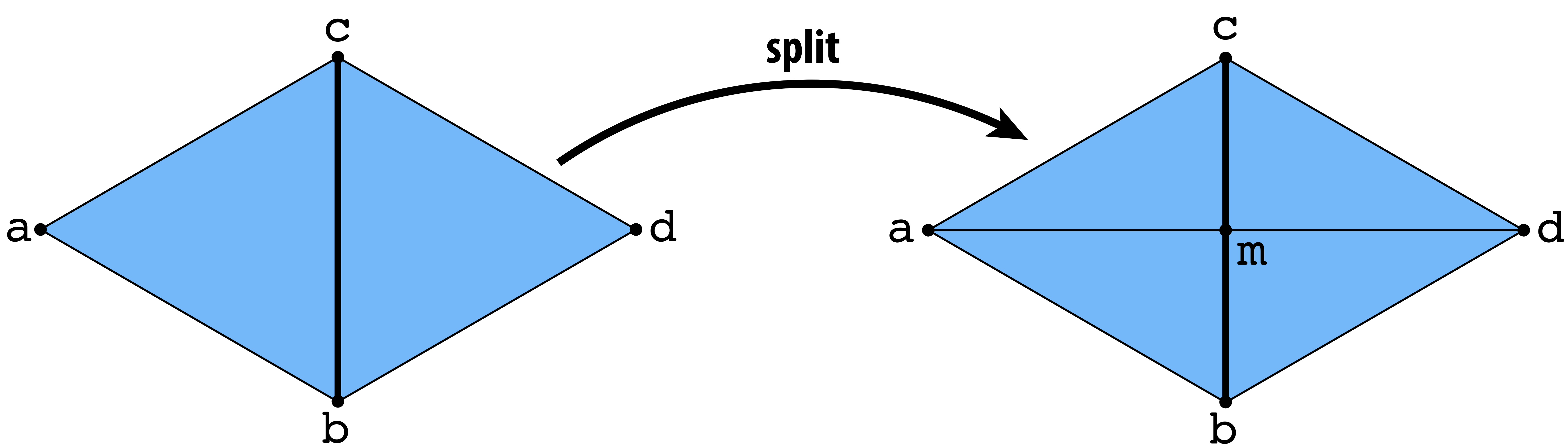
Triangles (a,b,c), (b,d,c) become (a,d,c), (a,b,d):



- **Long list of half-edge pointer reassignments**
- **However, no mesh elements created/destroyed**

Half-edge – edge split

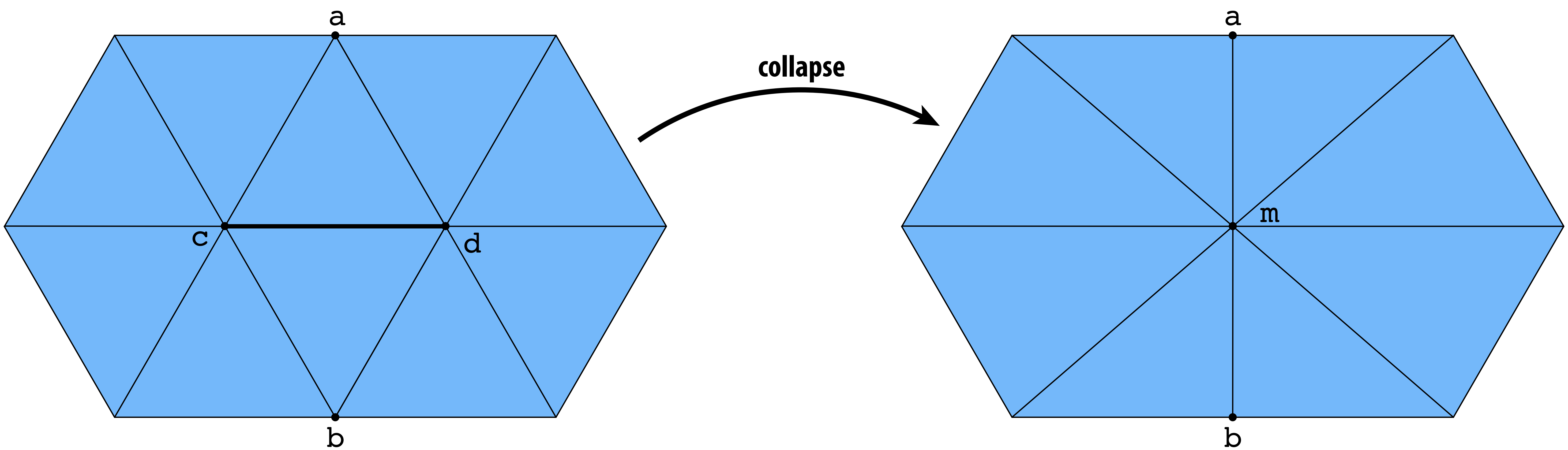
Insert midpoint m of edge (c,b), connect to get four triangles:



- **Must add elements to mesh (new vertex, faces, edges)**
- **Again, many half-edge pointer reassignments**

Half-edge – edge collapse

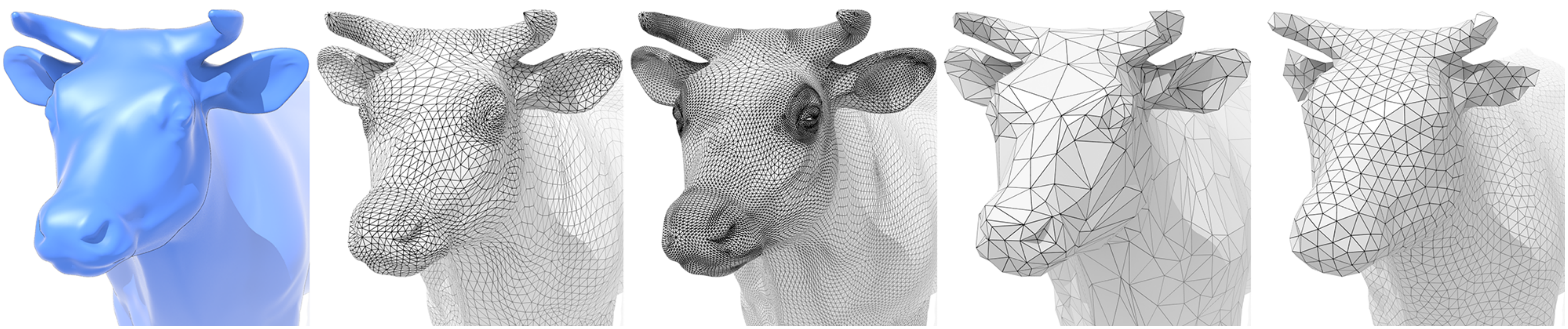
Replace edge (c,d) with a single vertex m:



- Must delete elements from the mesh
- Again, many half-edge pointer reassignments

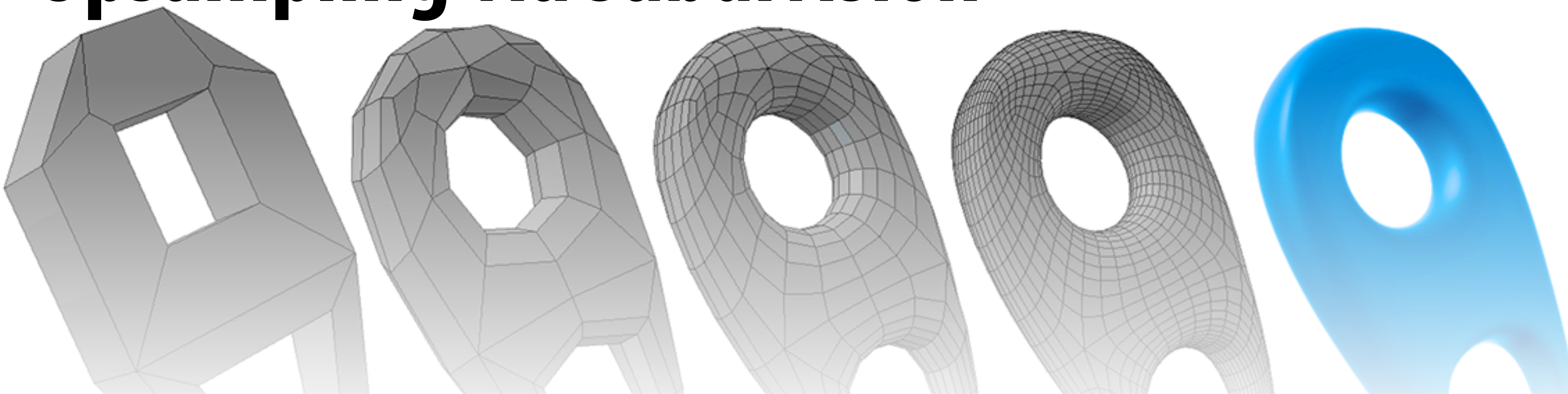
Global mesh operations: geometry processing

- Mesh subdivision (form of subsampling)
- Mesh simplification (form of downsampling)
- Mesh regularization (form of resampling)

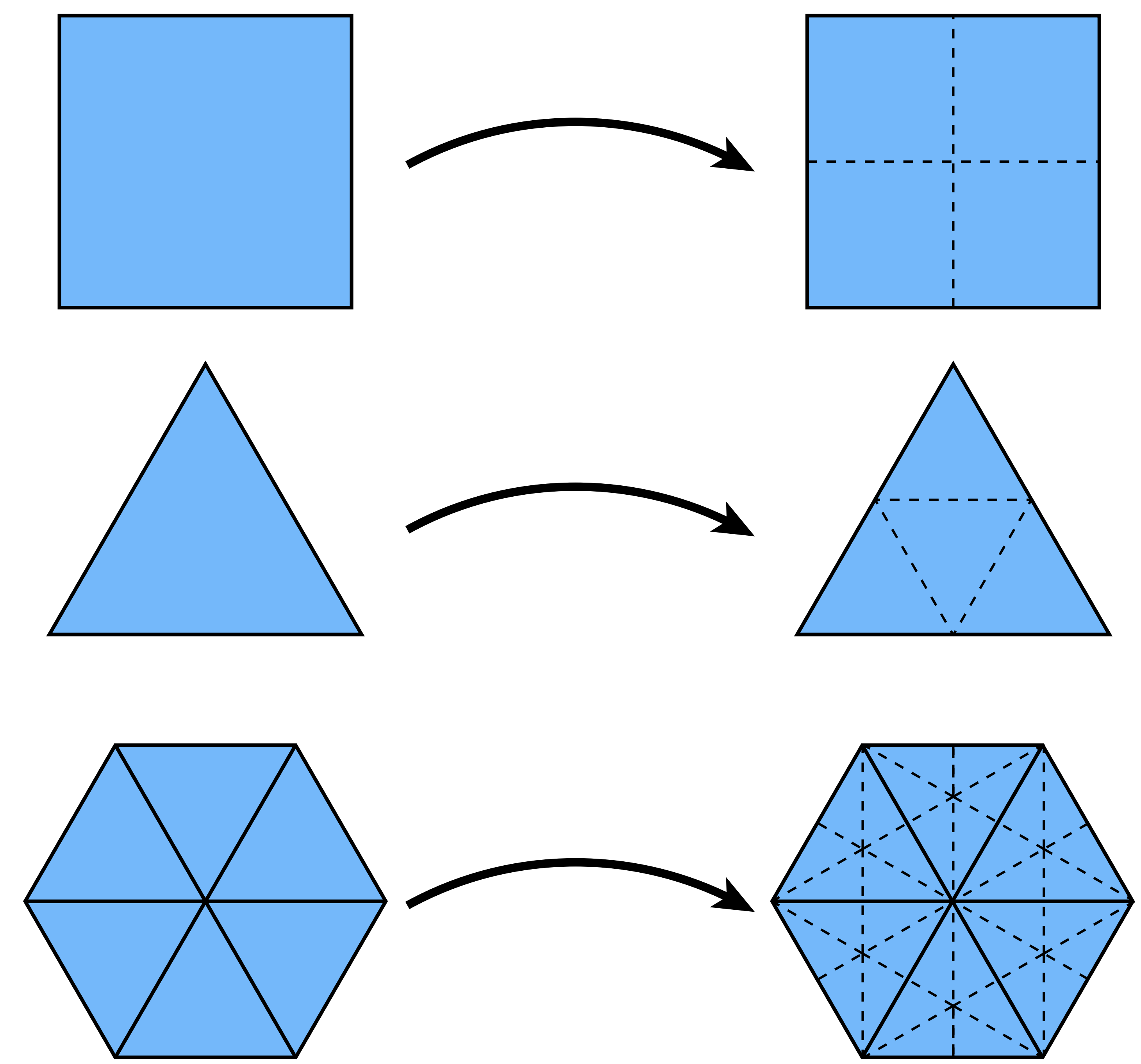


Upsampling a mesh — subdivision

Upsampling via subdivision



- Repeatedly split each element into smaller pieces
- Replace vertex positions with weighted average of neighbors
- Main considerations:
  - interpolating vs. approximating
  - limit surface continuity (C^1, C^2, ...)
  - behavior at irregular vertices
- Many options:
  - Quad: Catmull-Clark
  - Triangle: Loop, butterfly, sqrt(3)

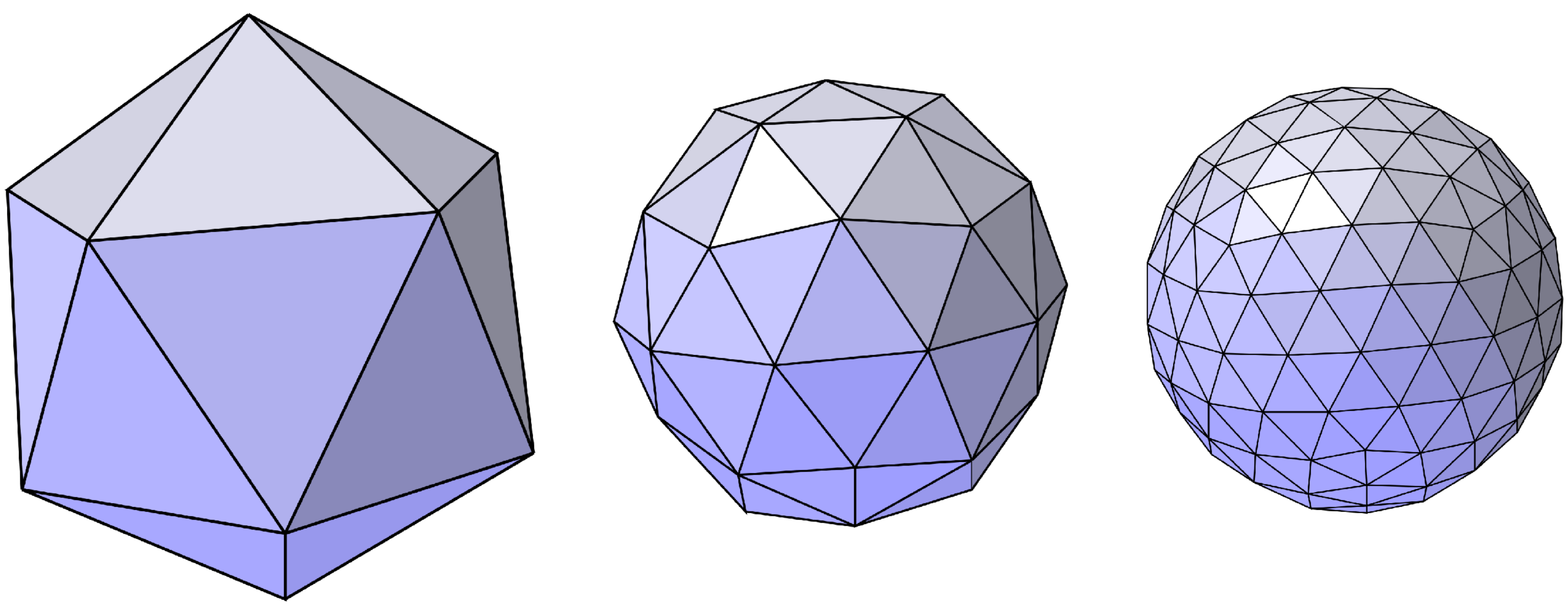


Loop subdivision

Common subdivision rule for triangle meshes

"C2" smoothness away from irregular vertices

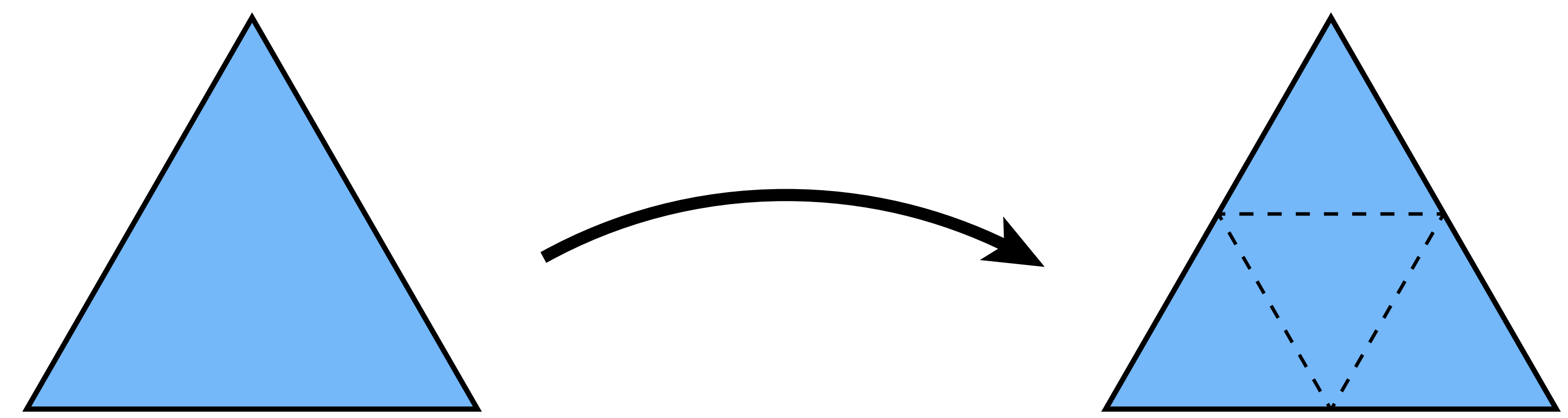
Approximating, not interpolating



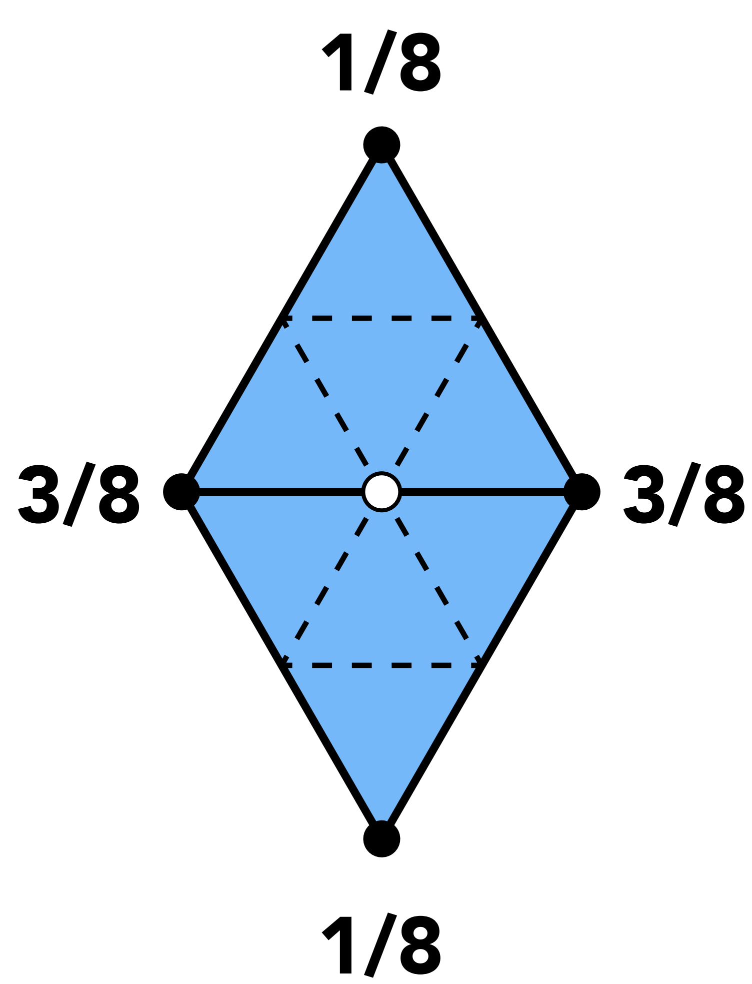
Simon Fuhrman

Loop subdivision algorithm

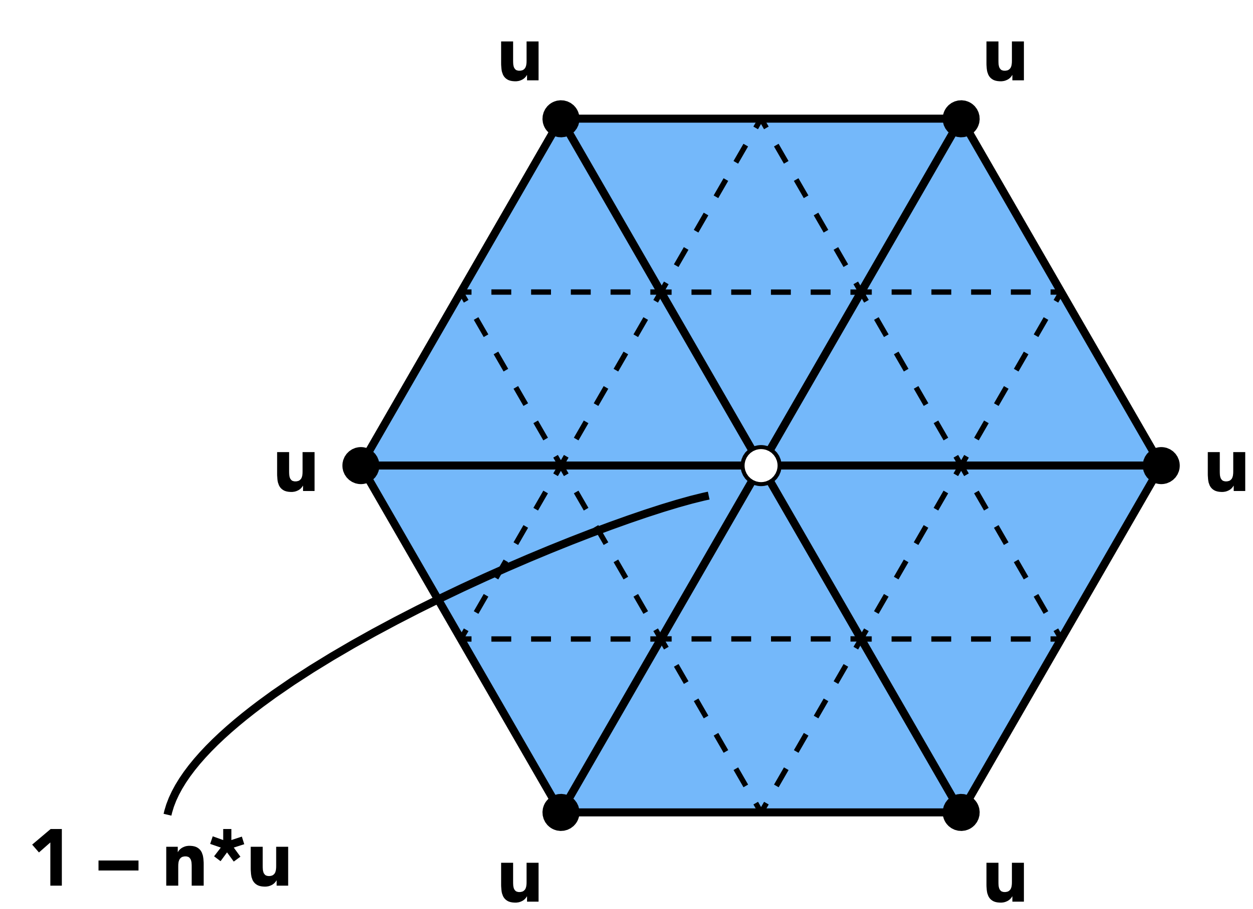
- **Split each triangle into four**



- **Compute new vertex positions using weighted sum of prior vertex positions:**



New vertices
(weighted sum of vertices on split edge, and vertices “across from” edge)



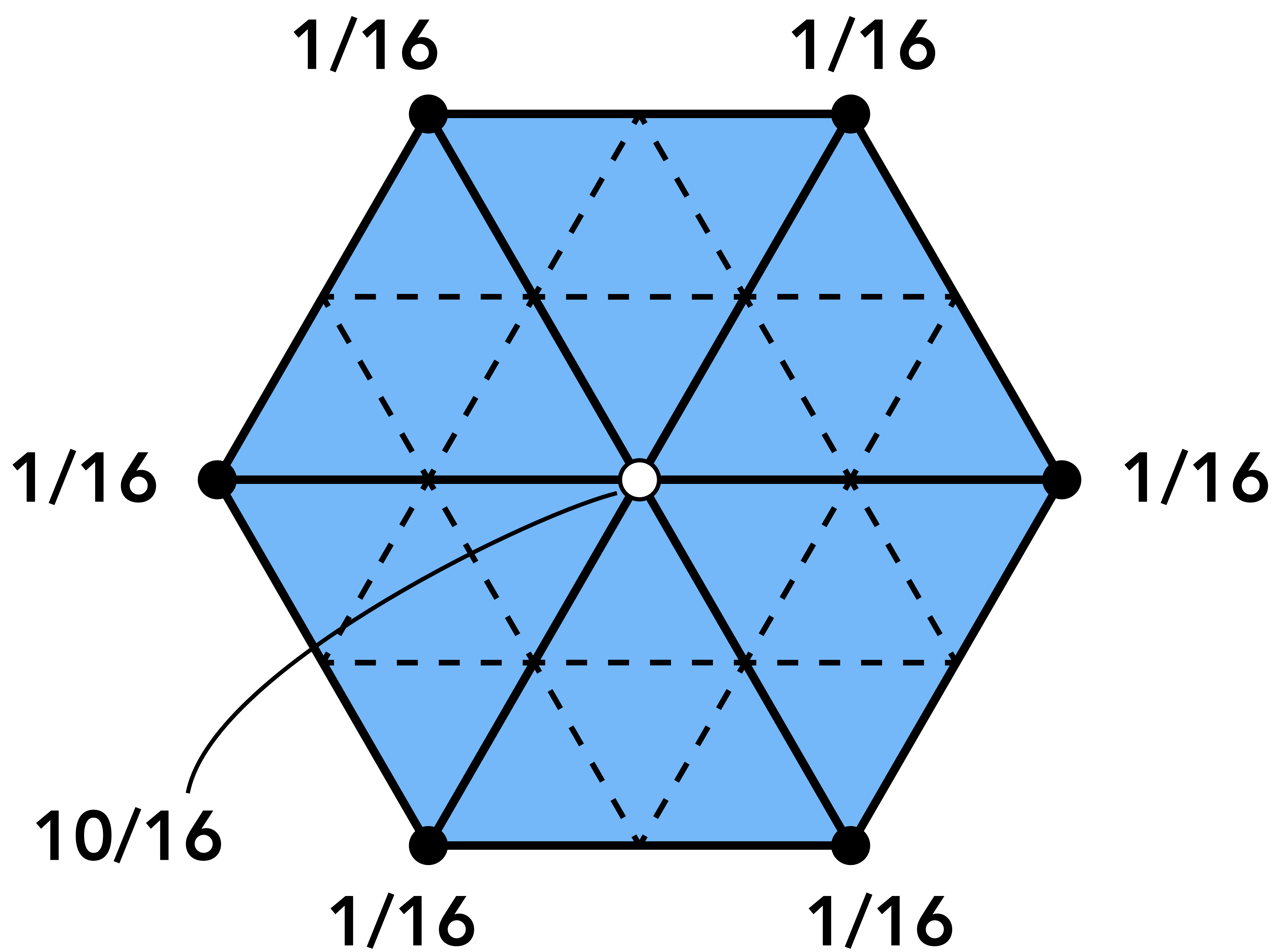
Old vertices
(weighted sum of edge adjacent vertices)

n = vertex degree

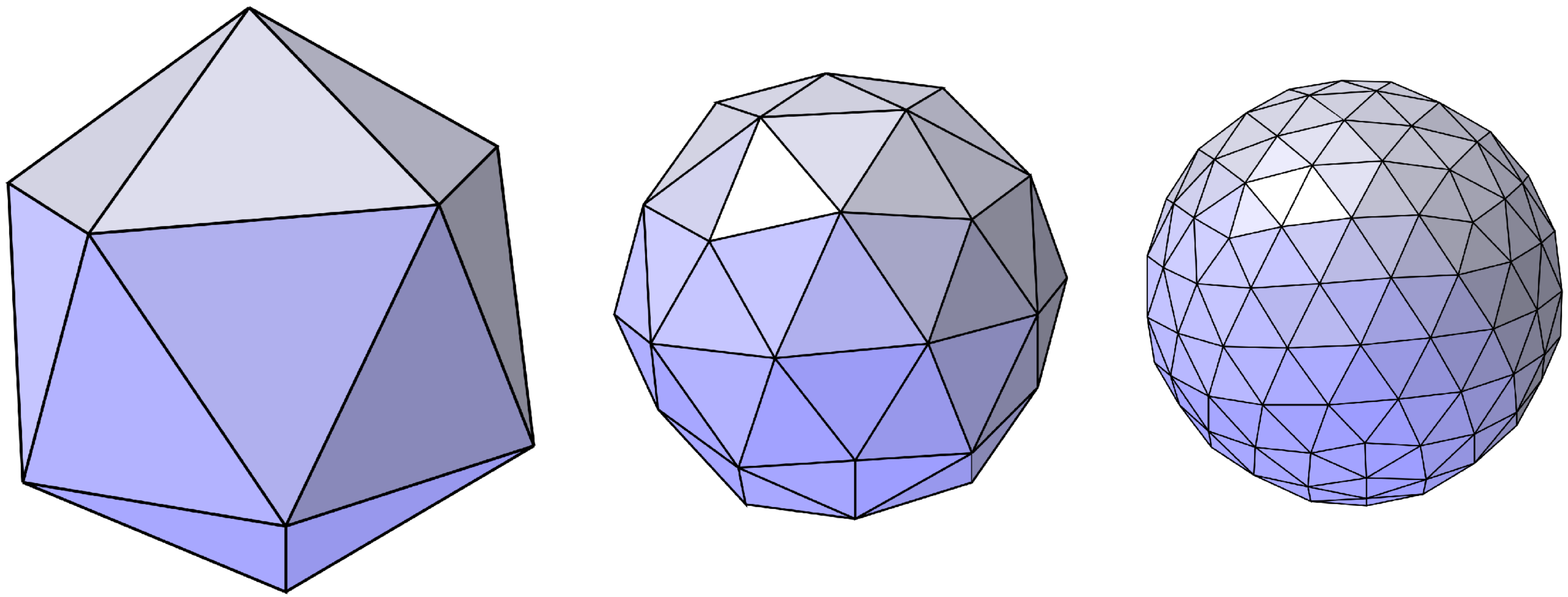
u = 3/16 if n=3, 3/(8n) otherwise

Loop subdivision algorithm

- **Example, for degree 6 vertices ("regular" vertices)**



Loop subdivision results



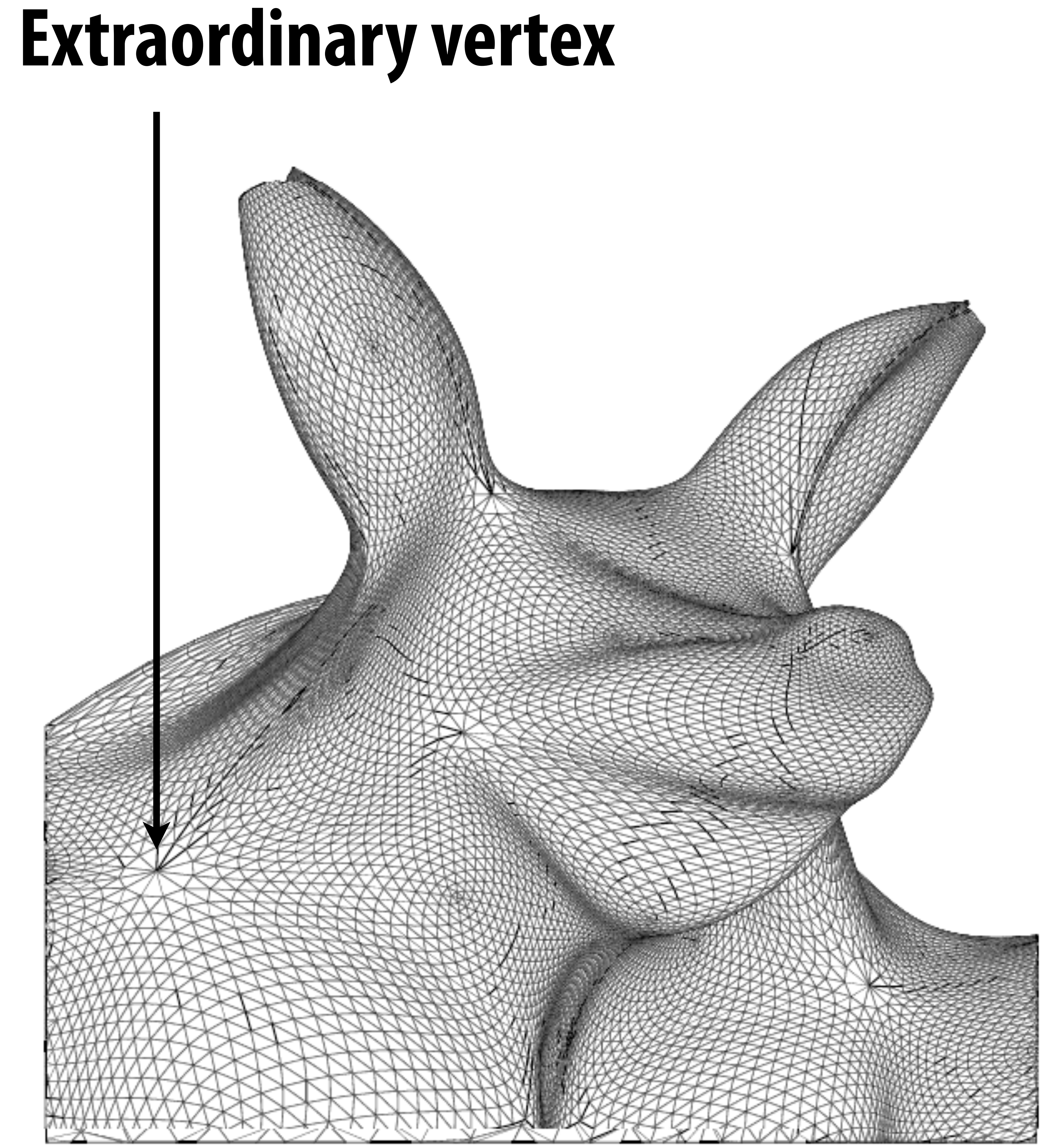
Credit: Simon Fuhrman

Semi-regular meshes

Most of the mesh has vertices with degree 6

But if the mesh is topologically equivalent to a sphere, then not all the vertices can have degree 6

Must have a few extraordinary points (degree not equal to 6)



Proof: always an extraordinary vertex

Our triangle mesh (topologically equivalent to sphere) has V vertices, E edges, and T triangles

E = 3/2 T

- There are 3 edges per triangle, and each edge is part of 2 triangles
- Therefore E = 3/2T

T = 2V – 4

- Euler Convex Polyhedron Formula: T – E + V = 2
- => V = 3/2 T – T + 2 => T = 2V – 4

If all vertices had 6 triangles, T = 2V

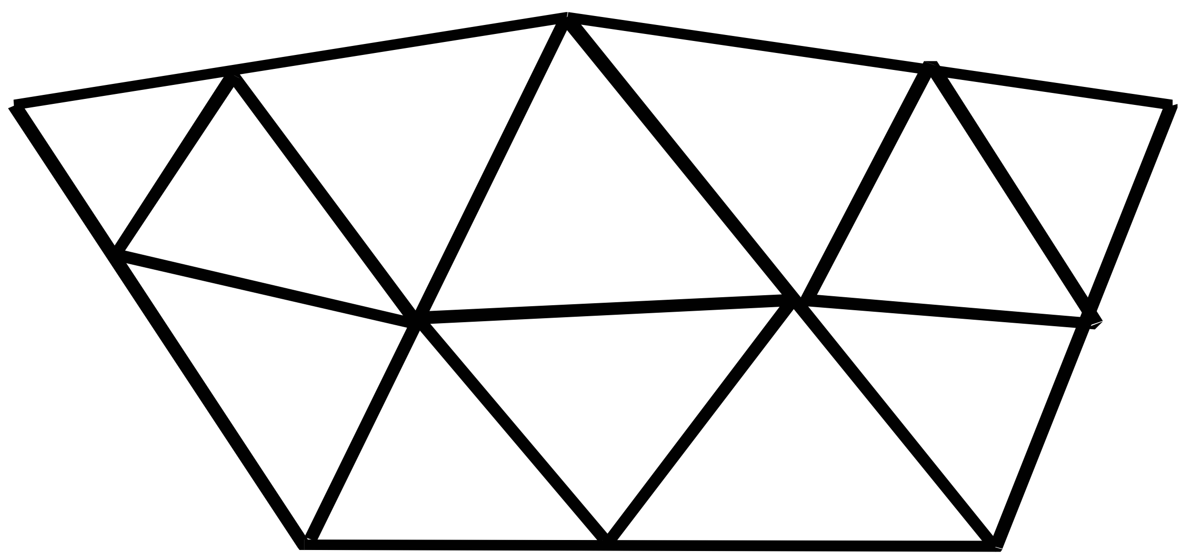
- There are 6 edges per vertex, and every edge connects 2 vertices
- Therefore, E = 6/2V => 3/2T = 6/2V => T = 2V

T cannot equal both 2V – 4 and 2V, a contradiction

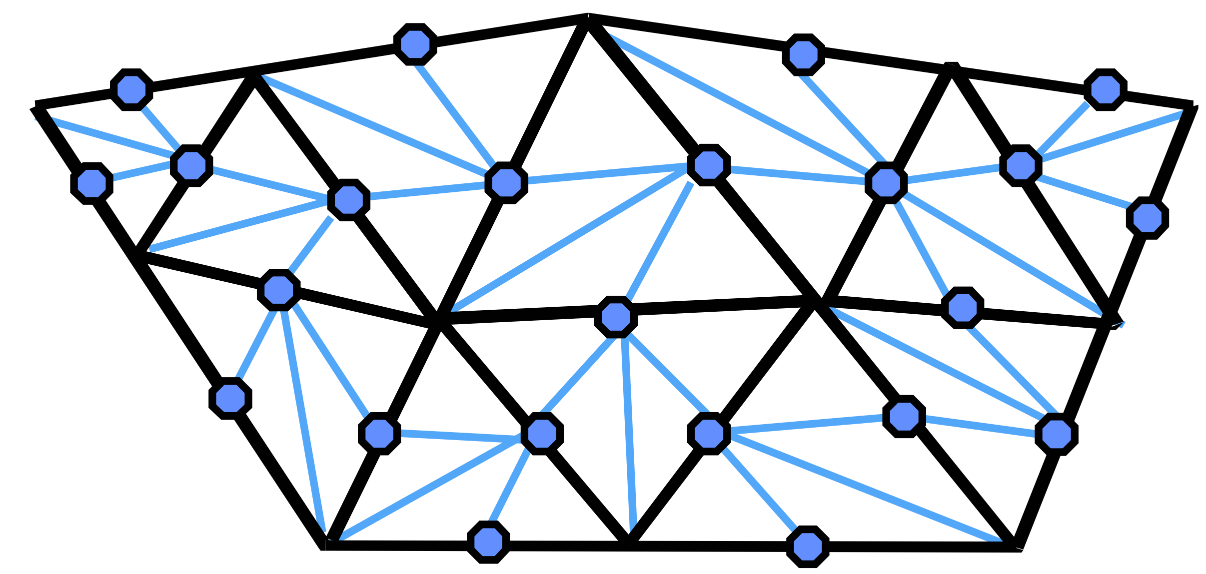
- Therefore, the mesh cannot have 6 triangles for every vertex

Loop subdivision via edge operations

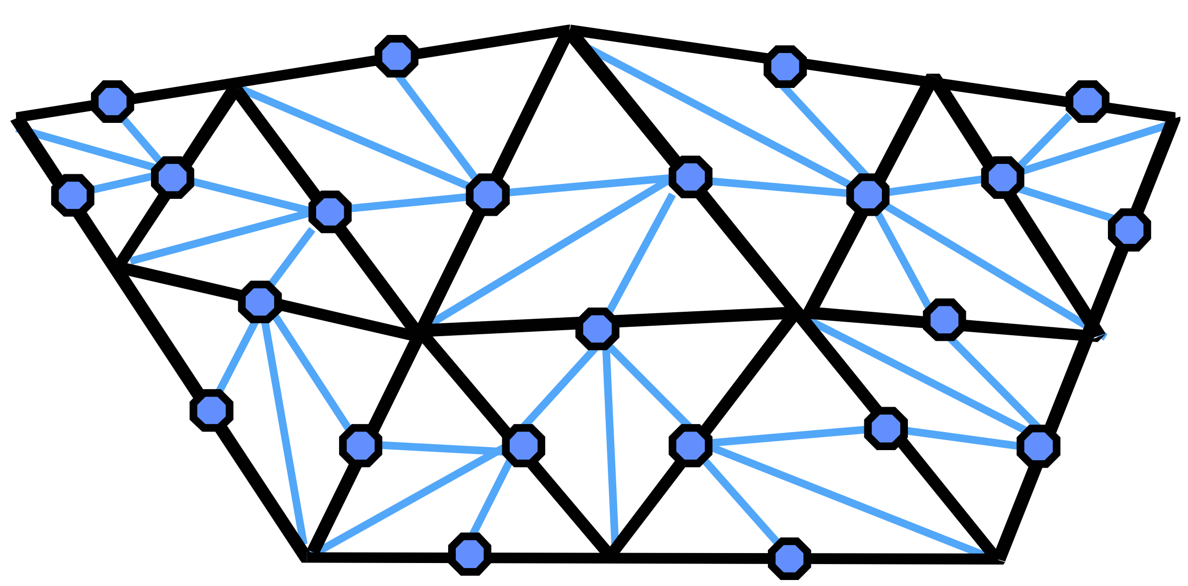
First, split edges of original mesh in any order:



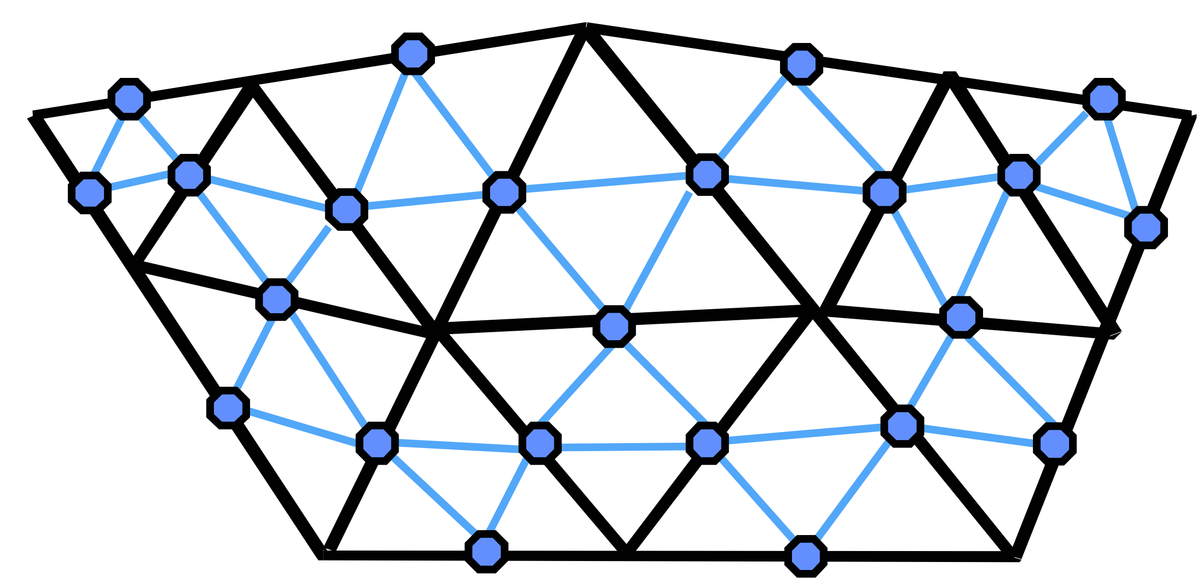
split



Next, flip new edges that touch a new and old vertex:



flip



(Don’t forget to update vertex positions!)

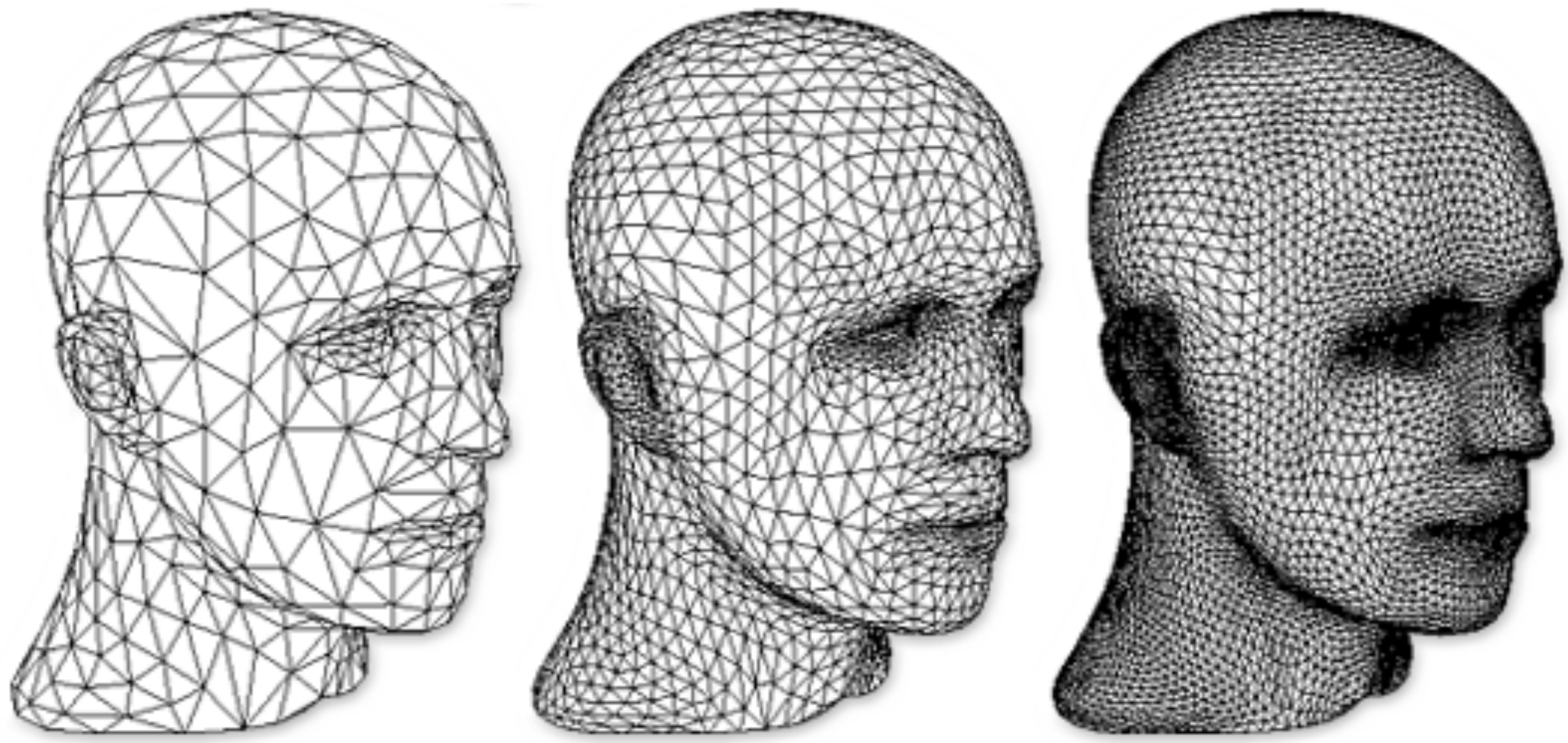
Images cribbed from Keenan Crane, cribbed from Denis Zorin

Continuity of loop subdivision surface

- **At extraordinary vertices**
  - **Surface is at least C^1 continuous**

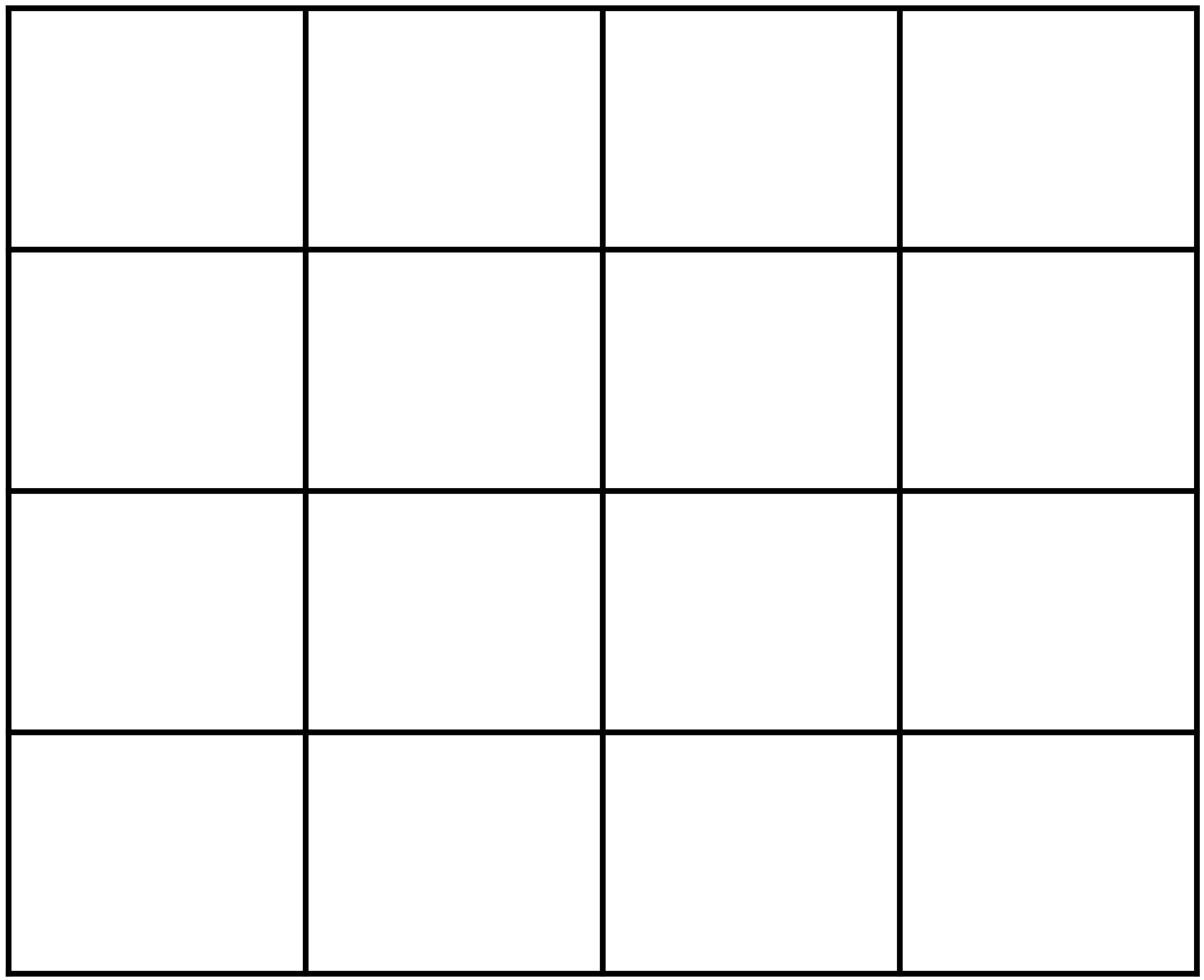
- **Everywhere else ("ordinary" regions)**
  - **Surface is C^2 continuous**

Loop subdivision results

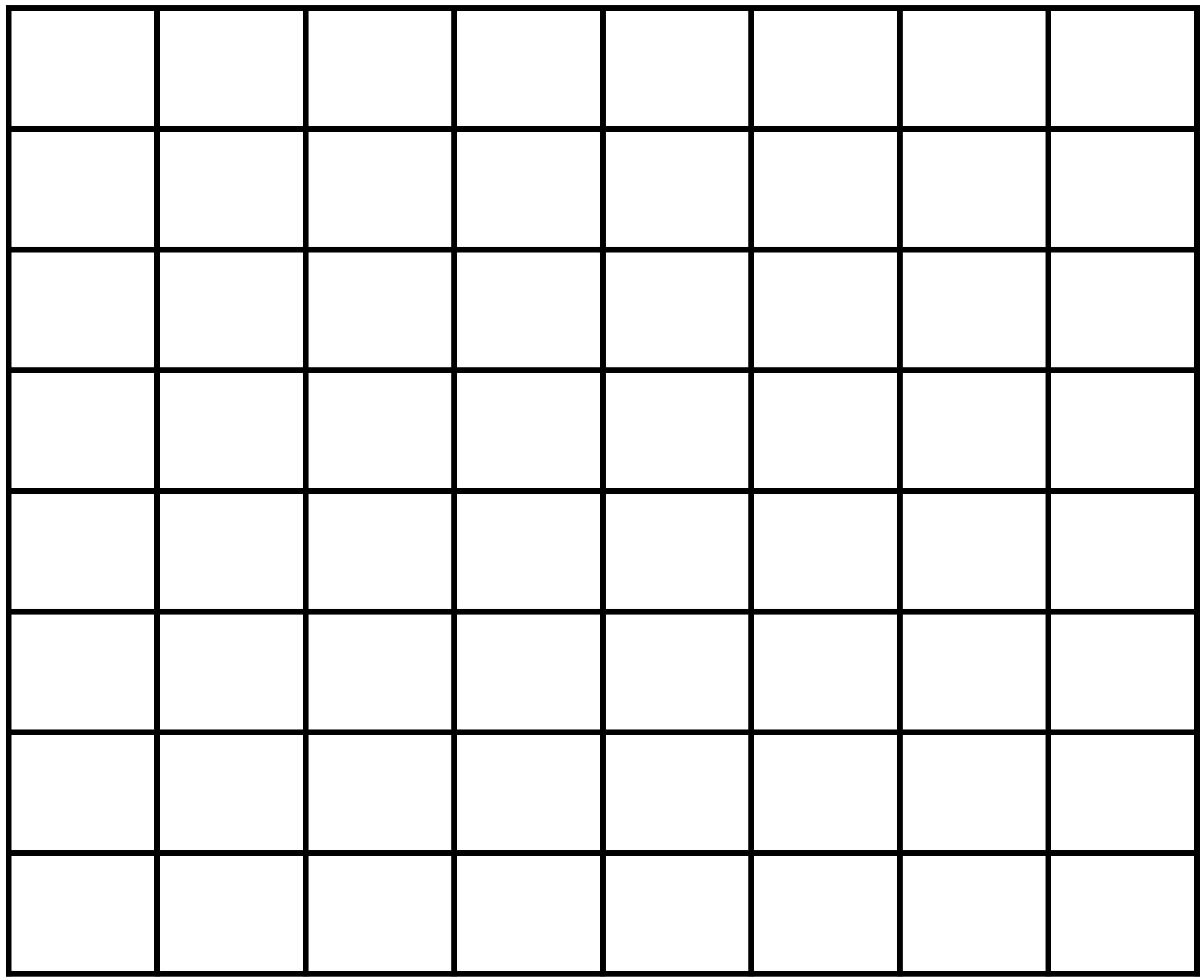


Catmull-Clark subdivision

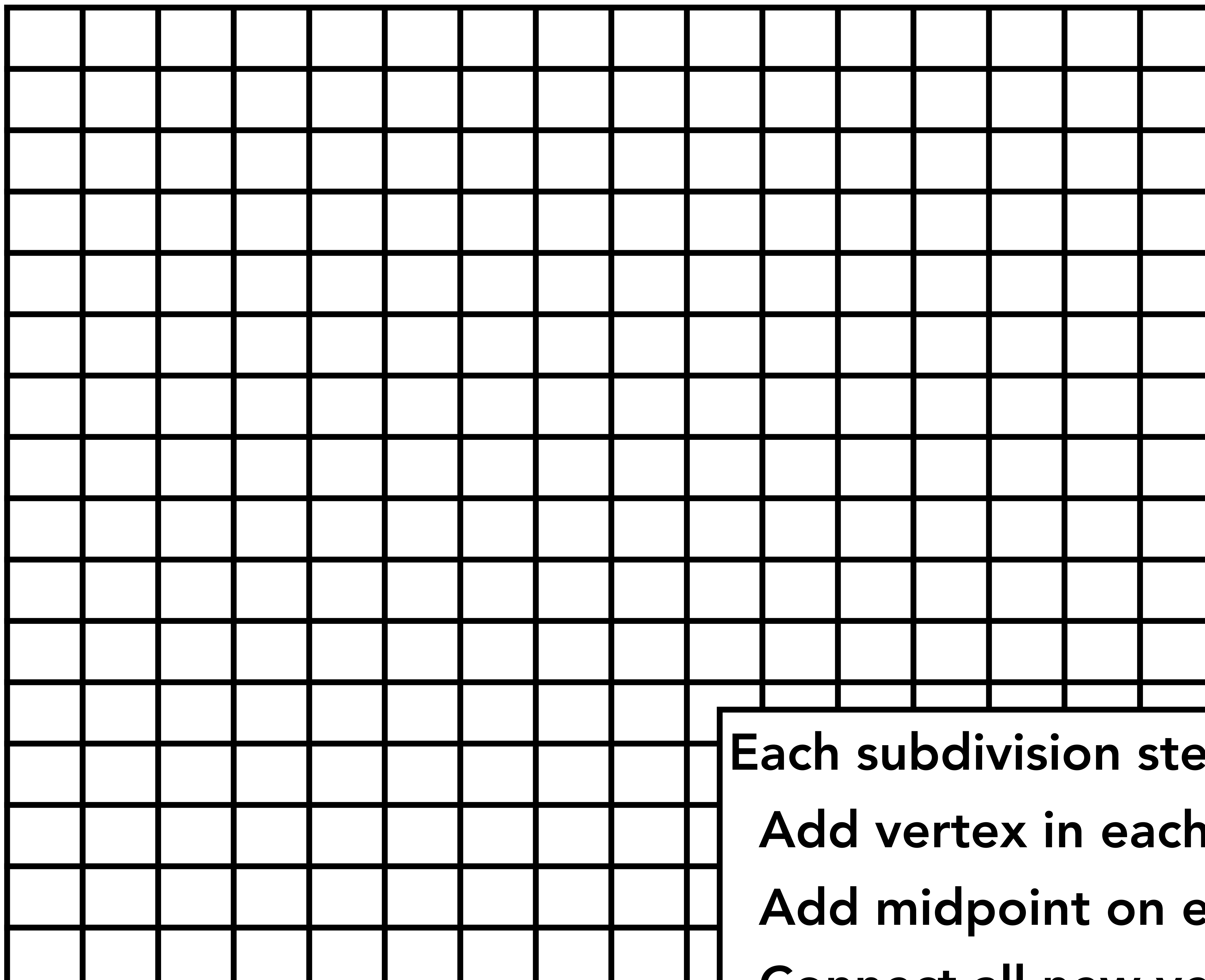
Catmull-Clark subdivision (regular quad mesh)



Catmull-Clark subdivision (regular quad mesh)



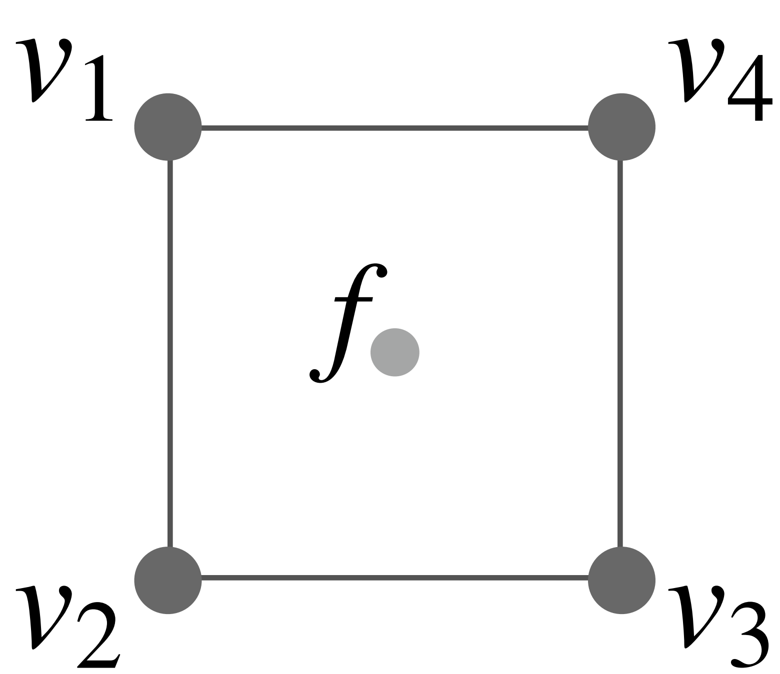
Catmull-Clark subdivision (regular quad mesh)



Each subdivision step:
- Add vertex in each face
- Add midpoint on each edge
- Connect all new vertices

Catmull-Clark vertex update rules (quad mesh)

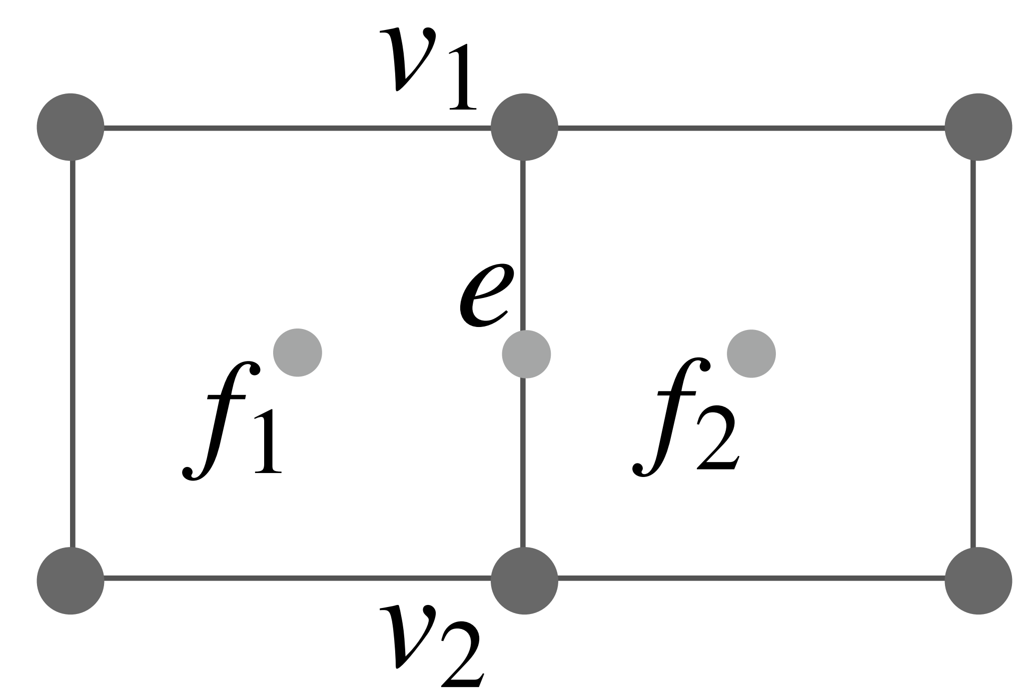
Face point

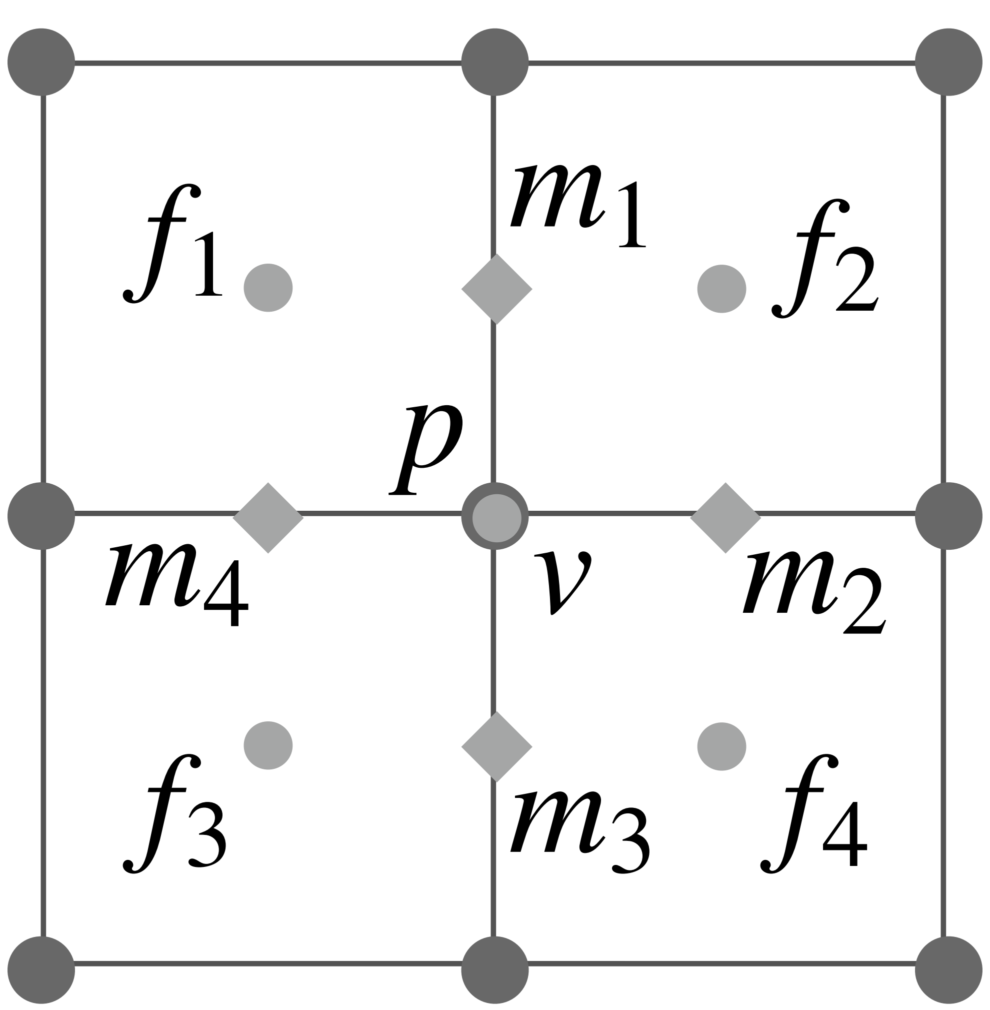


$$f = \frac{v_1 + v_2 + v_3 + v_4}{4}$$

$$e = \frac{v_1 + v_2 + f_1 + f_2}{4}$$

Edge point





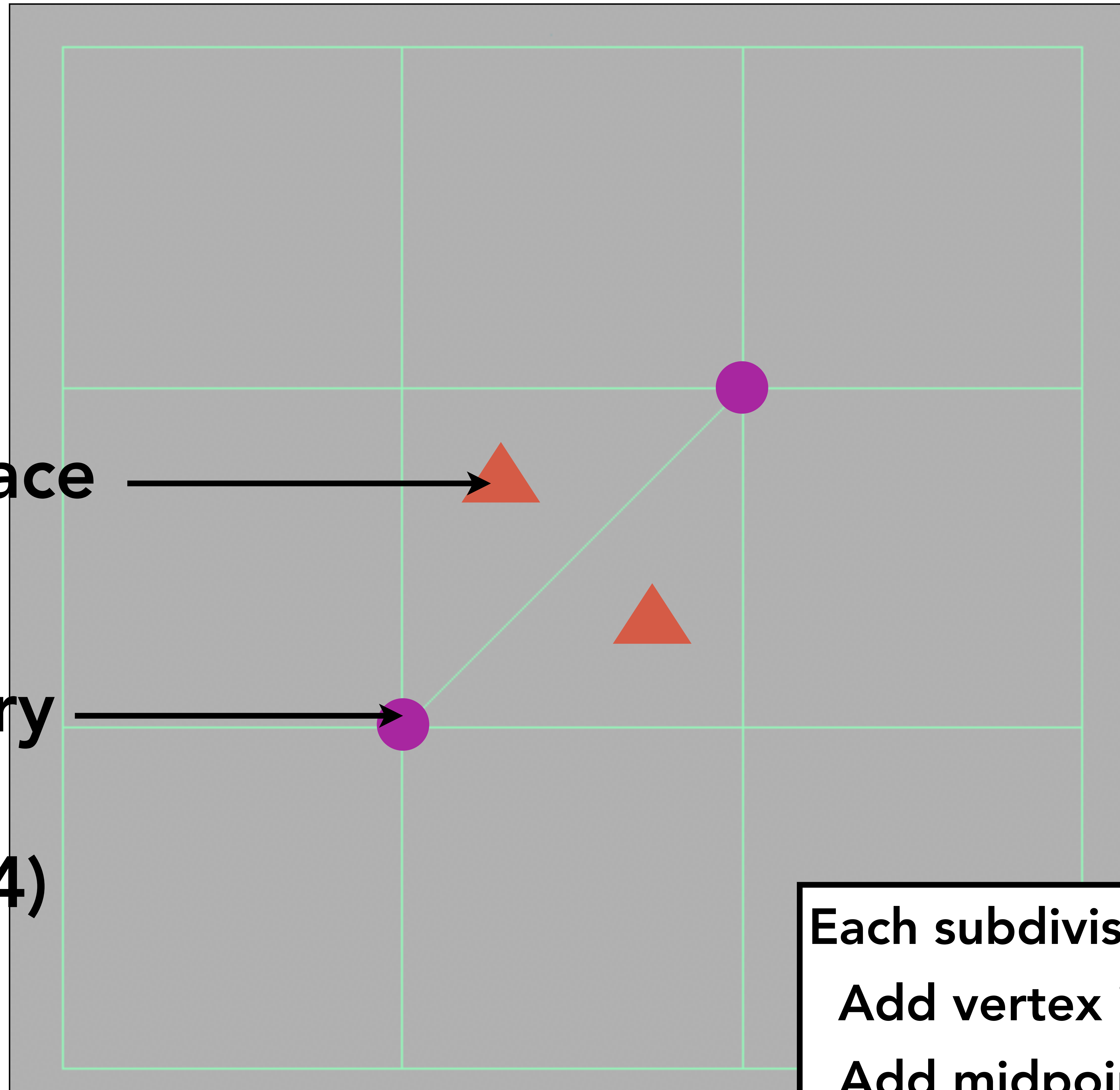
Vertex point

$$v = \frac{f_1 + f_2 + f_3 + f_4 + 2(m_1 + m_2 + m_3 + m_4) + 4p}{16}$$

m midpoint of edge, not "edge point"

p old "vertex point"

Catmull-Clark subdivision (general mesh)

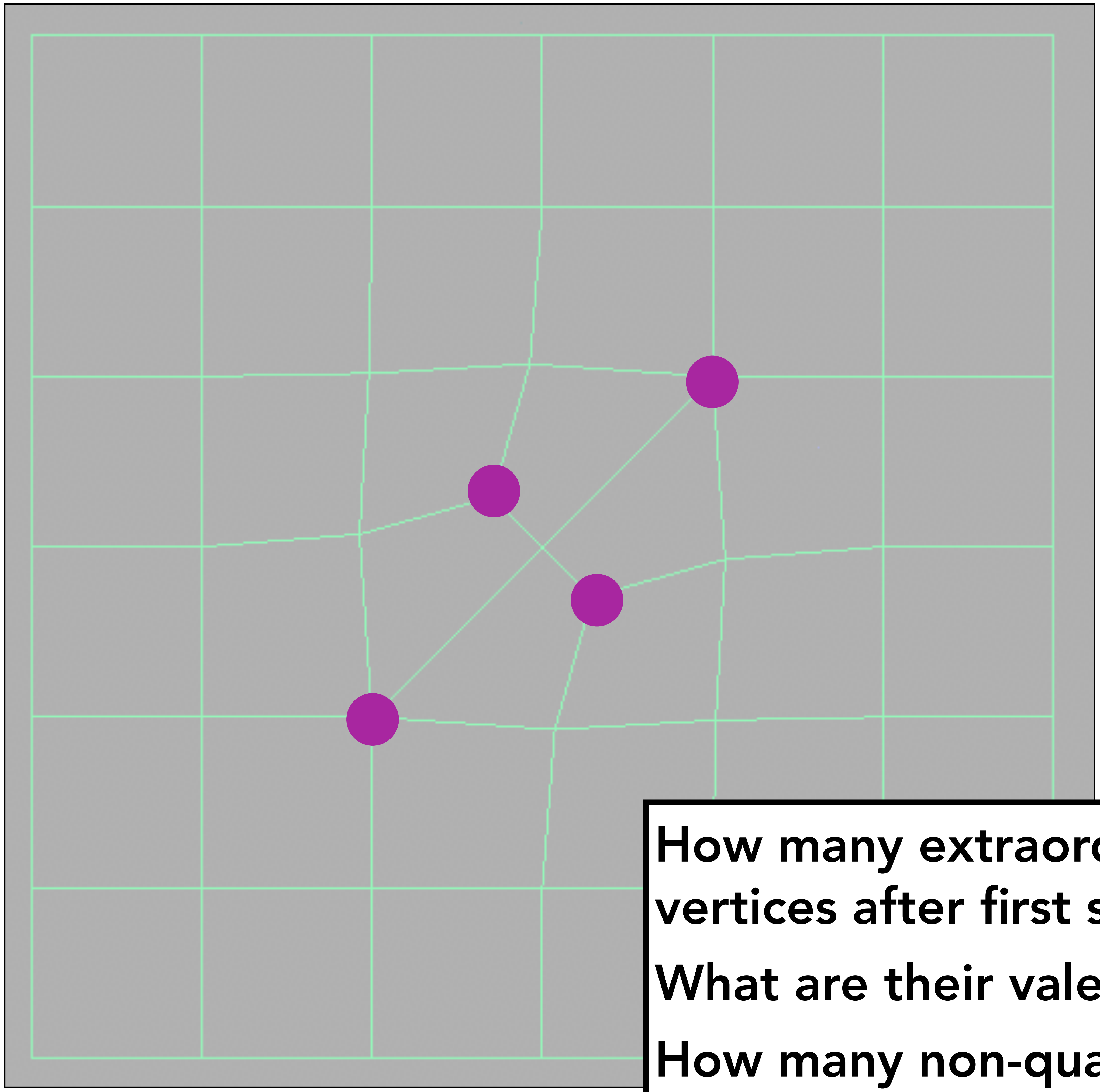


Non-quad face

Extraordinary vertex (valence != 4)

Each subdivision step:
- Add vertex in each face
- Add midpoint on each edge
- Connect all new vertices

Catmull-Clark subdivision (general mesh)

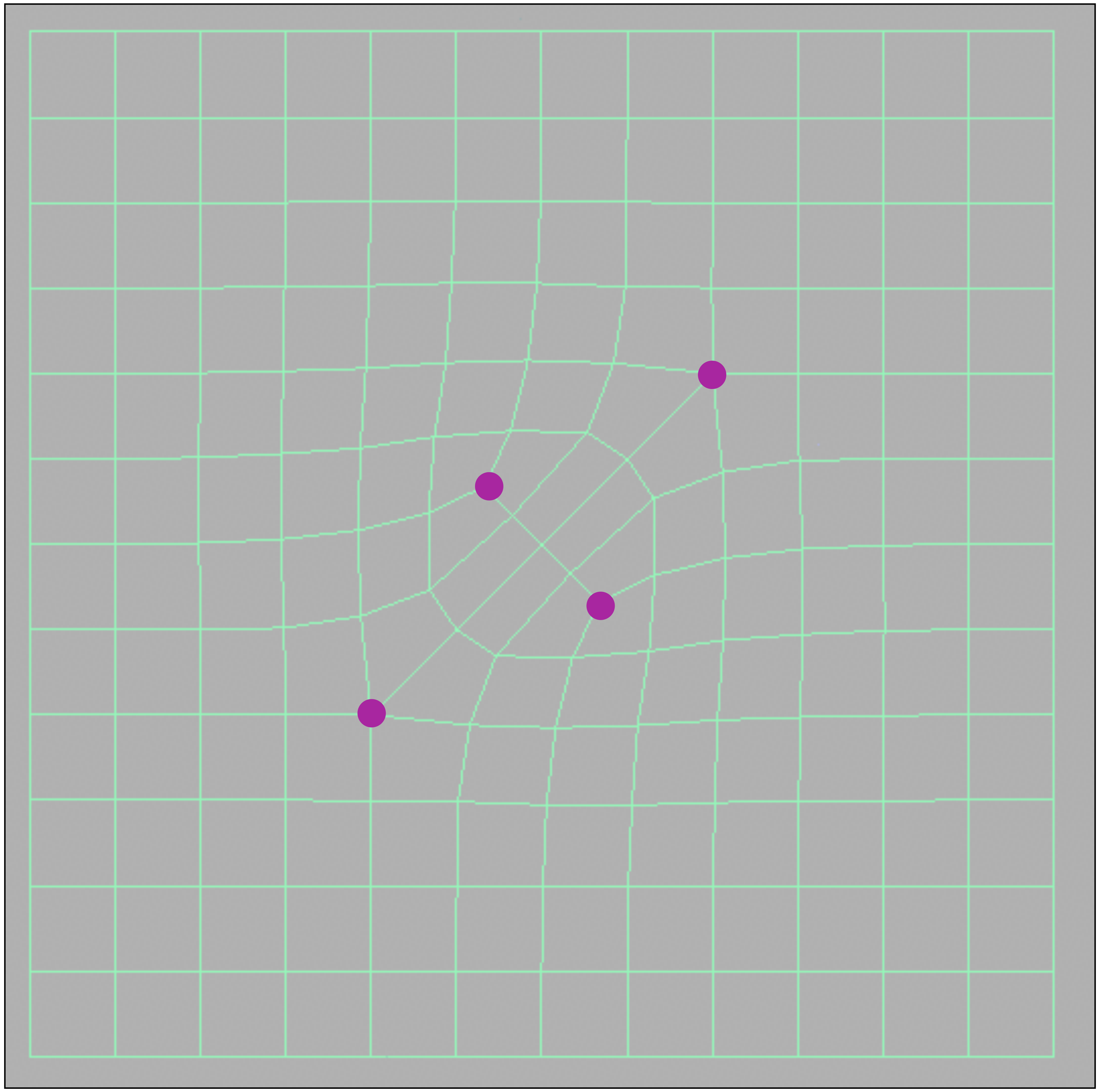


How many extraordinary vertices after first subdivision?

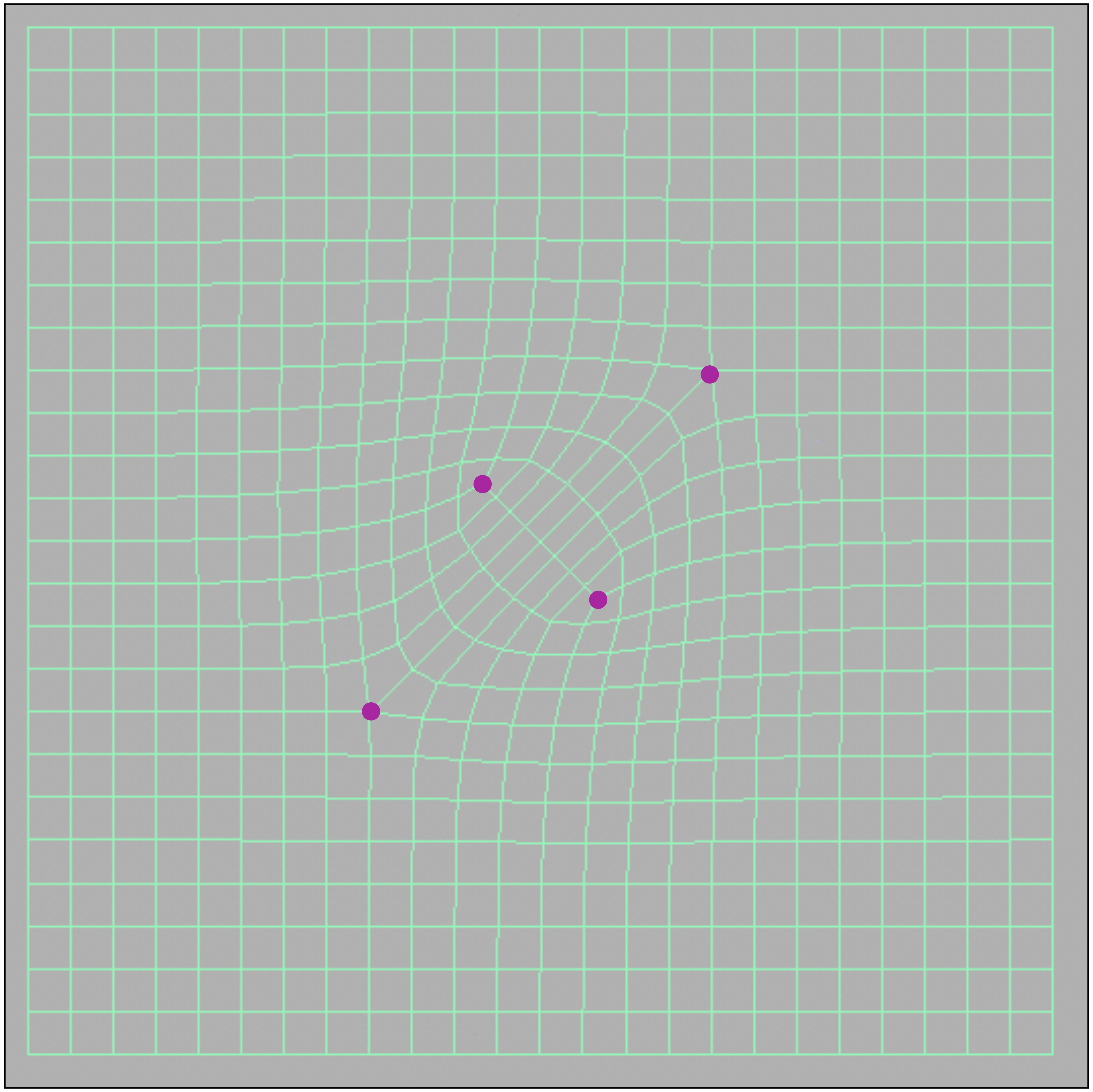
What are their valences?

How many non-quad faces?

Catmull-Clark subdivision (general mesh)



Catmull-Clark subdivision (general mesh)



Catmull-Clark vertex update rules (general mesh)

$$f = \text{average of surrounding vertices}$$

$$e = \frac{f_1 + f_2 + v_1 + v_2}{4}$$

These rules reduce to earlier quad rules for ordinary vertices / faces

$$v = \frac{\bar{f}}{n} + \frac{2\bar{m}}{n} + \frac{p(n-3)}{n}$$

$$\bar{m} = \text{average of adjacent midpoints}$$
$$\bar{f} = \text{average of adjacent face points}$$
$$n = \text{valence of vertex}$$
$$p = \text{old "vertex" point}$$

Continuity of Catmull-Clark surface

- At extraordinary points
  - Surface is at least C^1 continuous

- Everywhere else ("ordinary" regions)
  - Surface is C^2 continuous

What about sharp creases?

From Pixar Short, "Geri's Game"

Hand is modeled as a Catmull Clark surface with creases between skin and fingernail

What about sharp creases?

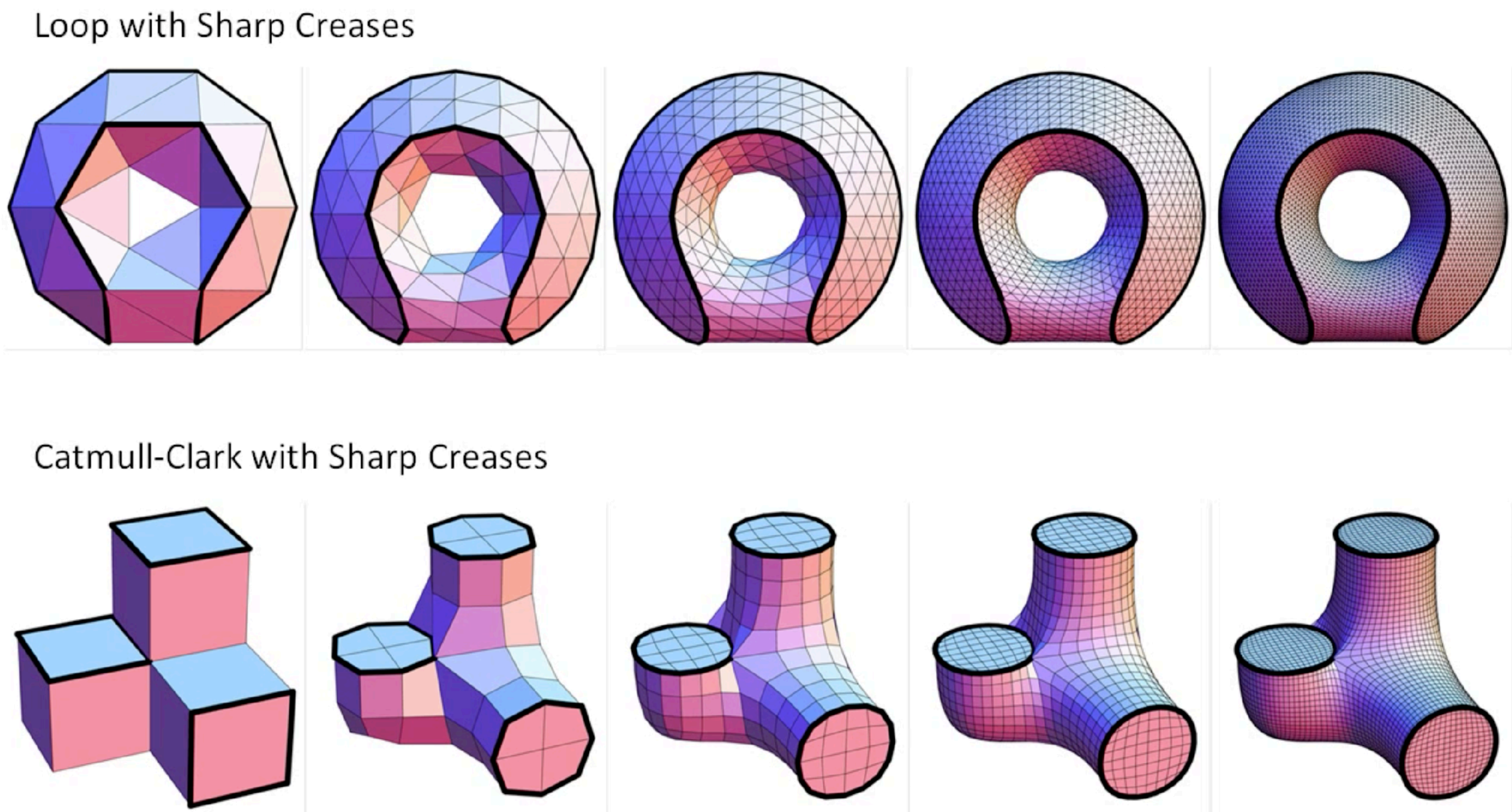
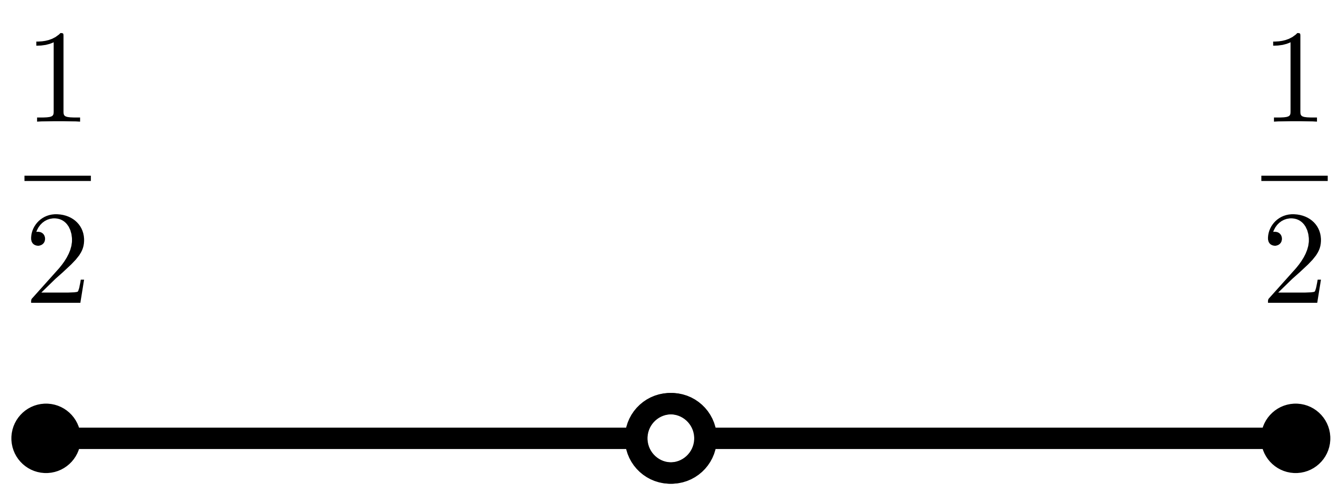


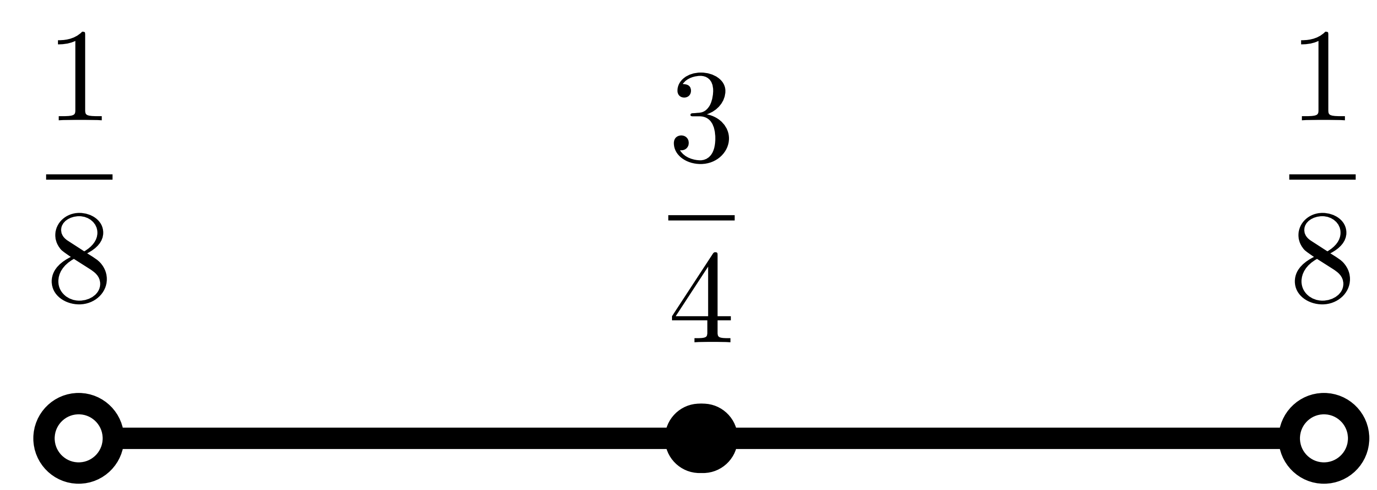
Figure from: Hakenberg et al. Volume Enclosed by Subdivision Surfaces with Sharp Creases

Creases and boundaries

- Can create creases in subdivision surfaces by marking certain edges as "sharp". Surface boundary edges can be handled the same way
  - Use different subdivision rules for vertices along these "sharp" edges



Insert new midpoint vertex, weights as shown



Update existing vertices, weights as shown

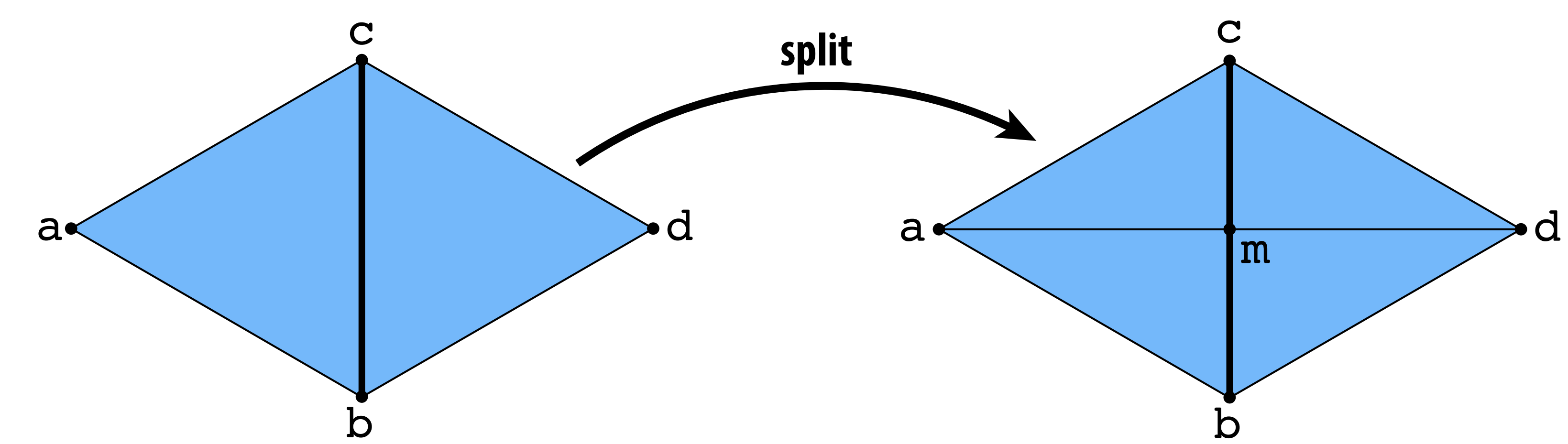
Subdivision in action (“Geri’s Game”, Pixar)

- **Subdivision used for entire character:**
  - **Hands and head**
  - **Clothing, tie, shoes**

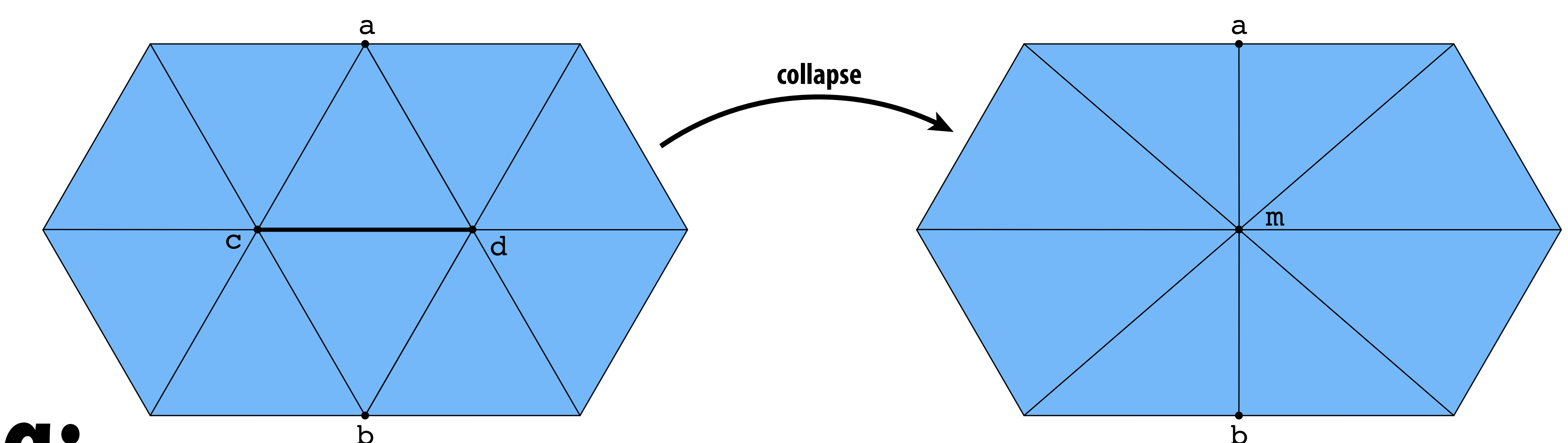
Mesh simplification — downsampling

How do we resample meshes? (reminder)

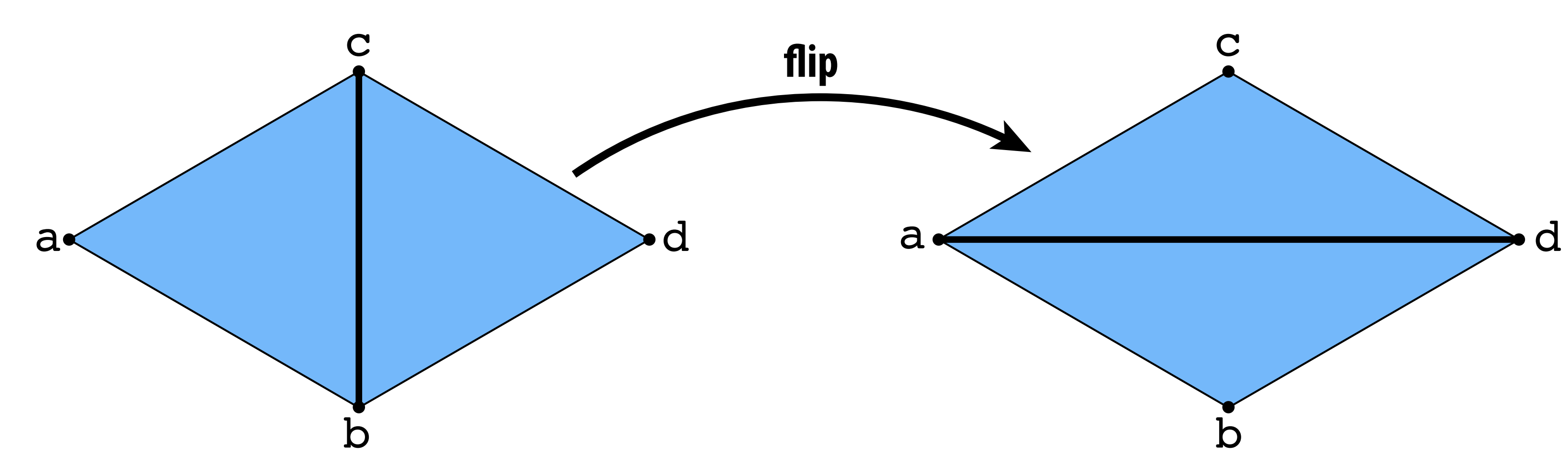
- **Edge split is (local) upsampling:**



- **Edge collapse is (local) downsampling:**



- **Edge flip is (local) resampling:**



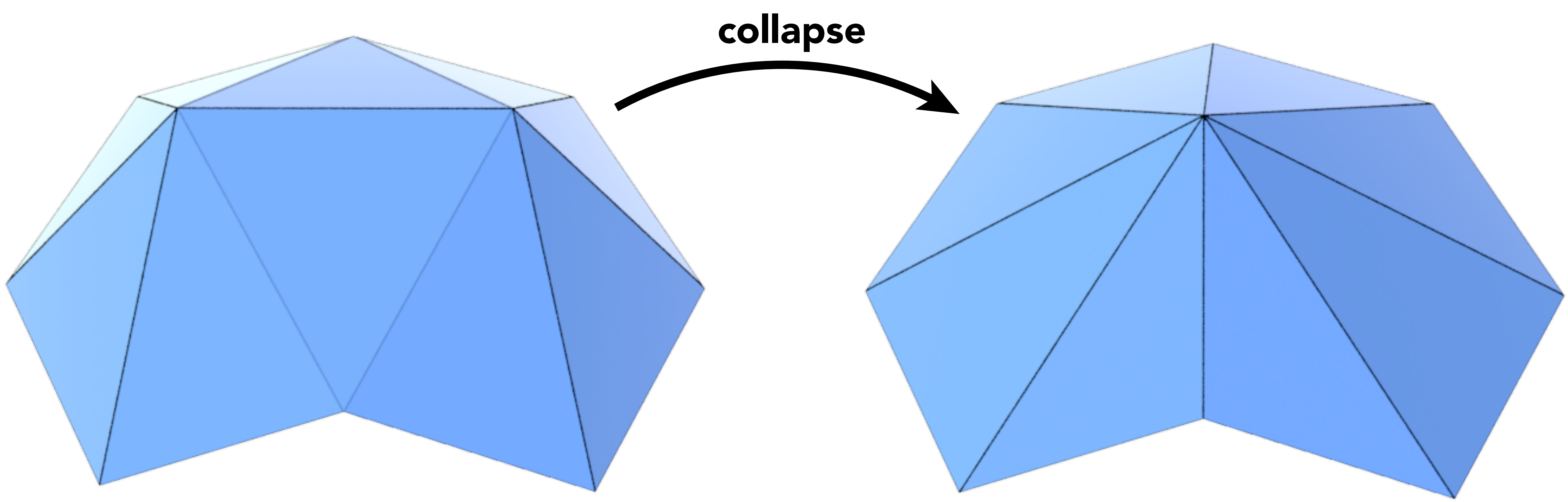
- **Still need to intelligently decide which edges to modify!**

Mesh simplification

- **Goal: reduce number of mesh elements while maintaining overall shape**

Estimate: error introduced by collapsing an edge?

How much geometric error is introduced by collapsing an edge?



Sketch of Quadric Error Mesh Simplification

Simplification via quadric error

- Iteratively collapse edges
- Which edges? Assign score with quadric error metric*
  - Approximate distance to surface as sum of squared distances to planes containing nearby triangles
  - Iteratively collapse edge with smallest score
  - Greedy algorithm... great results!

* (Garland & Heckbert 1997)

Distance from point to a line (and a plane)

Line is defined by:

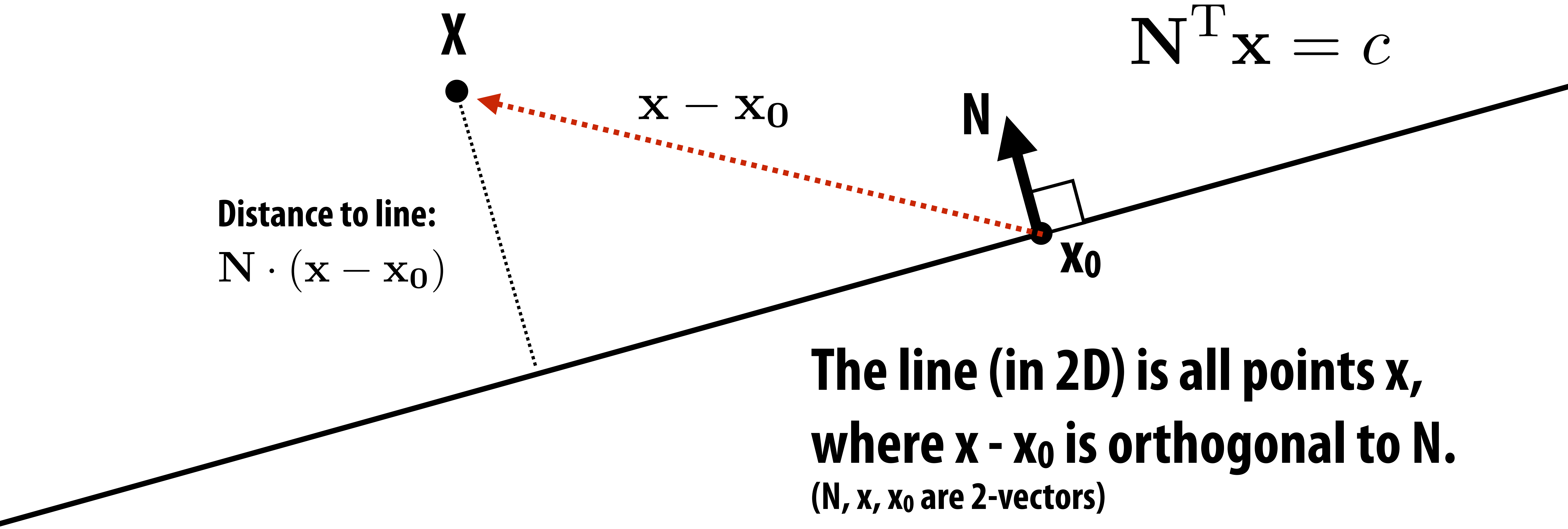
- Its normal: N
- A point x_0 on the line

$$\mathbf{N} \cdot (\mathbf{x} - \mathbf{x_0}) = 0$$

$$\mathbf{N}^{\mathrm{T}}(\mathbf{x} - \mathbf{x_0}) = 0$$

$$\mathbf{N}^{\mathrm{T}}\mathbf{x} = \mathbf{N}^{\mathrm{T}}\mathbf{x_0}$$

$$\mathbf{N}^{\mathrm{T}}\mathbf{x} = c$$



The line (in 2D) is all points x, where x - x_0 is orthogonal to N.

(N, x, x_0 are 2-vectors)

(And a plane (in 3D) is all points x where x - x_0 is orthogonal to N.)

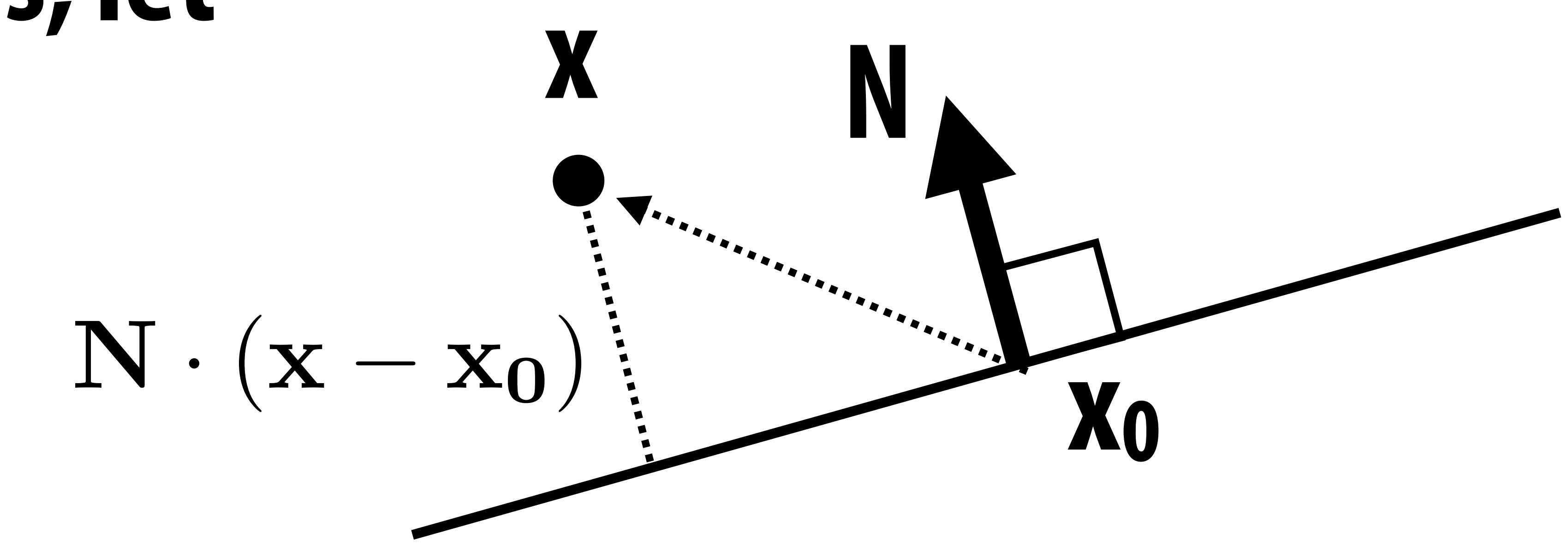
(N, x, x_0 are 3-vectors)

Quadric error matrix (encodes squared distance)

- Suppose we have:
  - a query point (x,y,z)
  - a plane normal N=(a,b,c)
  - a plane offset d := –(x_{0x},x_{0y},x_{0z}) • (a,b,c)

$$Q = \begin{bmatrix} a^2 & ab & ac & ad \\ ab & b^2 & bc & bd \\ ac & bc & c^2 & cd \\ ad & bd & cd & d^2 \end{bmatrix}$$

- Then in homogeneous coordinates, let
  - u := (x,y,z,1)
  - v := (a,b,c,d)



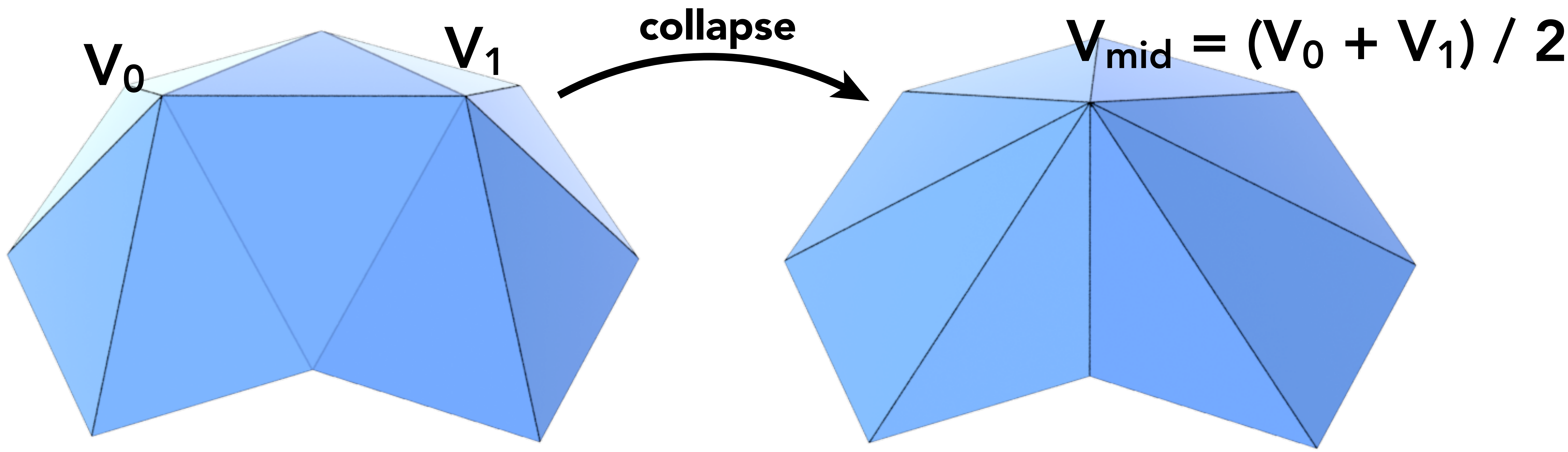
- Signed distance to plane is then
  $D = uv^T = vu^T = ax+by+cz+d$

- Squared distance is $D^2 = (uv^T)(vu^T) = u\,(v^Tv)\,u^T := u^TQu$

- Distance is 2nd degree (“quadric”) polynomial in x,y,z

Cost of edge collapse

- How much does it cost to collapse an edge?
- Idea: compute edge midpoint V_{mid}, measure quadric error at this point
- Error at V_{mid} given by $v_{mid}^T(Q_0 + Q_1)v_{mid}$ See next slide for Q_i
- Intuition: cost is sum of squared differences to original position of triangles now touching V_{mid}

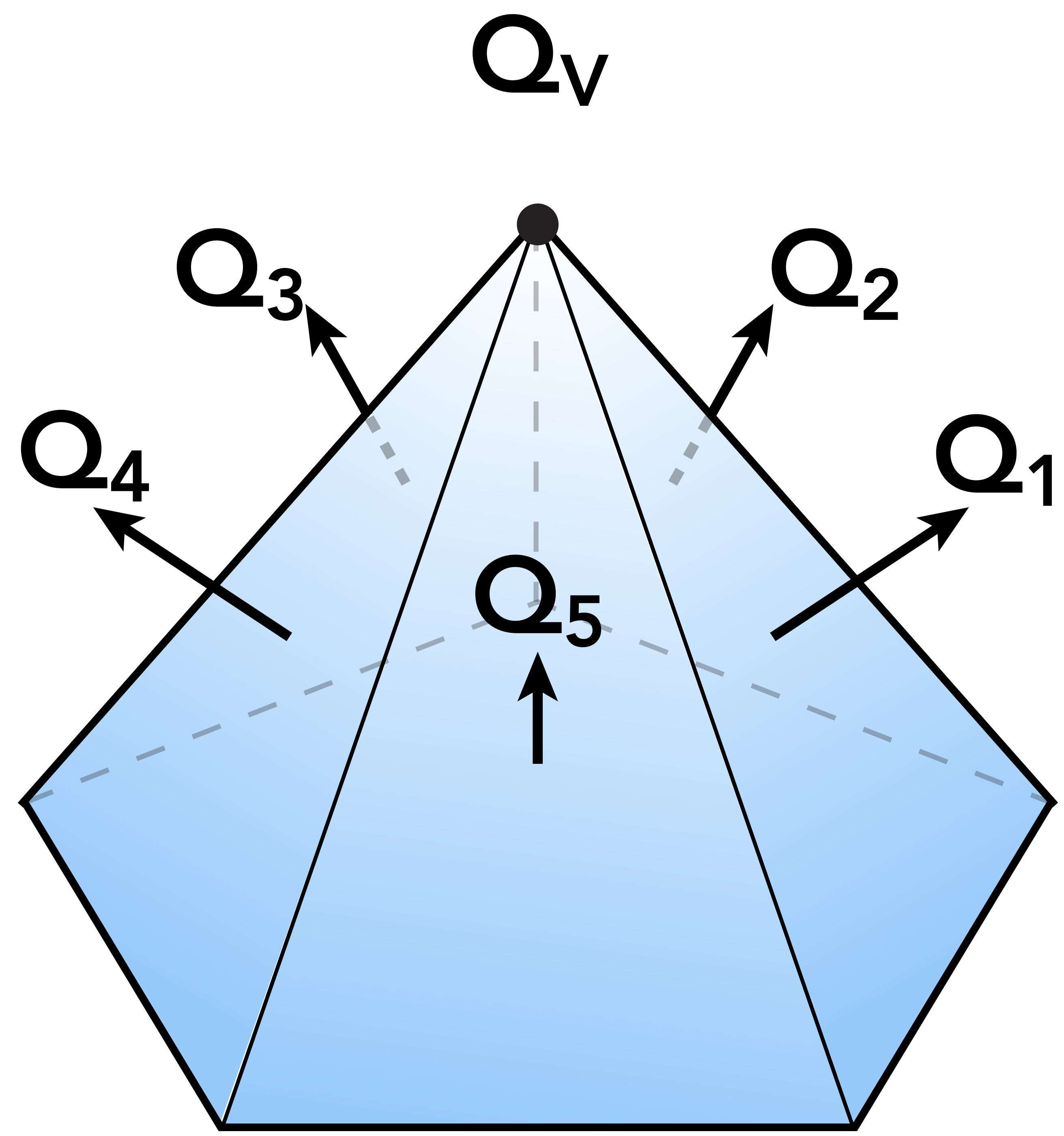


- Better idea: choose point on edge (not necessarily the midpoint) that minimizes quadric error
- More details: Garland & Heckbert 1997

“Quadric error metric at mesh vertex”

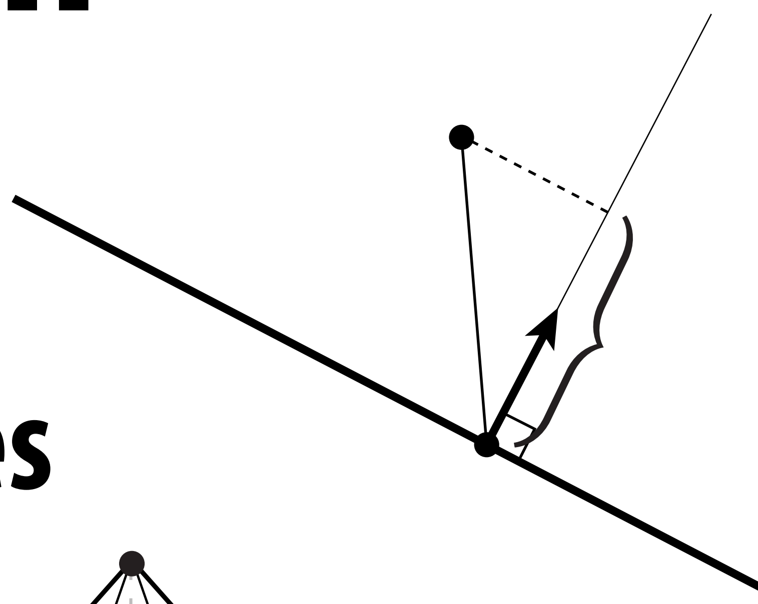
Heuristic: “error metric at vertex V” is sum of squared distances to triangles connected to V

Encode this as a single quadric matrix per vertex that is the sum of quadric error matrices for all triangles

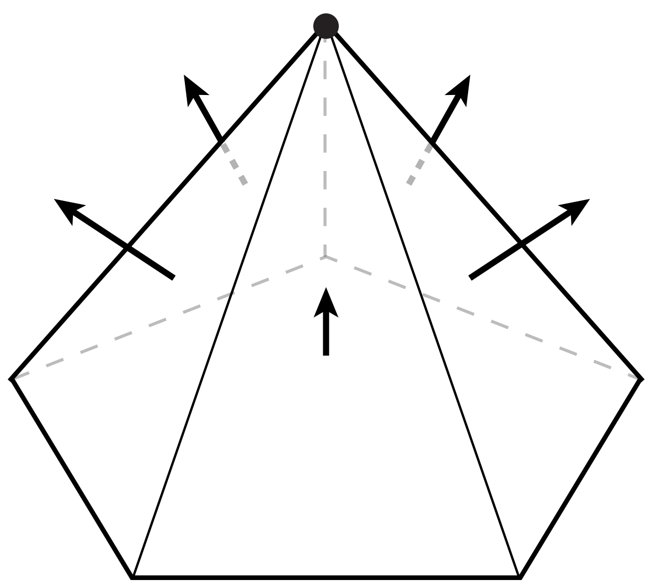


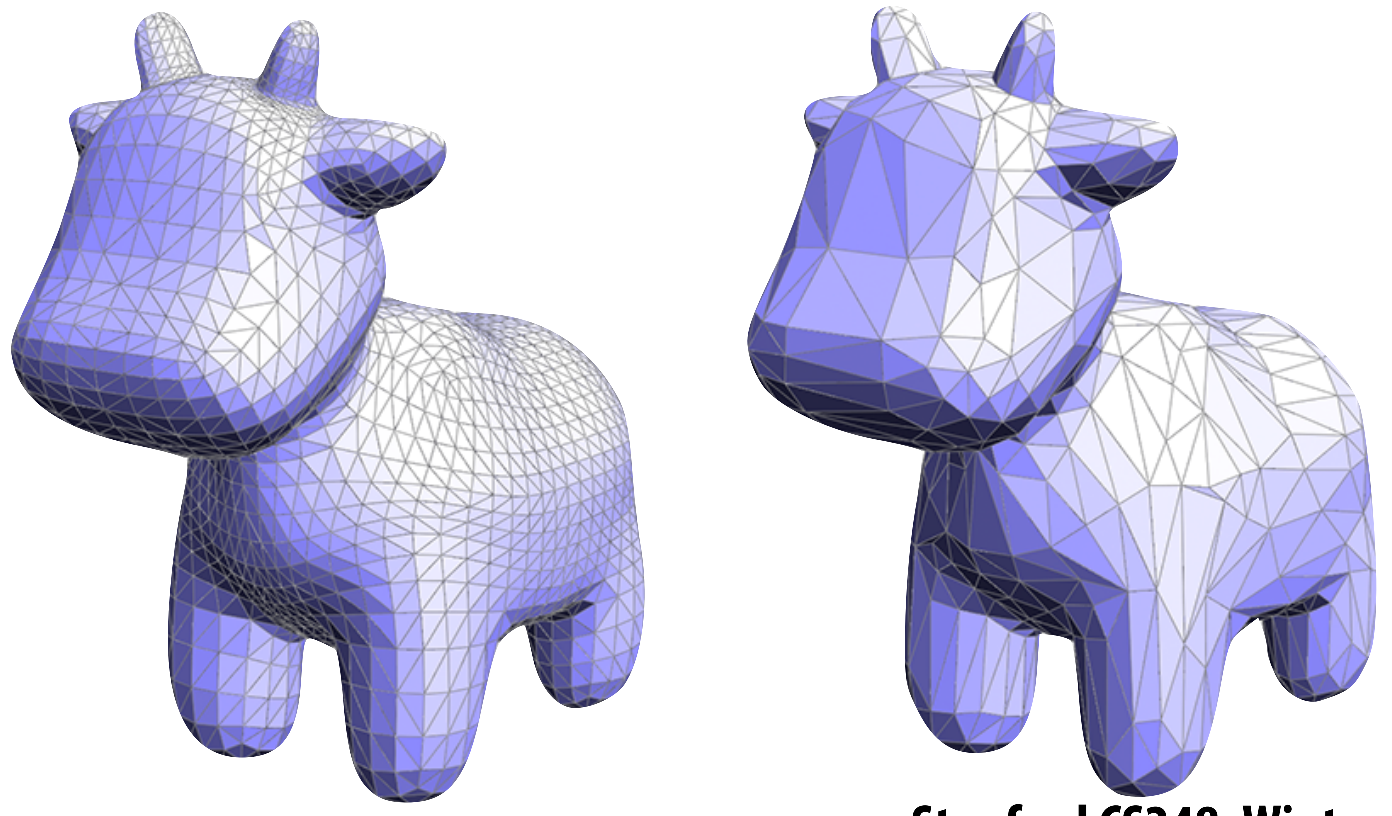
$$Q_V = \sum_{i=1}^{N} Q_i$$

Quadric error simplification: algorithm

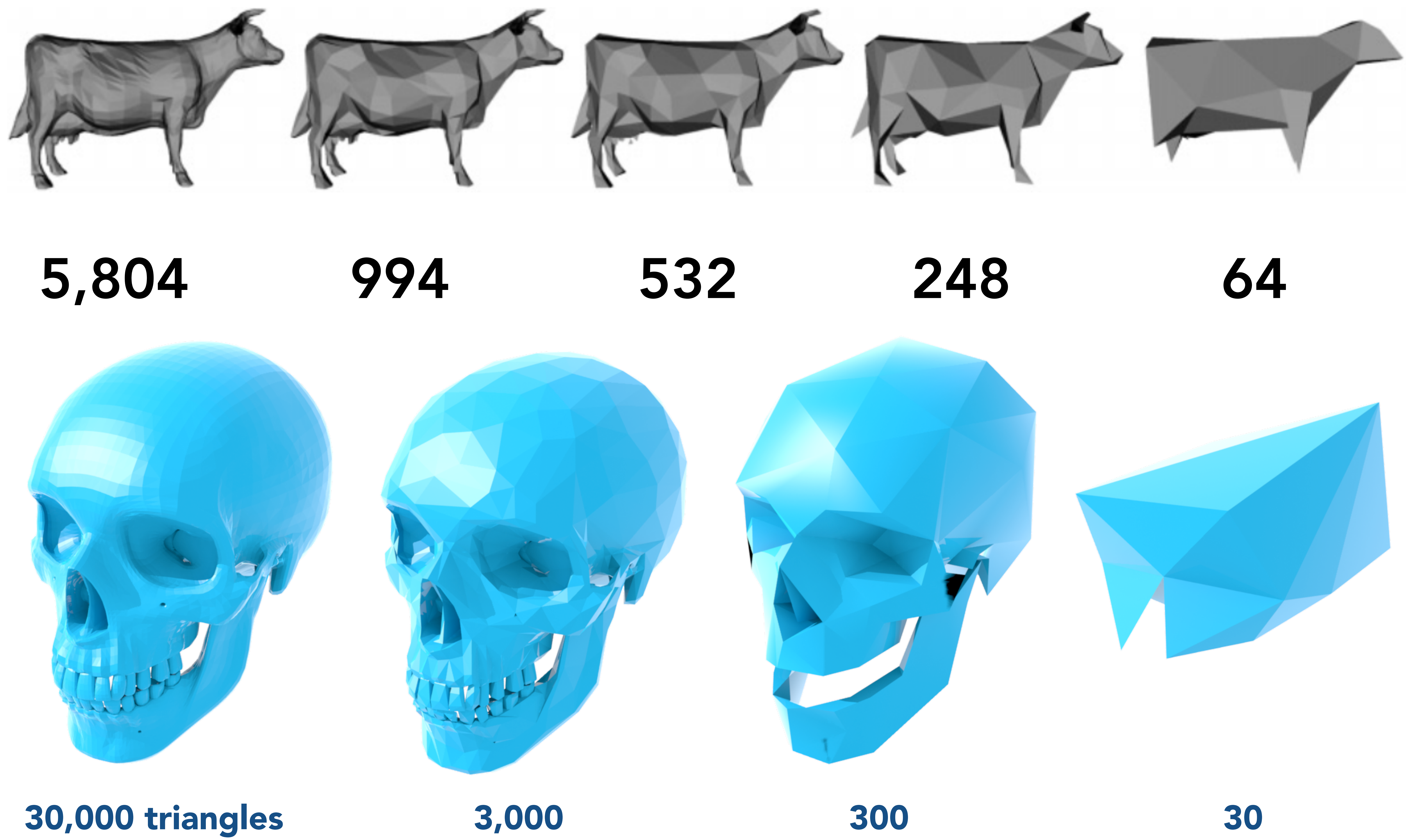


- Compute quadric error matrix Q for each triangle's plane
- Set Q at each vertex to sum of Q's from neighbor triangles
- Set Q at each edge to sum of Q's at endpoints
- Find point at each edge minimizing quadric error
- Until we reach target # of triangles:
    - collapse edge (i,j) with smallest cost to get new vertex m
    - add Q_i and Q_j to get quadric Q_m at vertex m
    - update cost of edges touching vertex m





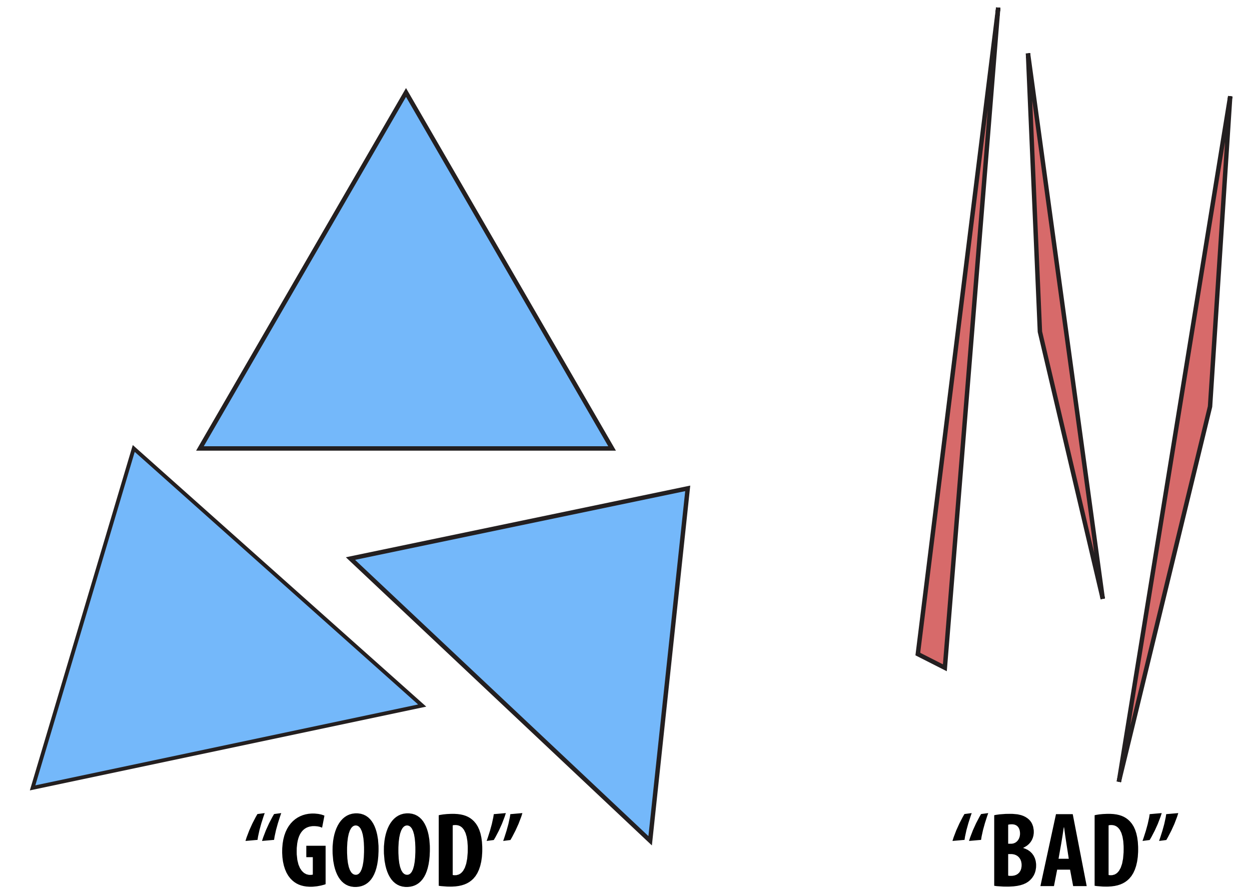
Quadric error mesh simplification



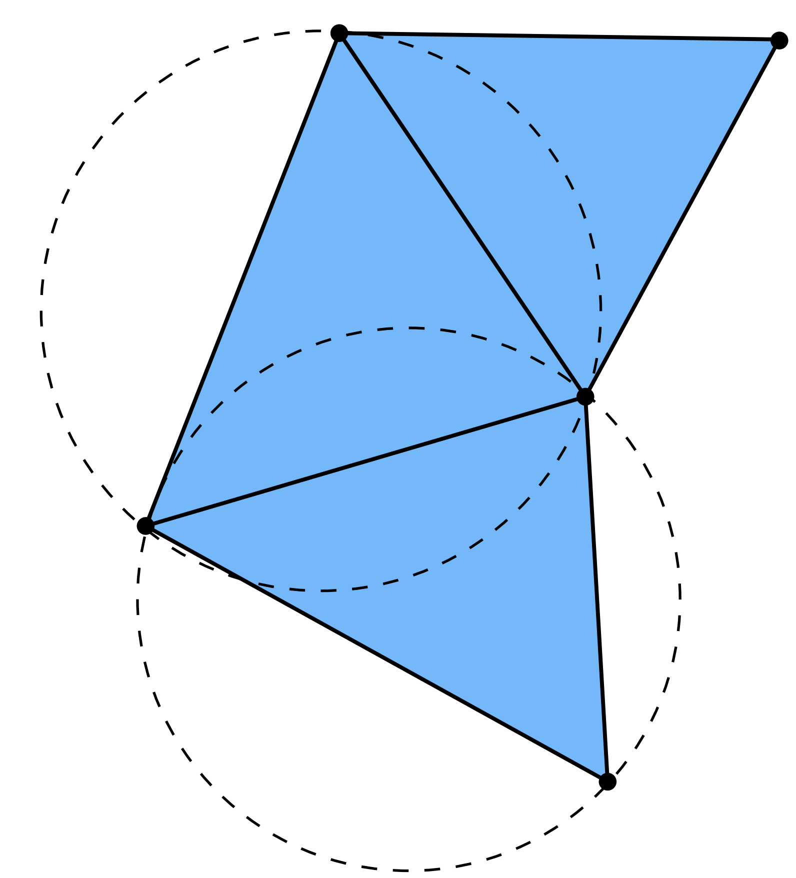
Garland and Heckbert '97

Mesh Regularization

What makes a “good” triangle mesh?



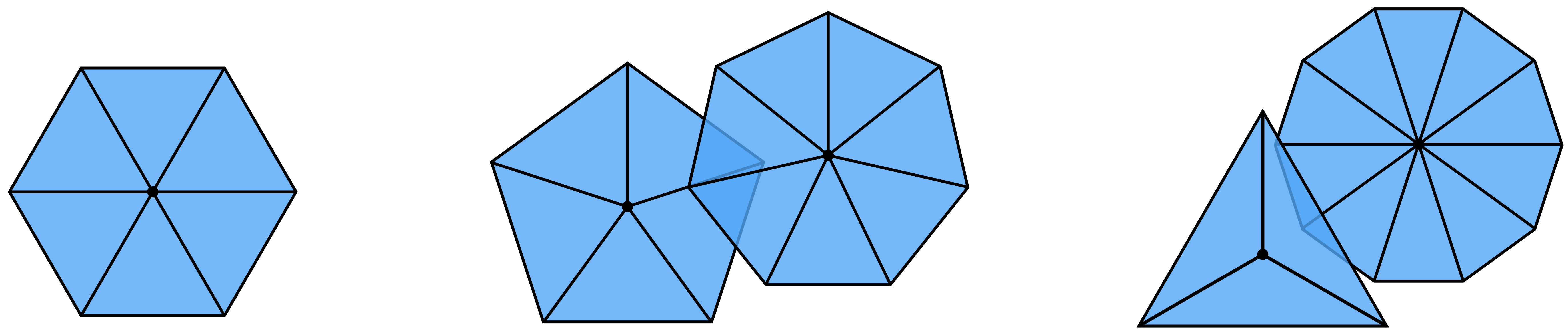
- One rule of thumb: triangle shape
- More specific condition: Delaunay
  - “Circumcircle interiors contain no vertices.”
- Not always a good condition, but often*
  - Good for simulation
  - Not always best for shape approximation



*See Shewchuk, “What is a Good Linear Element”

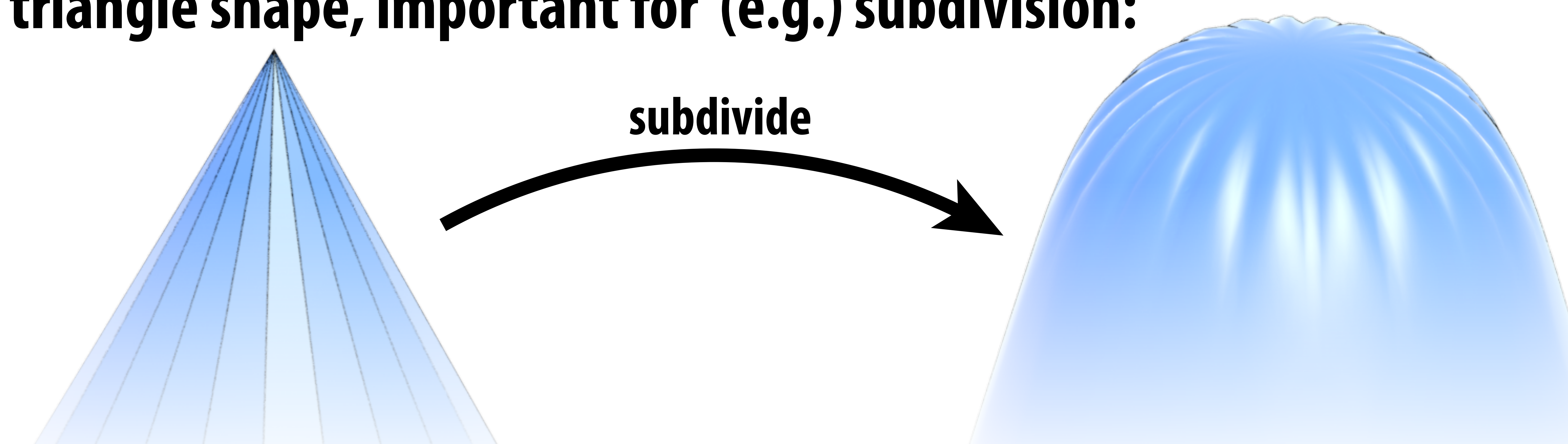
What else constitutes a good mesh?

- **Rule of thumb: regular vertex degree**
- **Triangle meshes: ideal is every vertex with valence 6:**



“GOOD” “OK” “BAD”

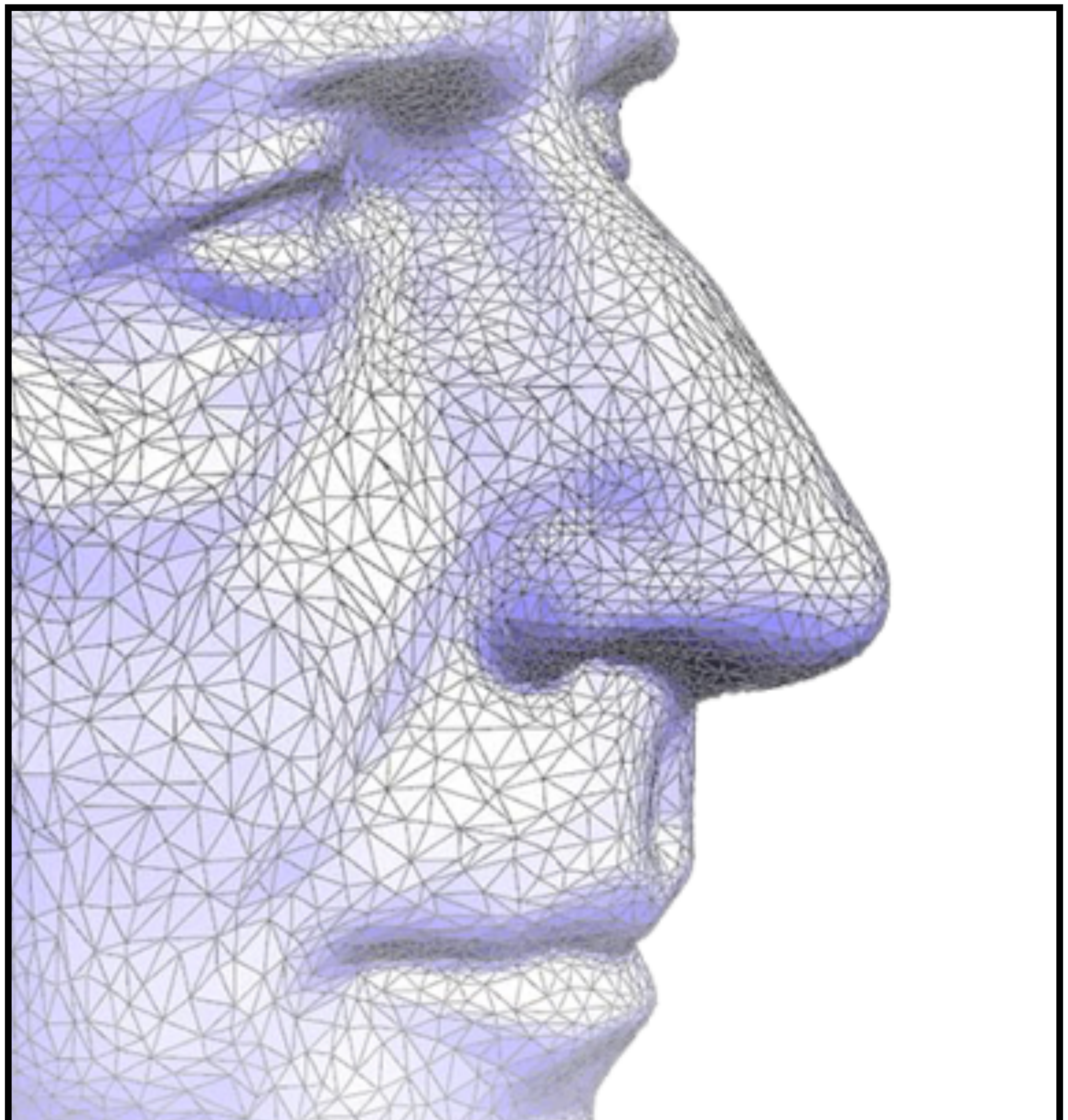
Why? Better triangle shape, important for (e.g.) subdivision:

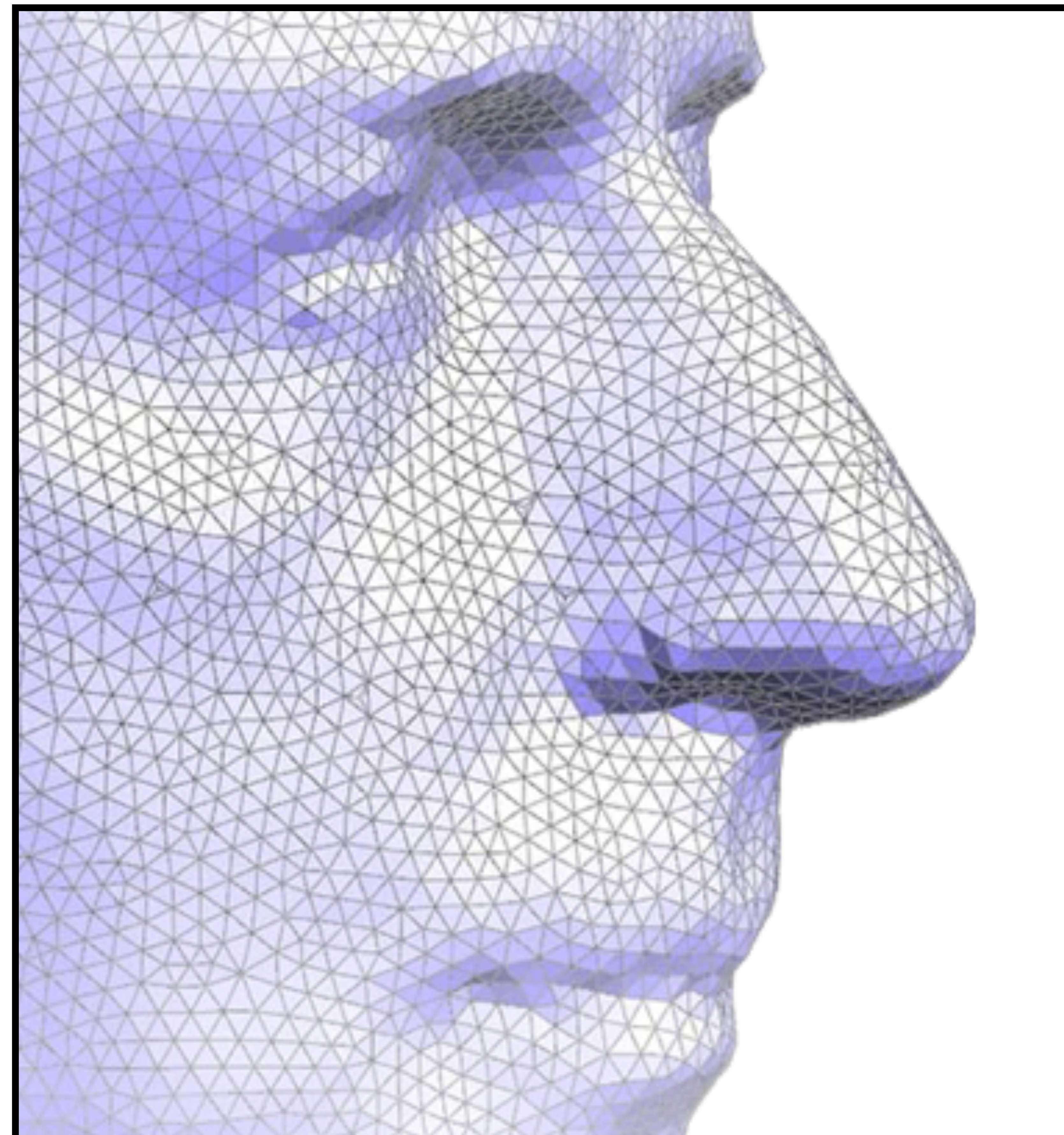


*See Shewchuk, “What is a Good Linear Element”

Isotropic remeshing

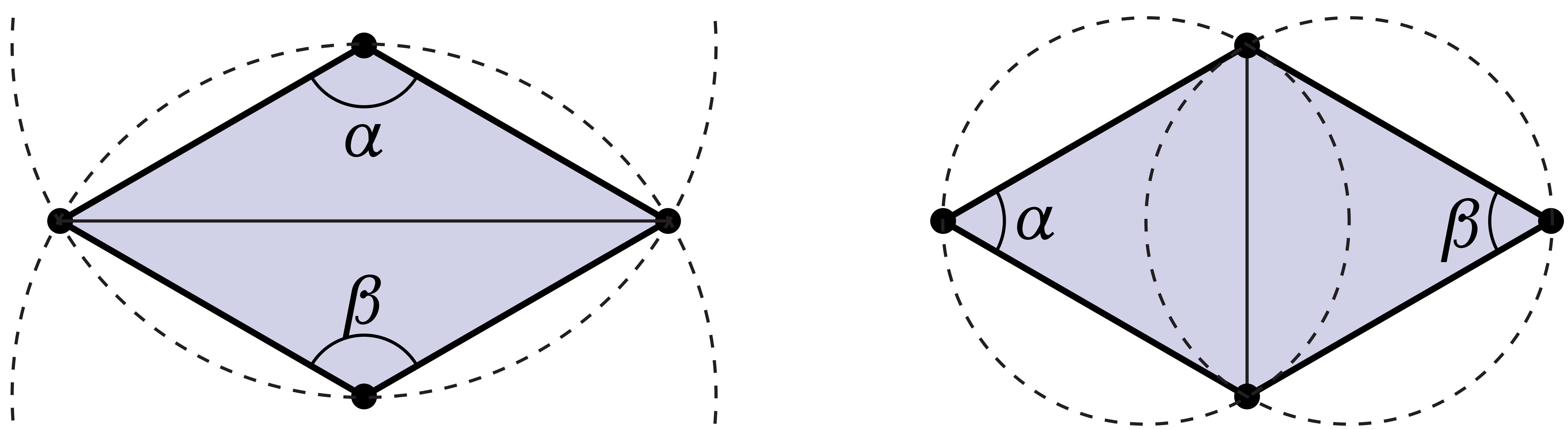
Goal: try to make triangles uniform in shape and size





How do we make a mesh “more delaunay”?

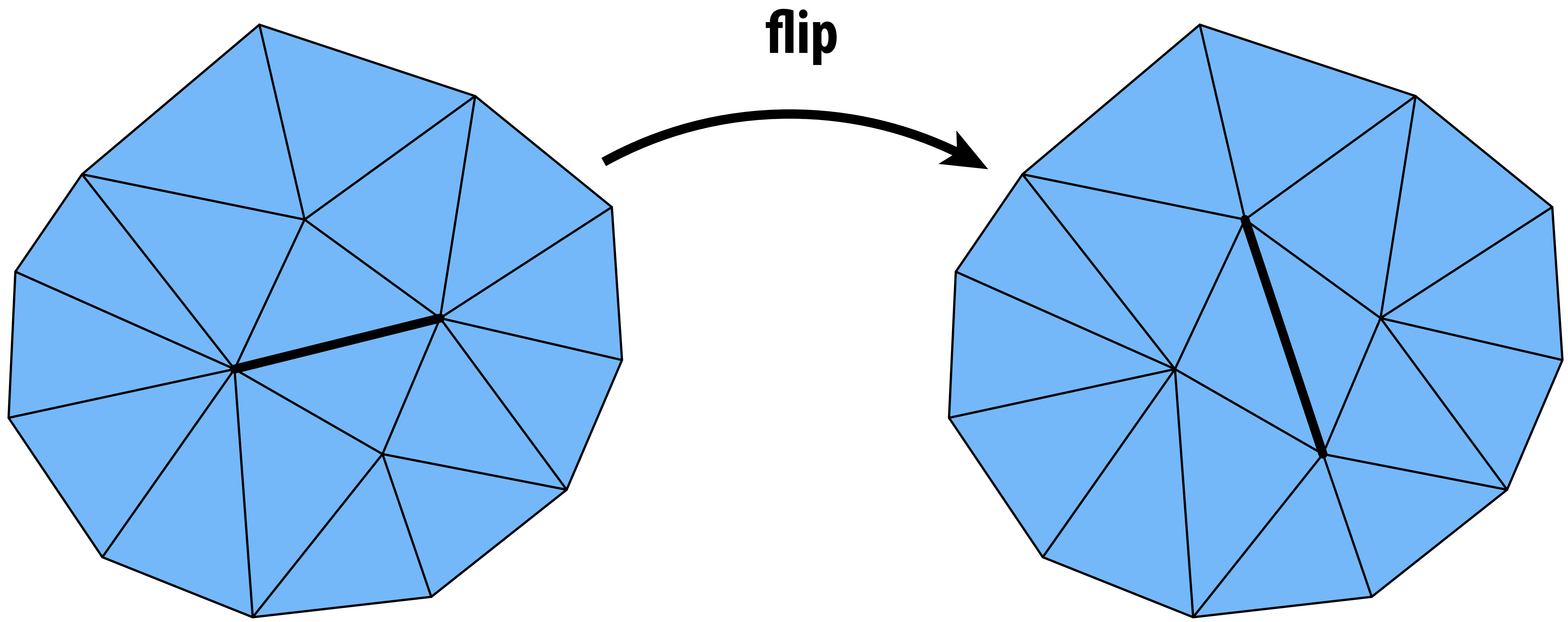
- Already have a good tool: edge flips!
- If $\alpha+\beta > \pi$, flip it!



- In practice: a simple, effective way to improve mesh quality

How do we improve degree?

- Edge flips!
- If total deviation from degree 6 gets smaller, flip it!

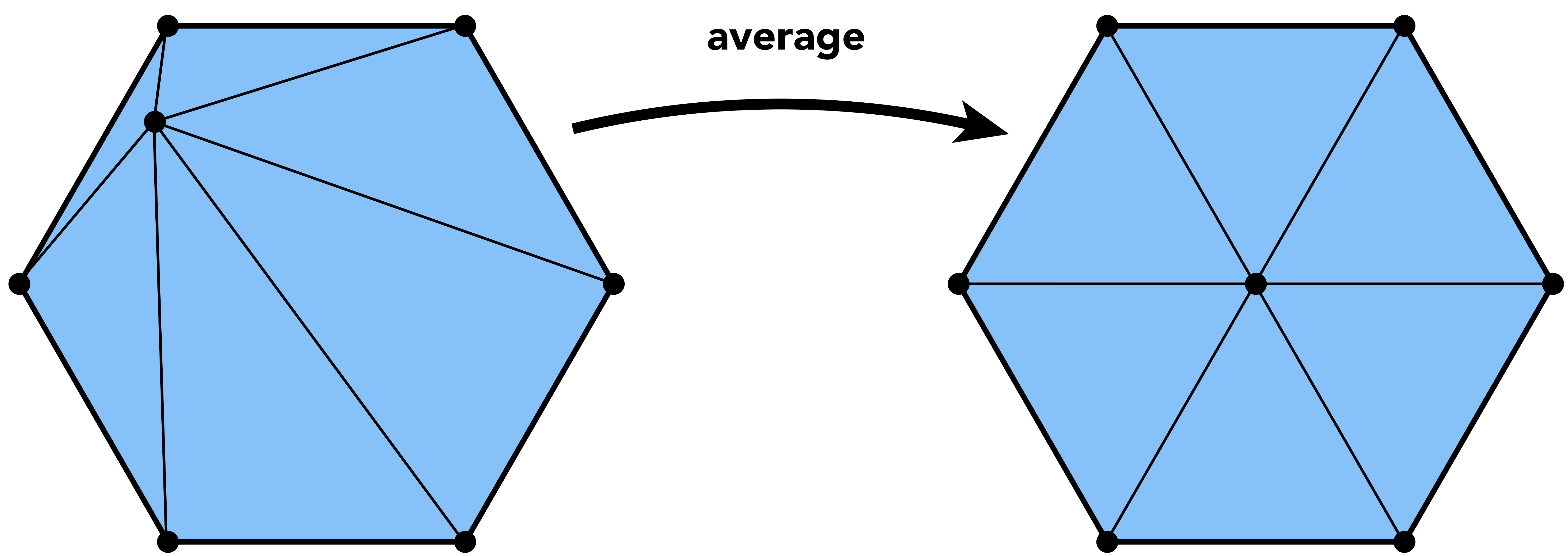


Iterative edge flipping acts like "discrete diffusion" of degree

No (known) guarantees; works well in practice

How do we make triangles "more round"?

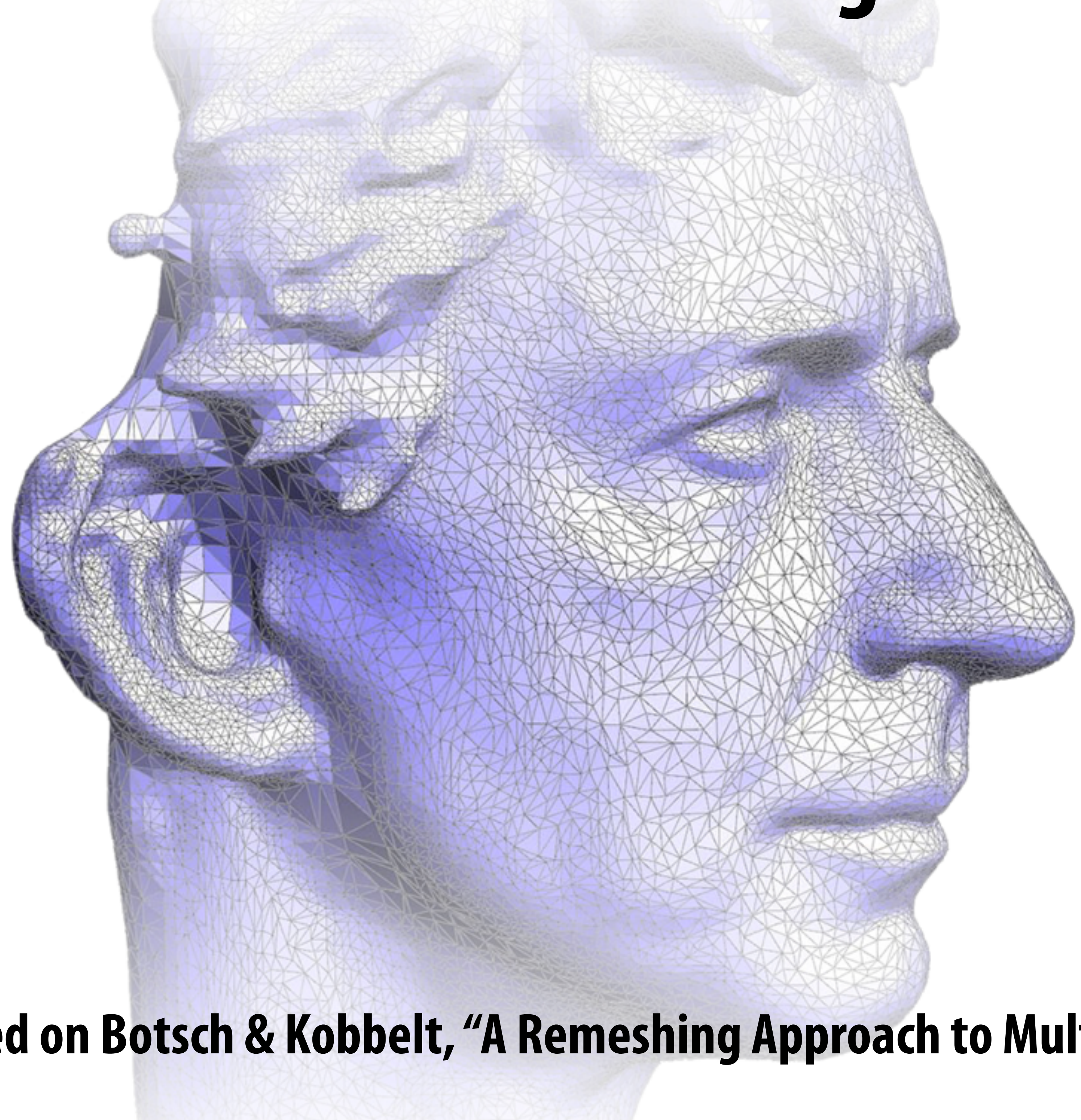
- Delaunay doesn't mean equilateral triangles
- Can often improve shape by centering vertices:

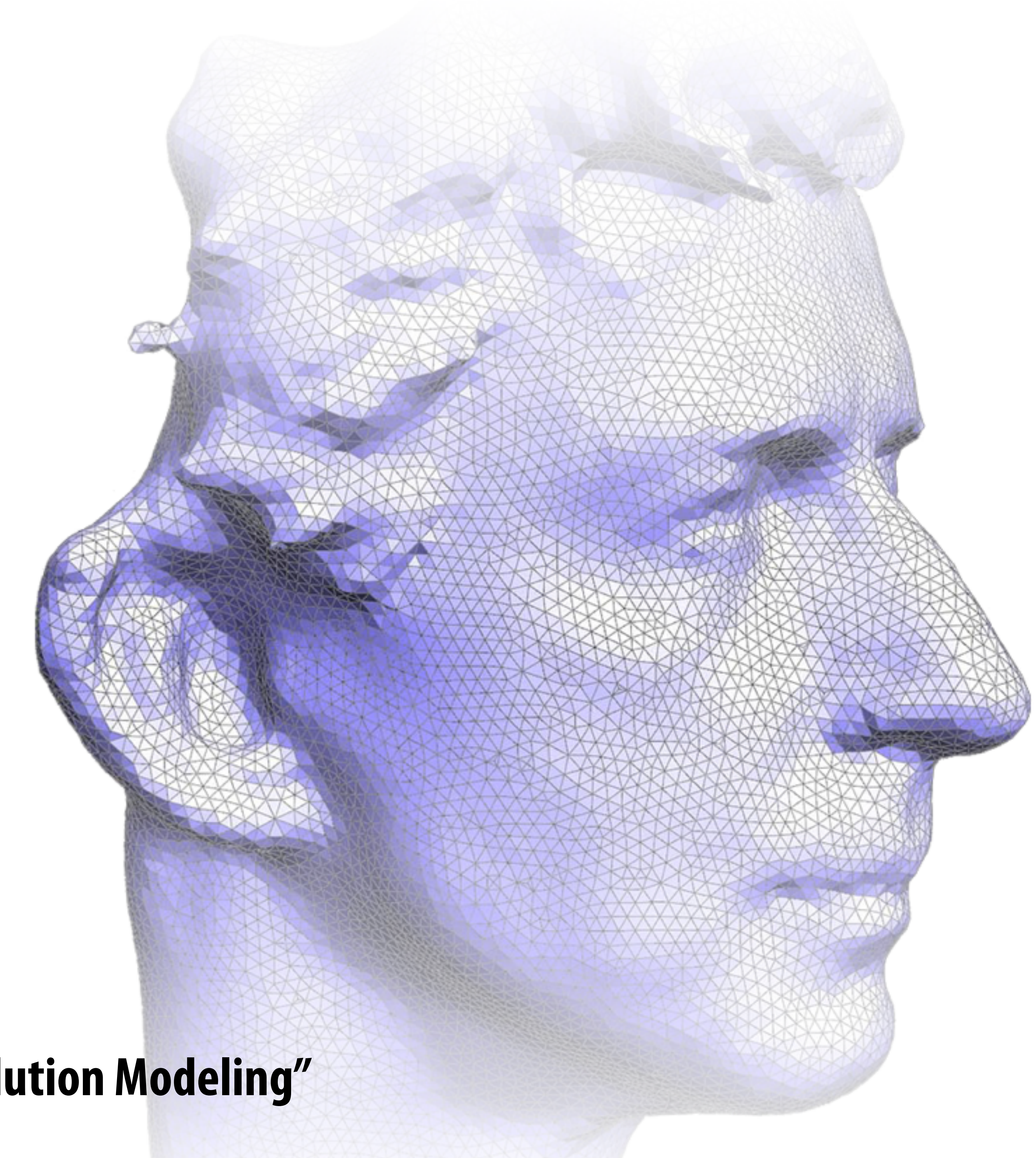


[See Crane, "Digital Geometry Processing with Discrete Exterior Calculus"]

Isotropic remeshing algorithm*

- **Repeat four steps:**
  - **Split edges over 4/3rds mean edge length**
  - **Collapse edges less than 4/5ths mean edge length**
  - **Flip edges to improve vertex degree**
  - **Center vertices tangentially**





*** Based on Botsch & Kobbelt, "A Remeshing Approach to Multiresolution Modeling"**

Things to remember

- **Triangle mesh representations**
  - Triangles vs points+triangles
  - Half-edge structure for mesh traversal and editing

- **Geometry processing basics**
  - Local operations: flip, split, and collapse edges
  - Upsampling by subdivision (Loop, Catmull-Clark)
  - Downsampling by simplification (Quadric error)
  - Regularization by isotropic remeshing

Acknowledgements

- Thanks to Keenan Crane, Ren Ng, Pat Hanrahan, James O'Brien, Steve Marschner for presentation resources