Lecture 2:

Drawing a Triangle
(+ the basics of sampling and anti-aliasing)

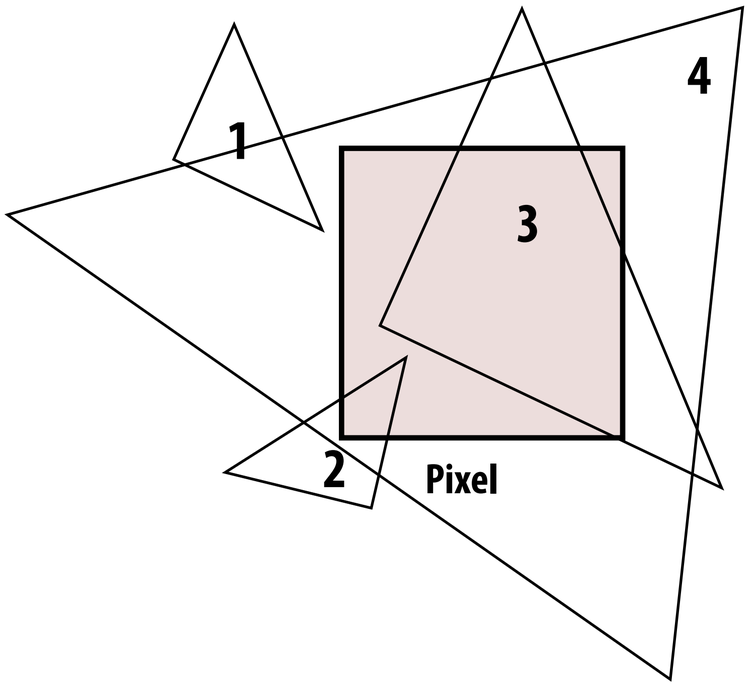
Interactive Computer Graphics
Stanford CS248, Winter 2022

Pre-class warm-up: introduce yourself to friends, and talk about these questions.

p.s. enjoy the boba, we'll start class at 1:35…

Question 1:
Last class we talked about the challenges of drawing a line. Now let's extend this to drawing triangles.

In your opinion which triangles "cover" this pixel? Imagine the triangles were bright red? What color should a drawing algorithm "paint" the pixel?

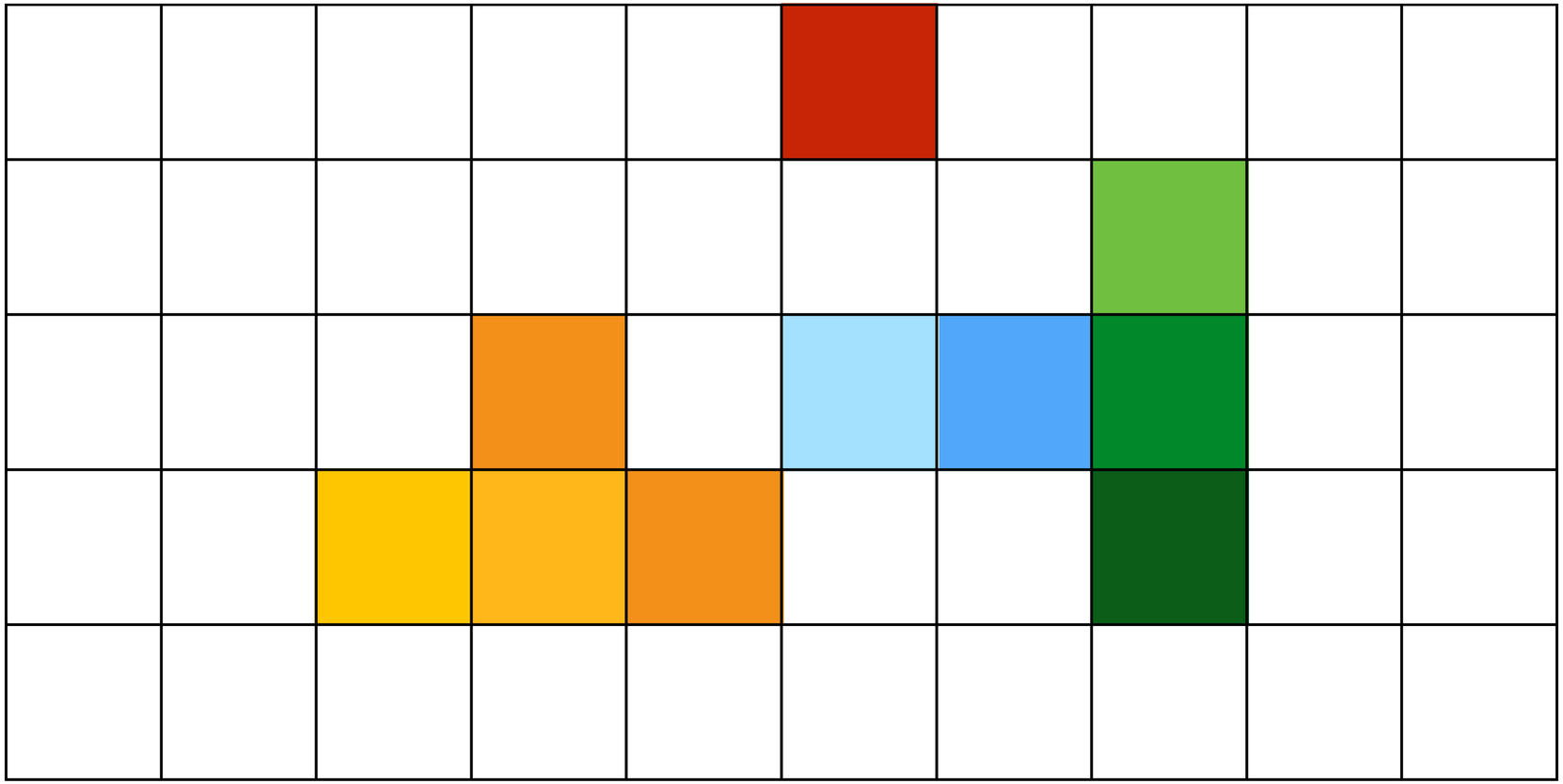


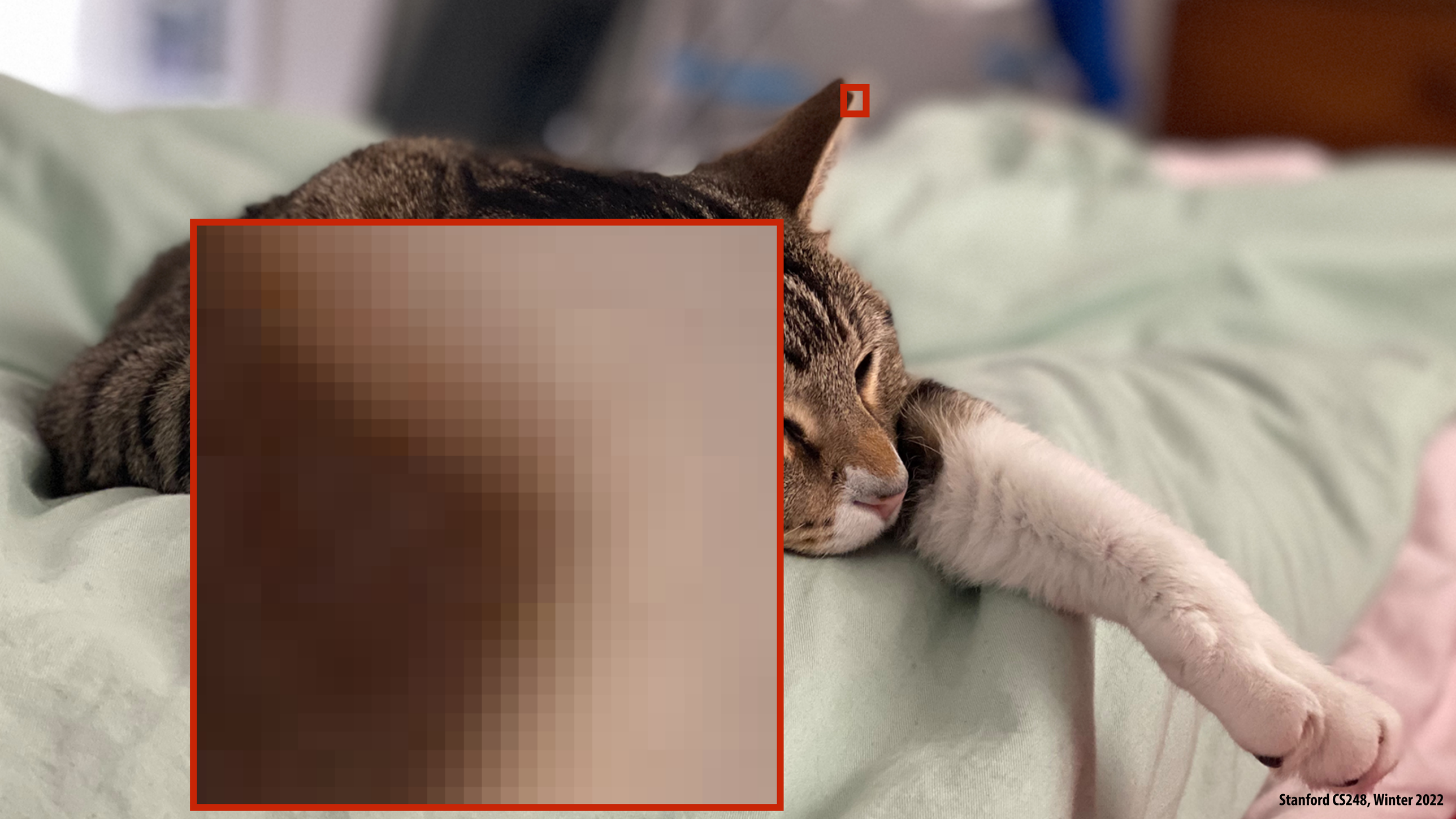
Question 2:
Consider the following C code that performs an operation on an input matrix… Think about the input matrix as an array of values defining a grayscale image of size WIDTHxHEIGHT. What does the code do?

```
float input[(WIDTH+2) * (HEIGHT+2)];
float output[WIDTH * HEIGHT];

float weights[] = {1./9, 1./9, 1./9,
                   1./9, 1./9, 1./9,
                   1./9, 1./9, 1./9};

for (int j=0; j<HEIGHT; j++) {
   for (int i=0; i<WIDTH; i++) {
      float tmp = 0.f;
      for (int jj=0; jj<3; jj++)
         for (int ii=0; ii<3; ii++) {
            int idx = (j+jj)*(WIDTH+2) + (i+ii);
            tmp += input[idx] * weights[jj*3 + ii];
         }
      output[j*WIDTH + i] = tmp;
  }
}
```

Last time

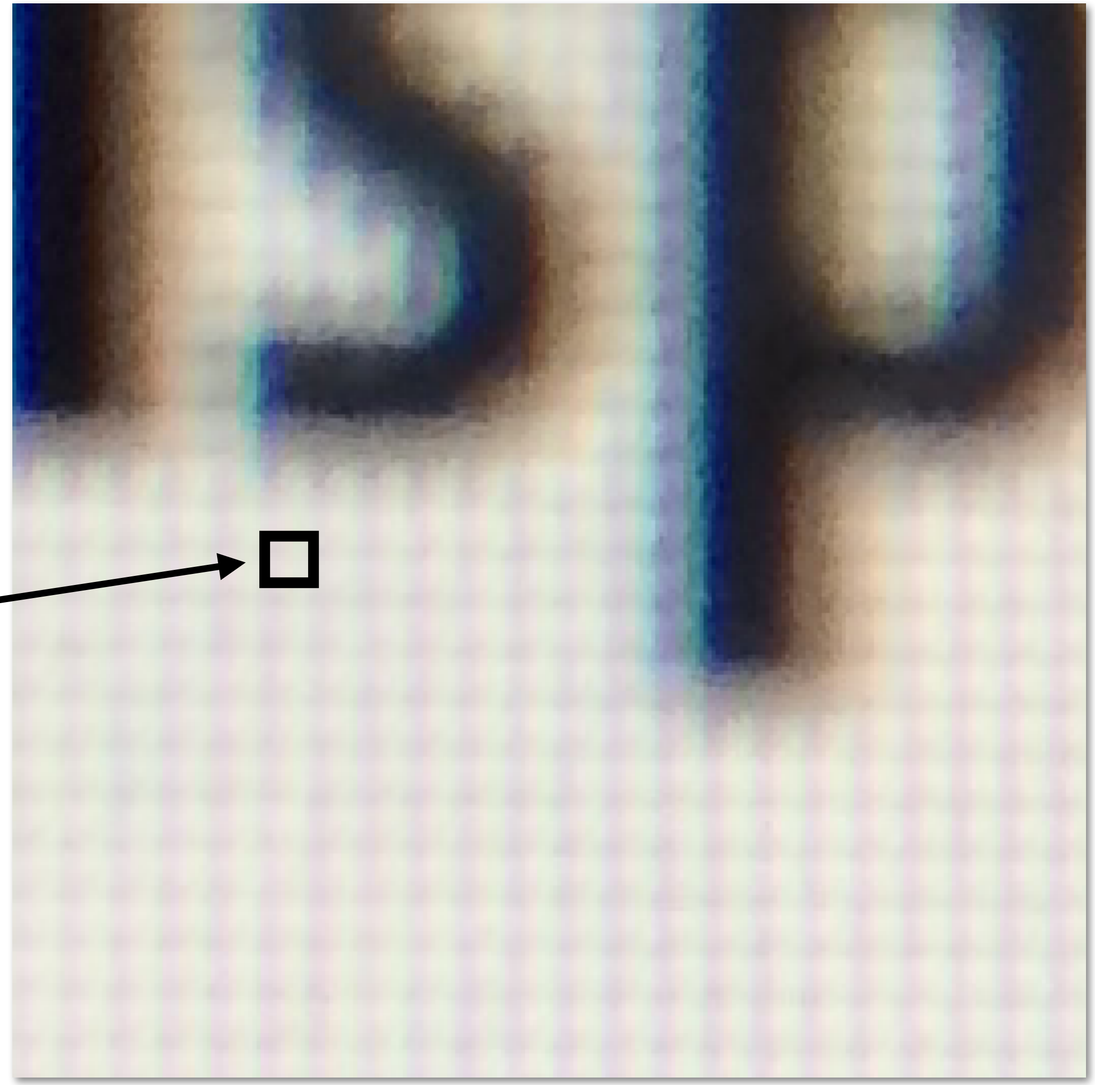
- A very simple notion of digital image representation (that we are about to challenge!)
- An image = a 2D array of color values





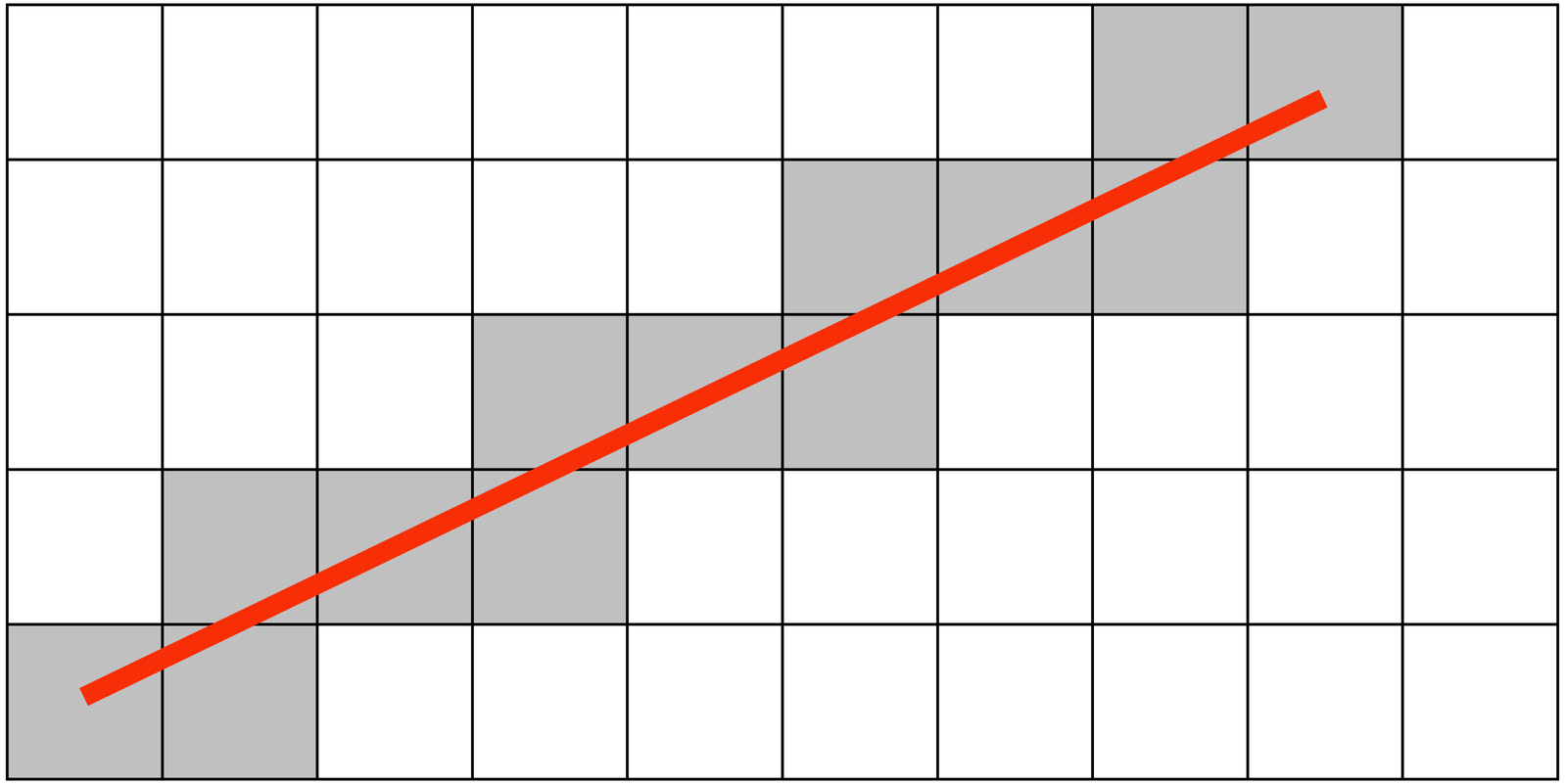
Last time

A display converts a color value at each pixel in an image to emitted light



Display pixel on my laptop
(close up photo)

Last time: what pixels should we color in to draw a line?



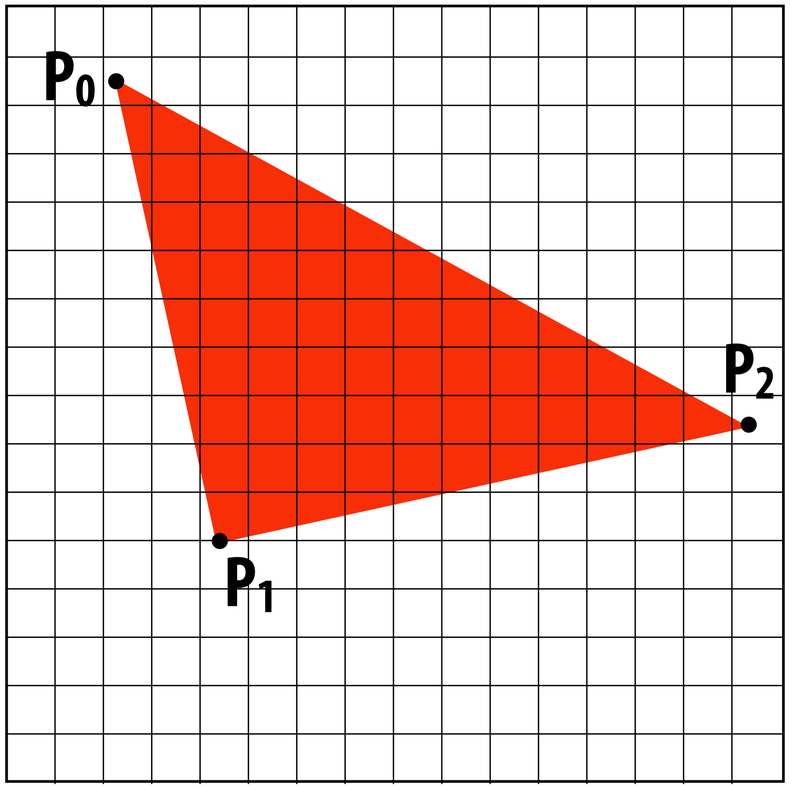
Light up all pixels intersected by the line?

Today: drawing a triangle

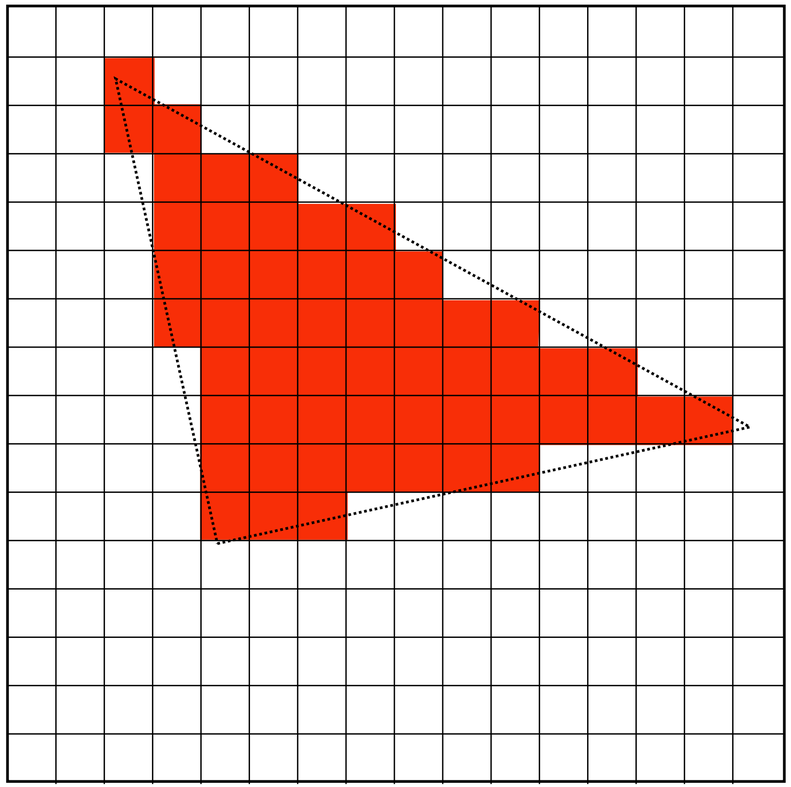
(Converting a representation of a triangle into an image)

"Triangle rasterization”

Input:
2D position of triangle vertices: P_0, P_1, P_2

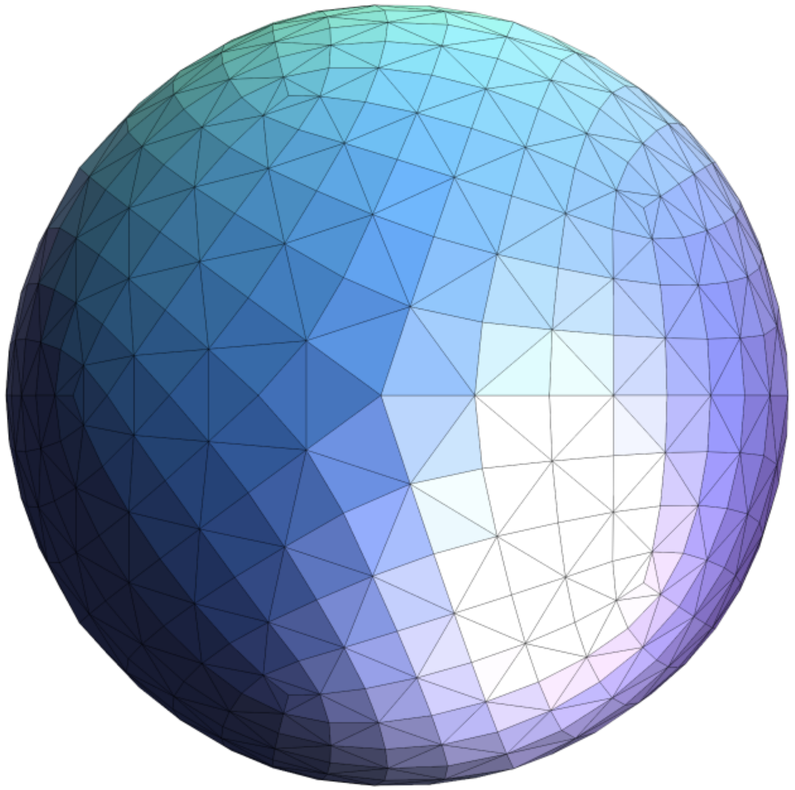


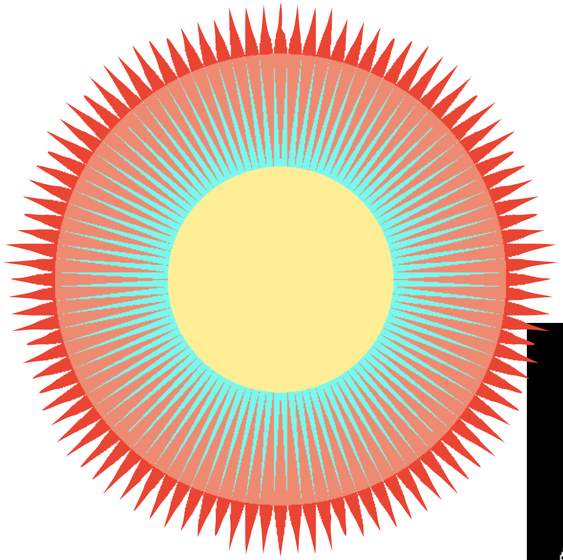
Output:
set of pixels “covered” by the triangle

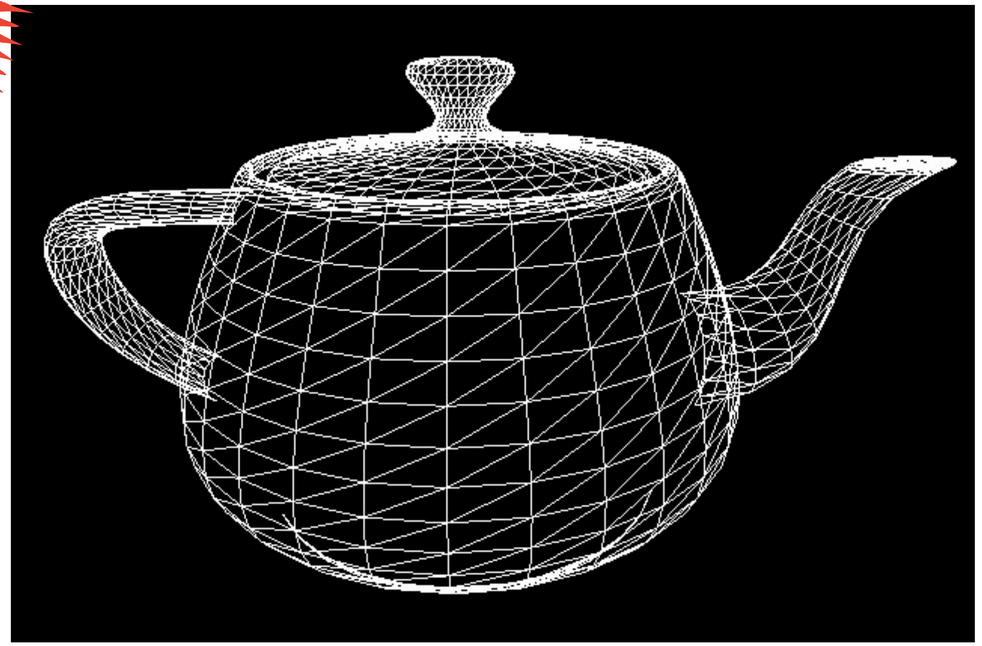


Why triangles?

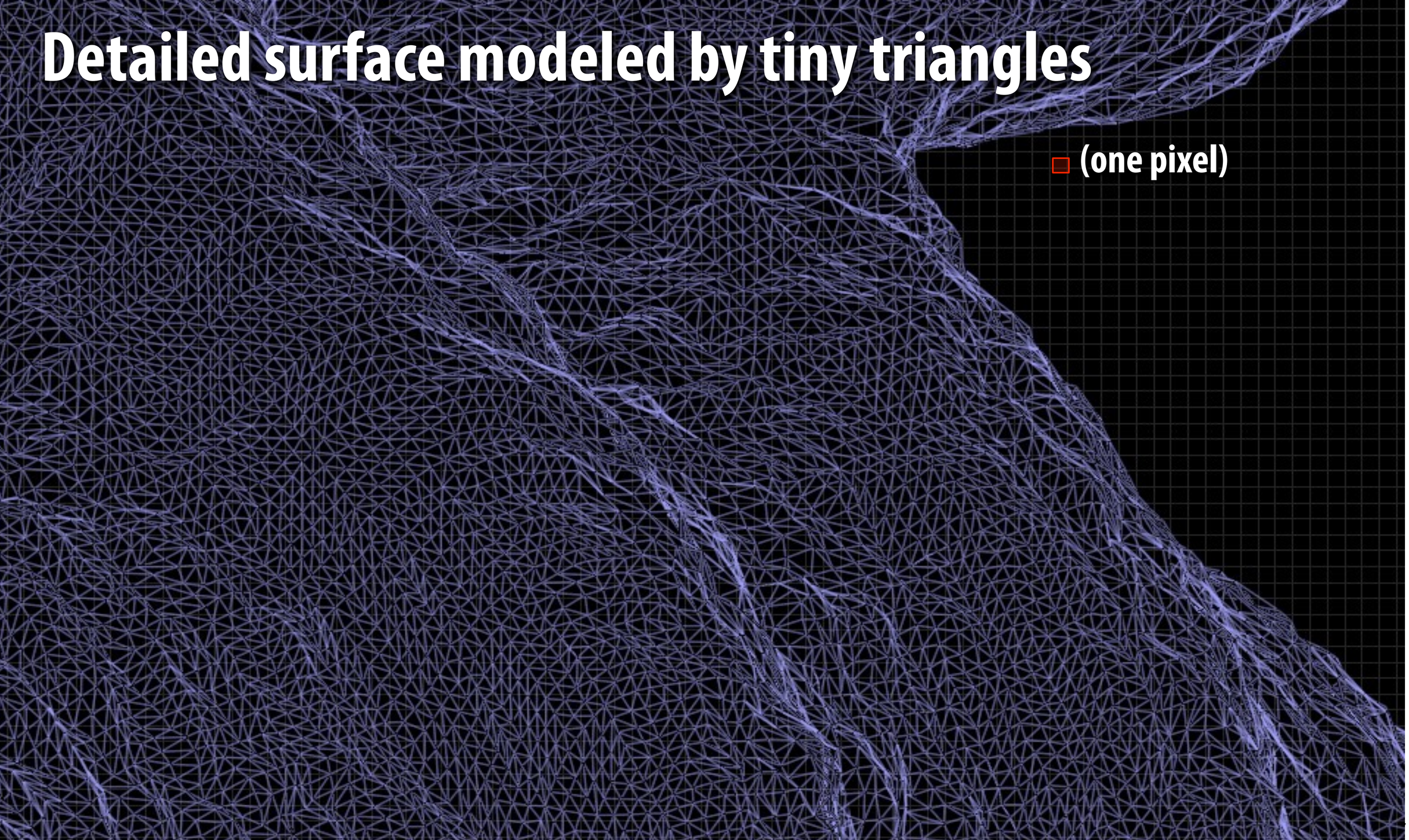
Triangles are a basic block for creating more complex shapes and surfaces





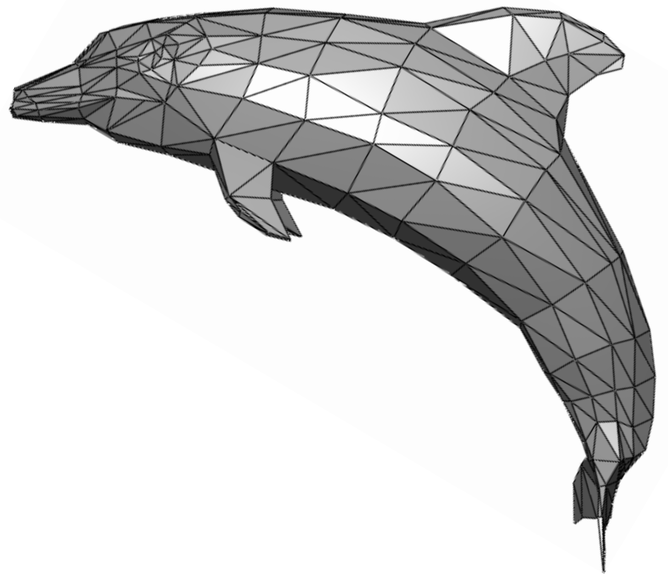


Detailed surface modeled by tiny triangles



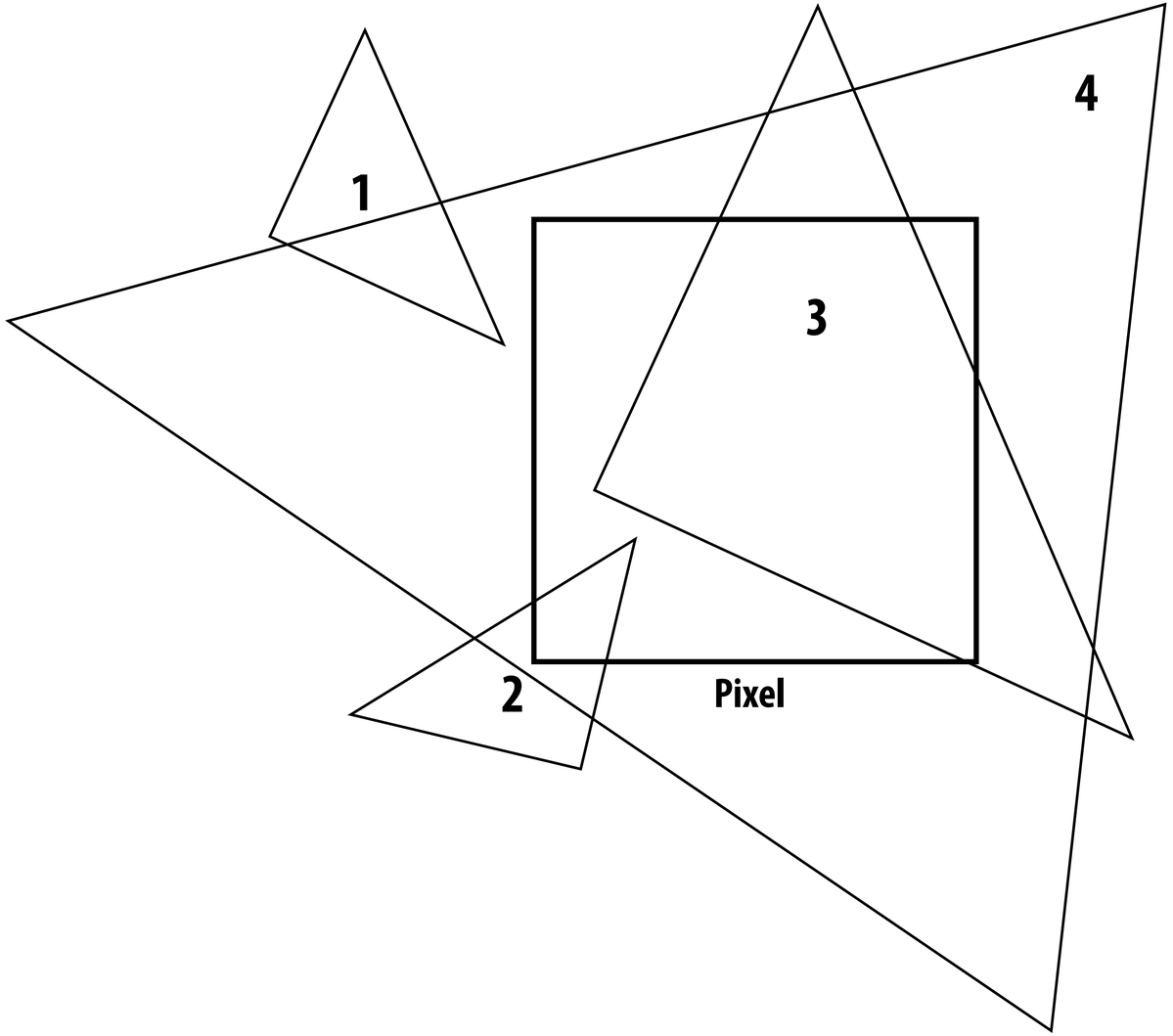
Triangles - a fundamental primitive

- Why triangles?
  - Most basic polygon
    - Can break up other polygons into triangles
    - Allows programs to optimize one implementation
  - Triangles have unique properties
    - Guaranteed to be planar
    - Well-defined interior
    - Well-defined method for interpolating values at vertices over triangle (a topic of a future lecture)

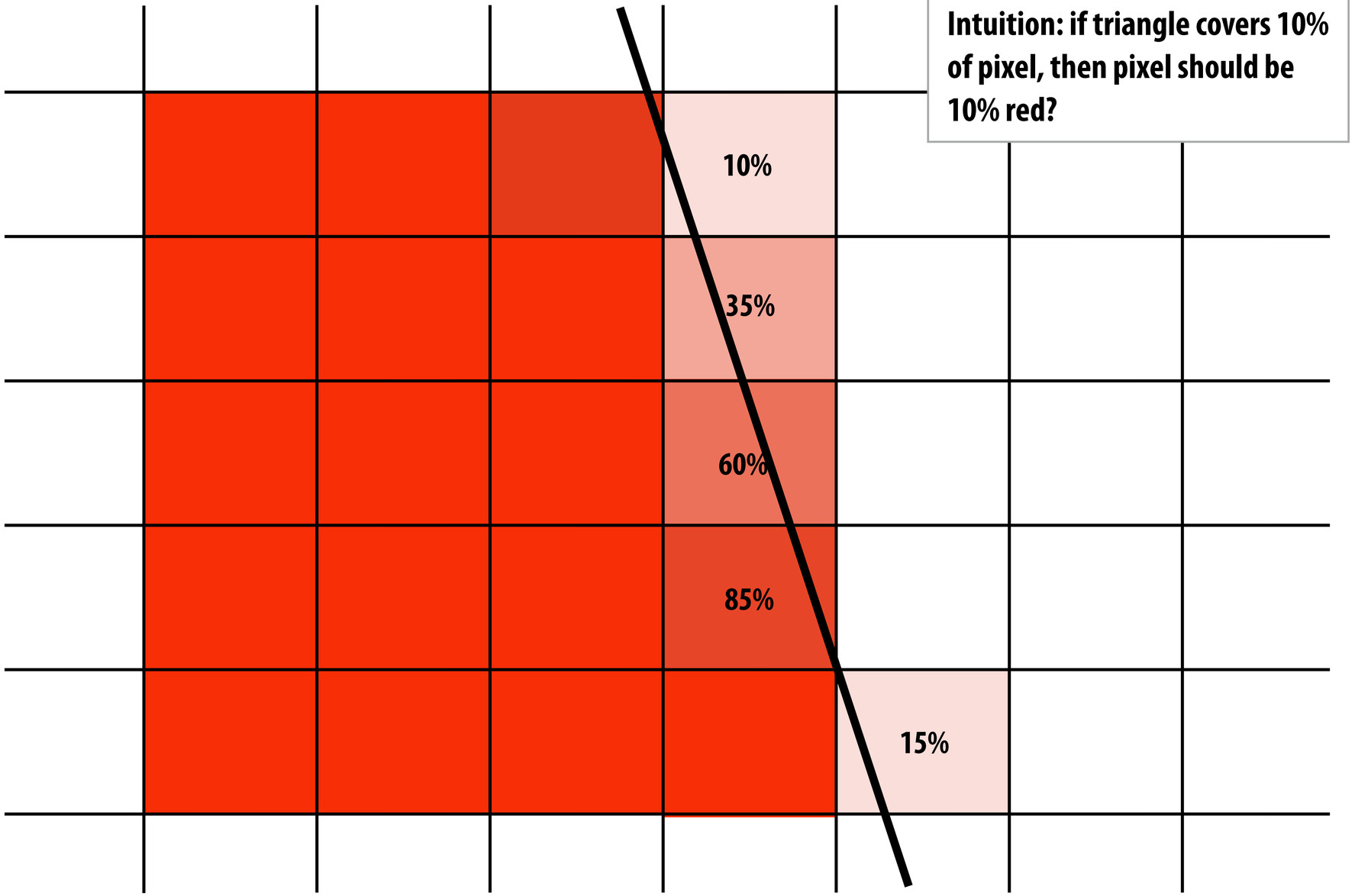


What does it mean for a pixel to be covered by a triangle?

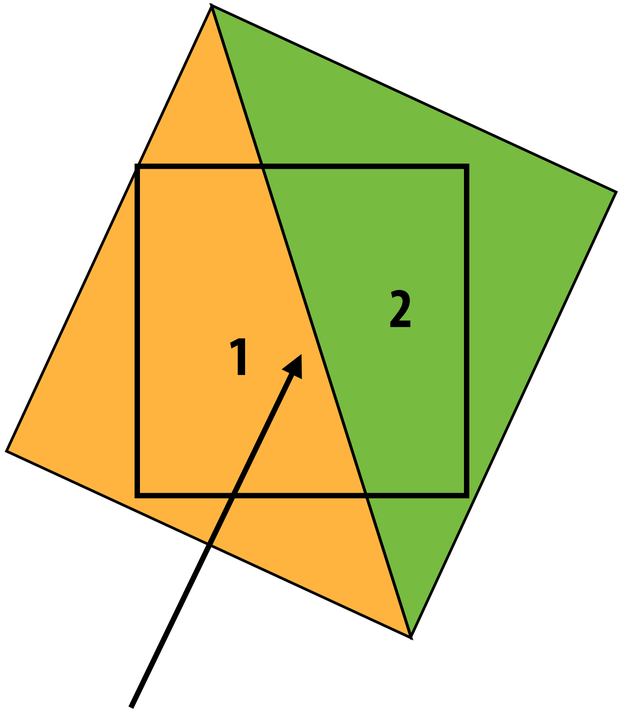
Question: which triangles "cover" this pixel?



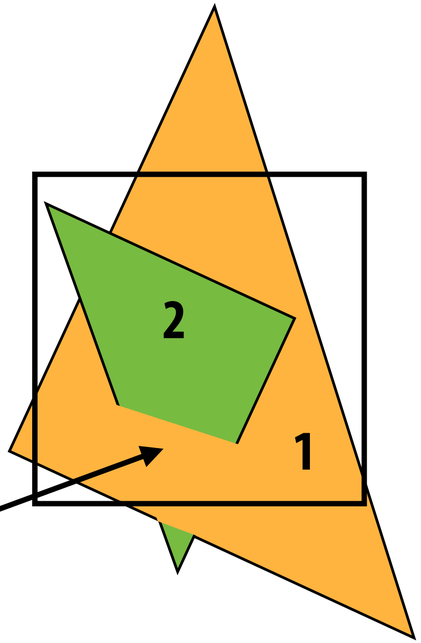
One option: compute fraction of pixel area covered by triangle, then color pixel according to this fraction.



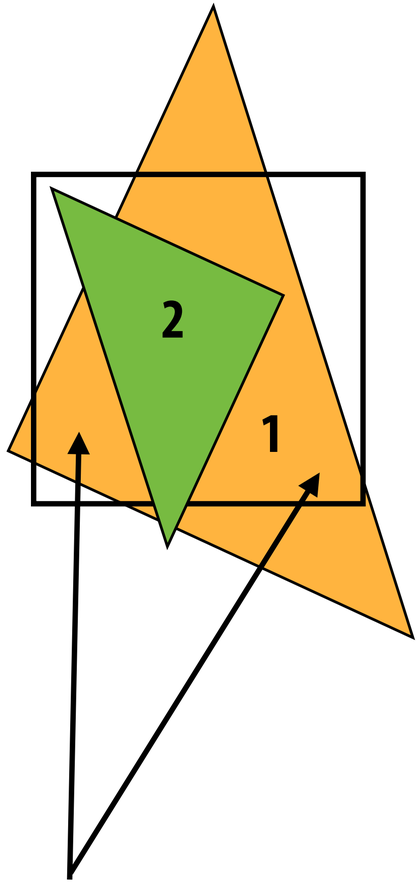
Analytical coverage schemes get tricky when considering occlusion of one triangle by another



Pixel covered by triangle 1, other half covered by triangle 2



Interpenetration of triangles: even trickier

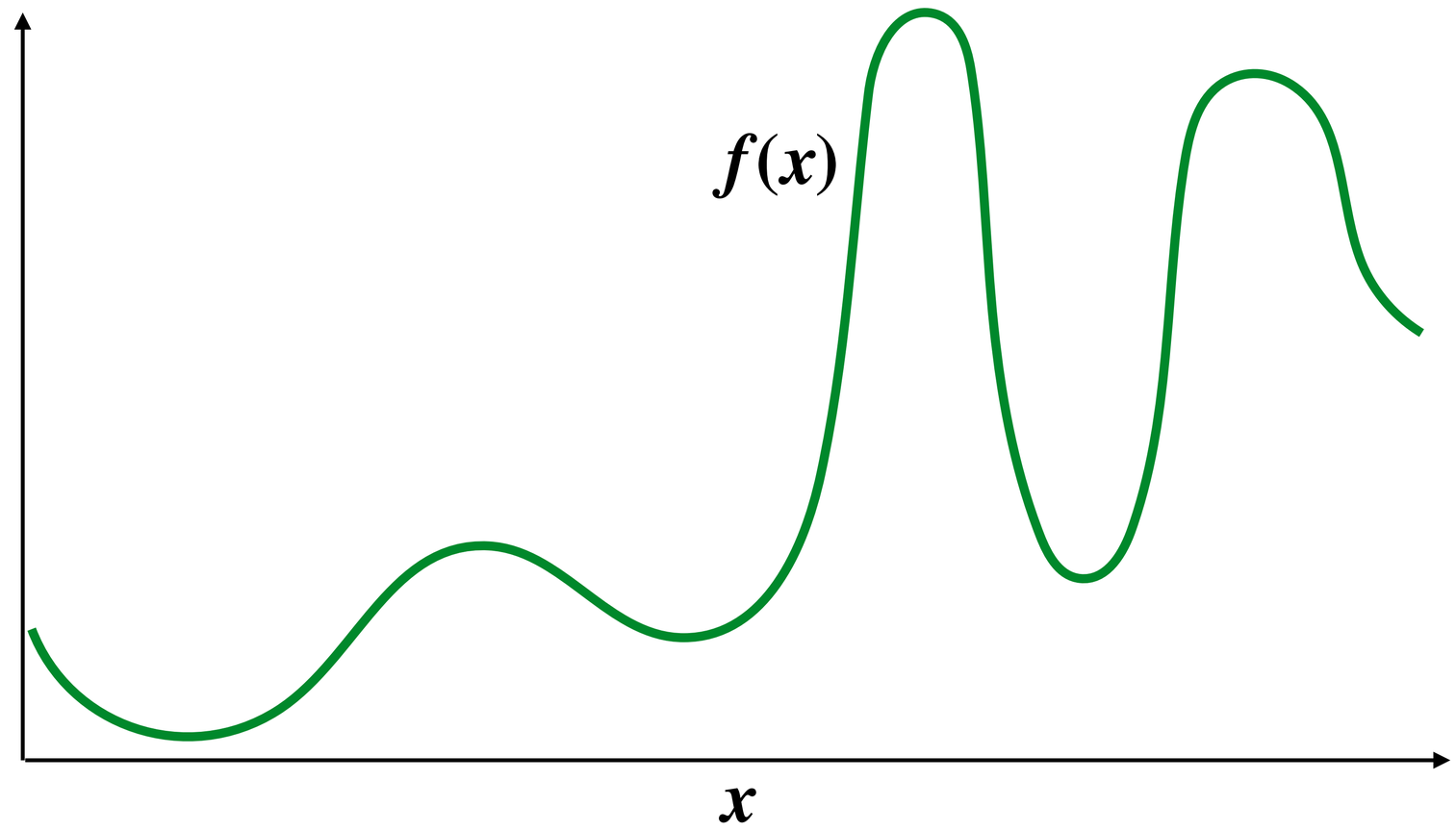


Two regions of triangle 1 contribute to pixel.
One of these regions is not even convex.

Today we will draw triangles using a simple method: point sampling

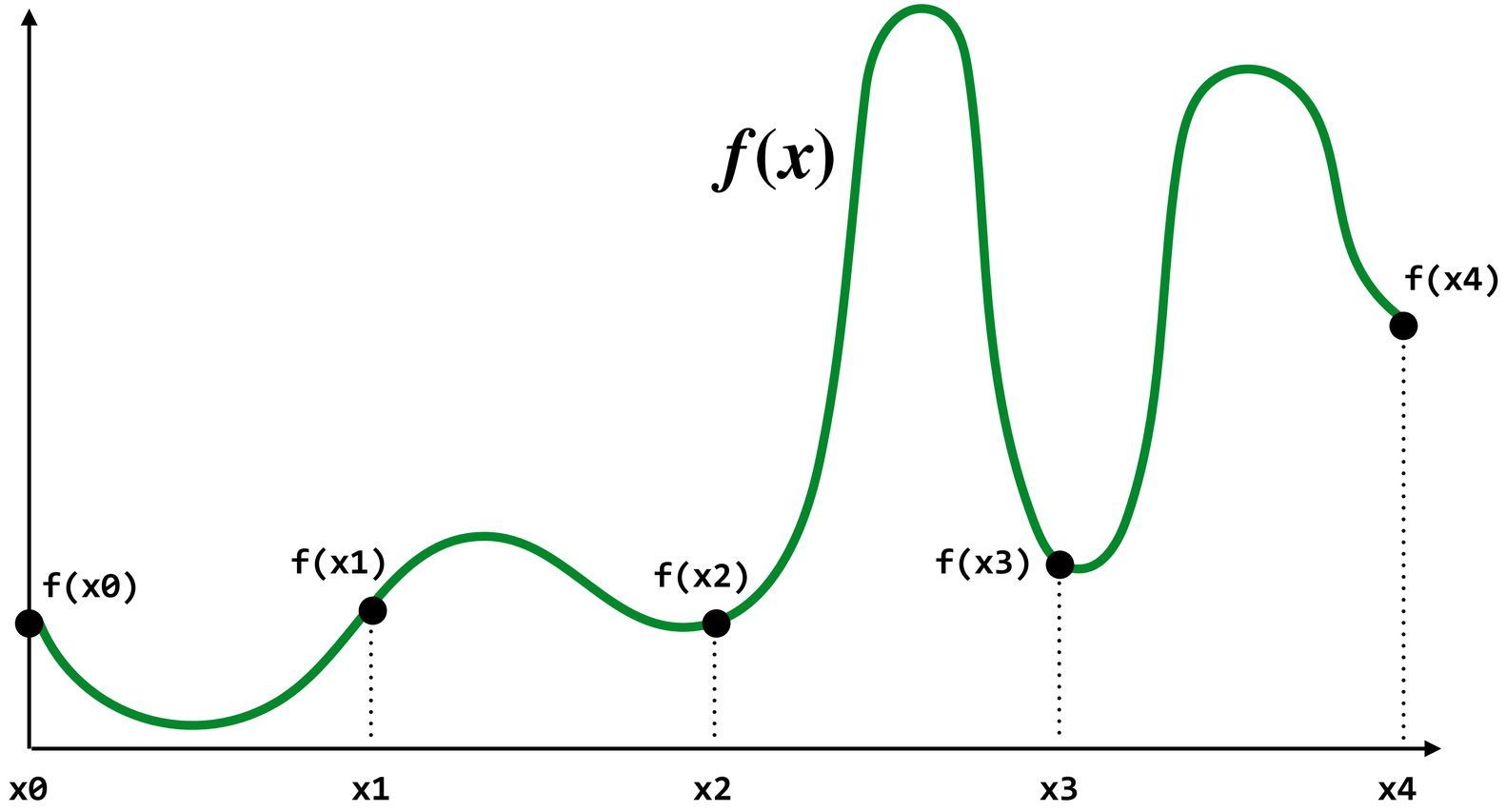
(let's consider sampling in 1D first)

Consider a 1D signal: f(x)



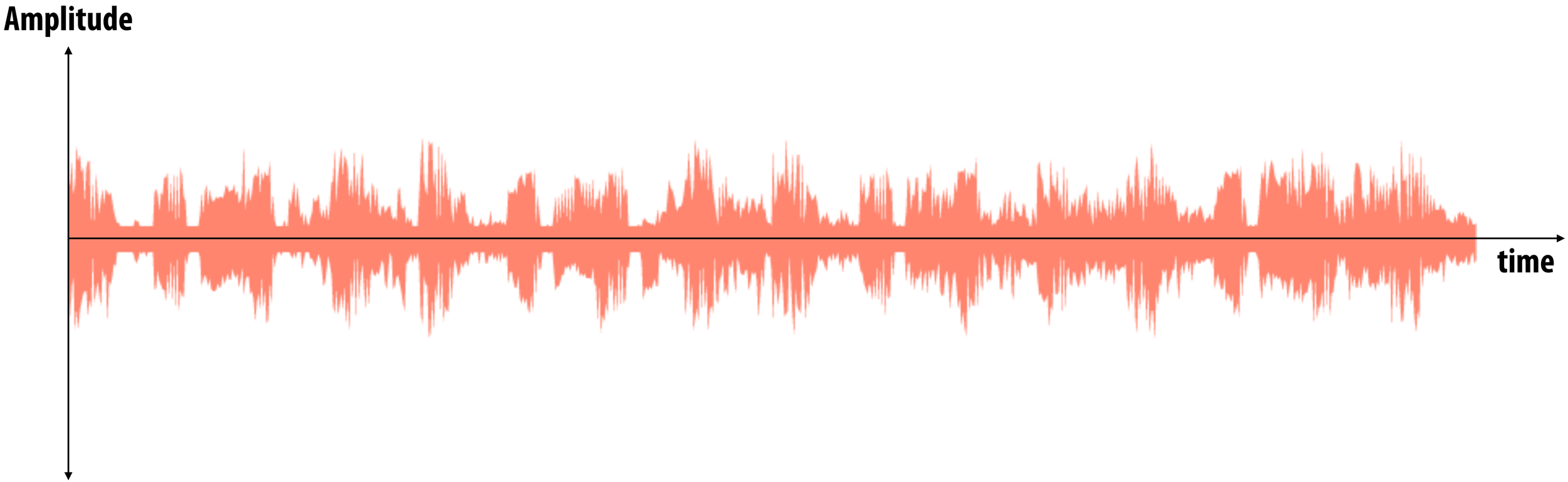
Sampling: taking measurements of a signal

Below: five measurements ("samples") of $f(x)$



Audio file: stores samples of a 1D signal

Audio is often sampled at 44.1 KHz



Sampling a function

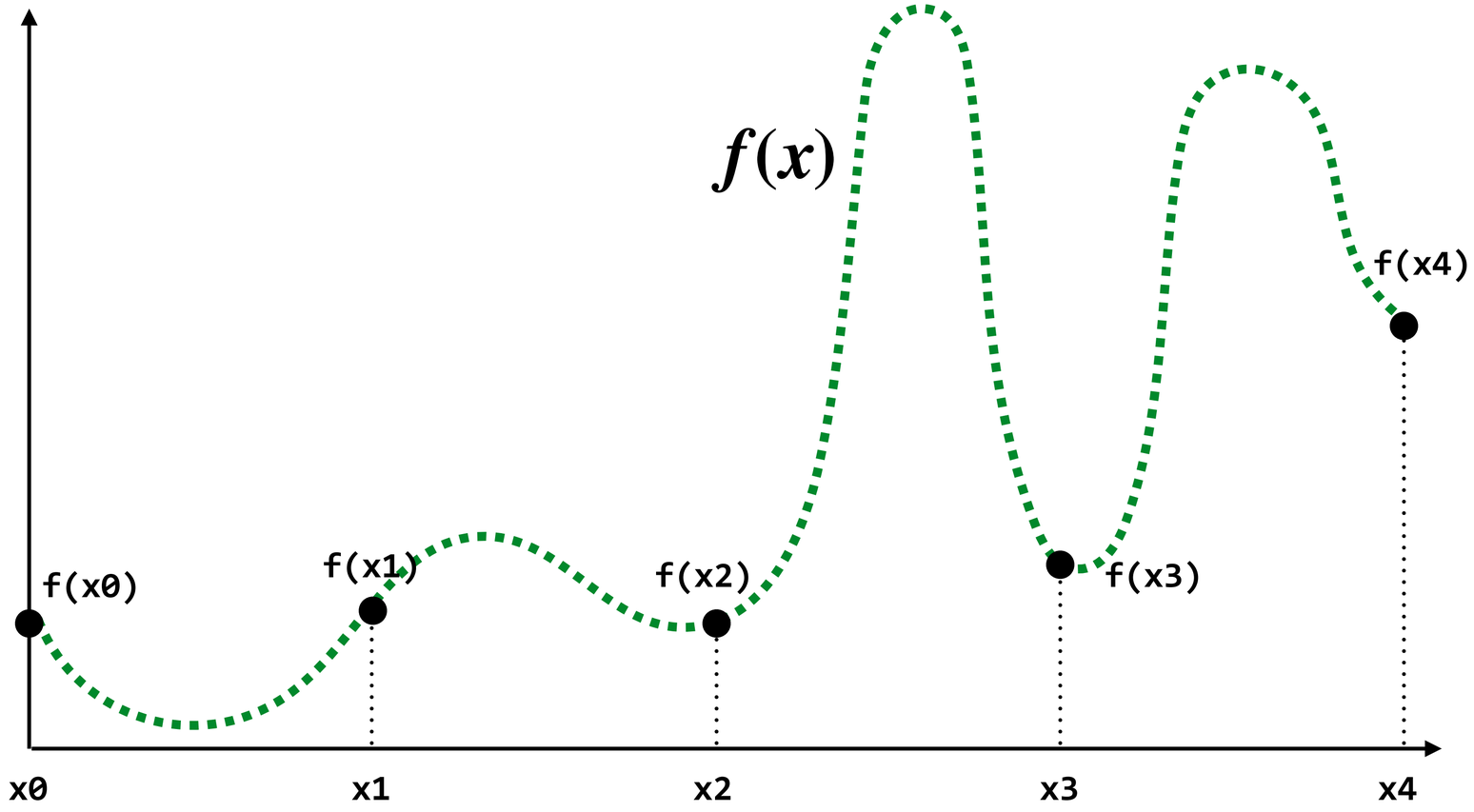
- Evaluating a function at a point is sampling the function's value

- We can discretize a function by periodic sampling

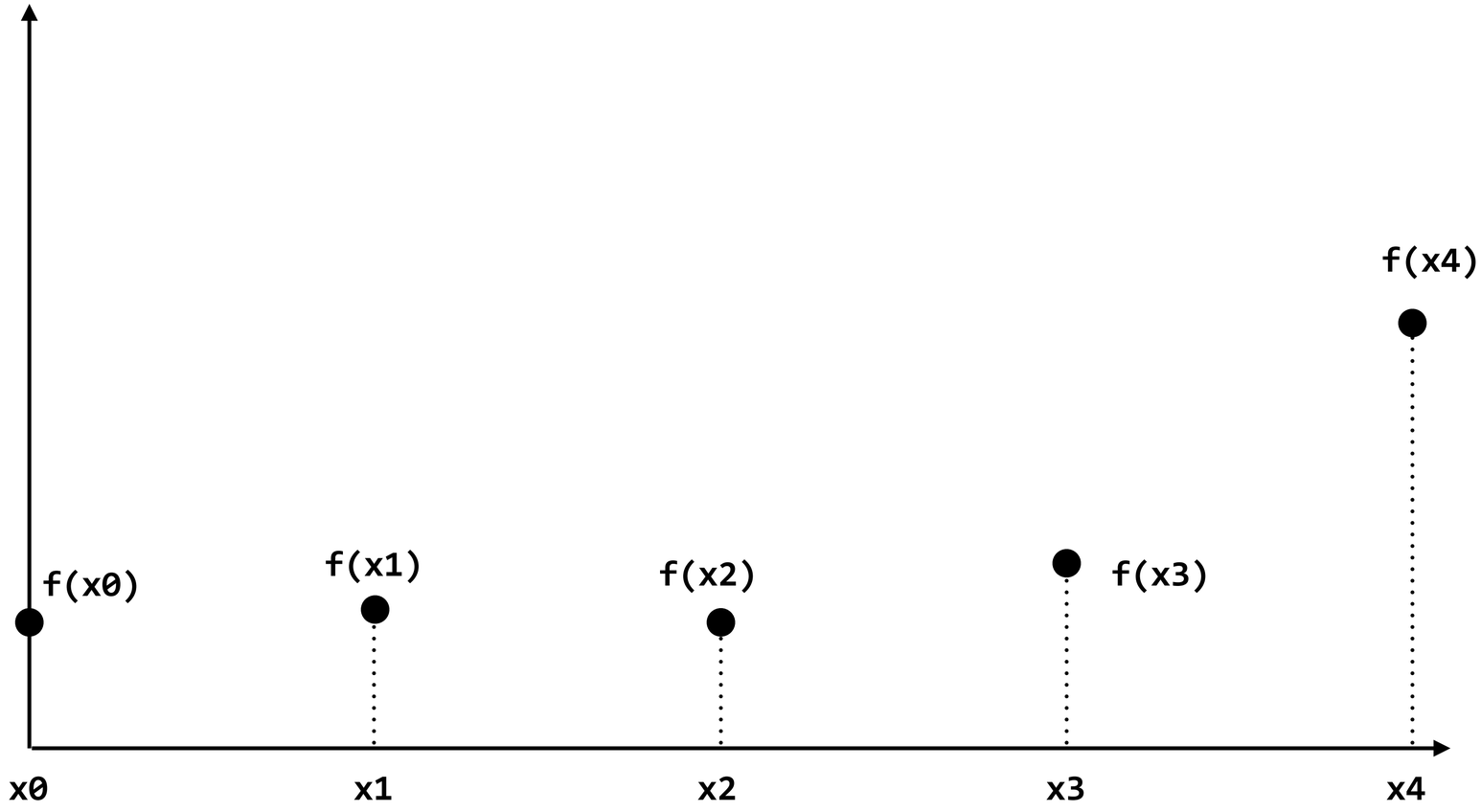
```
for(int x = 0; x < xmax; x++)
    output[x] = f(x);
```

- Sampling is a core idea in graphics. In this class we'll sample time (1D), area (2D), angle (2D), volume (3D), etc . . .

Reconstruction: given a set of samples, how might we attempt to reconstruct the original (continuous) signal $f(x)$?



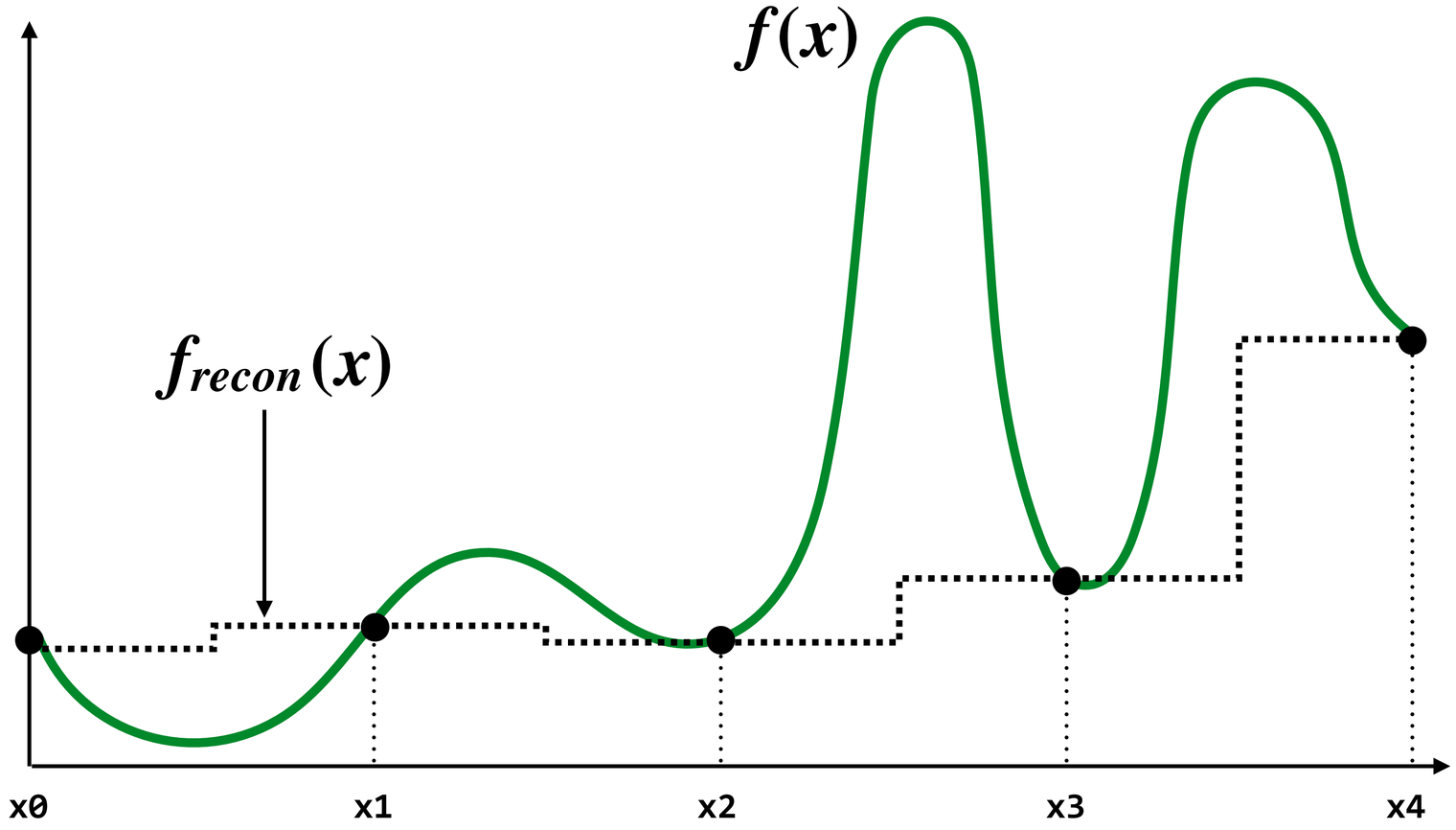
Reconstruction: given a set of samples, how might we attempt to reconstruct the original (continuous) signal $f(x)$?



Piecewise constant approximation

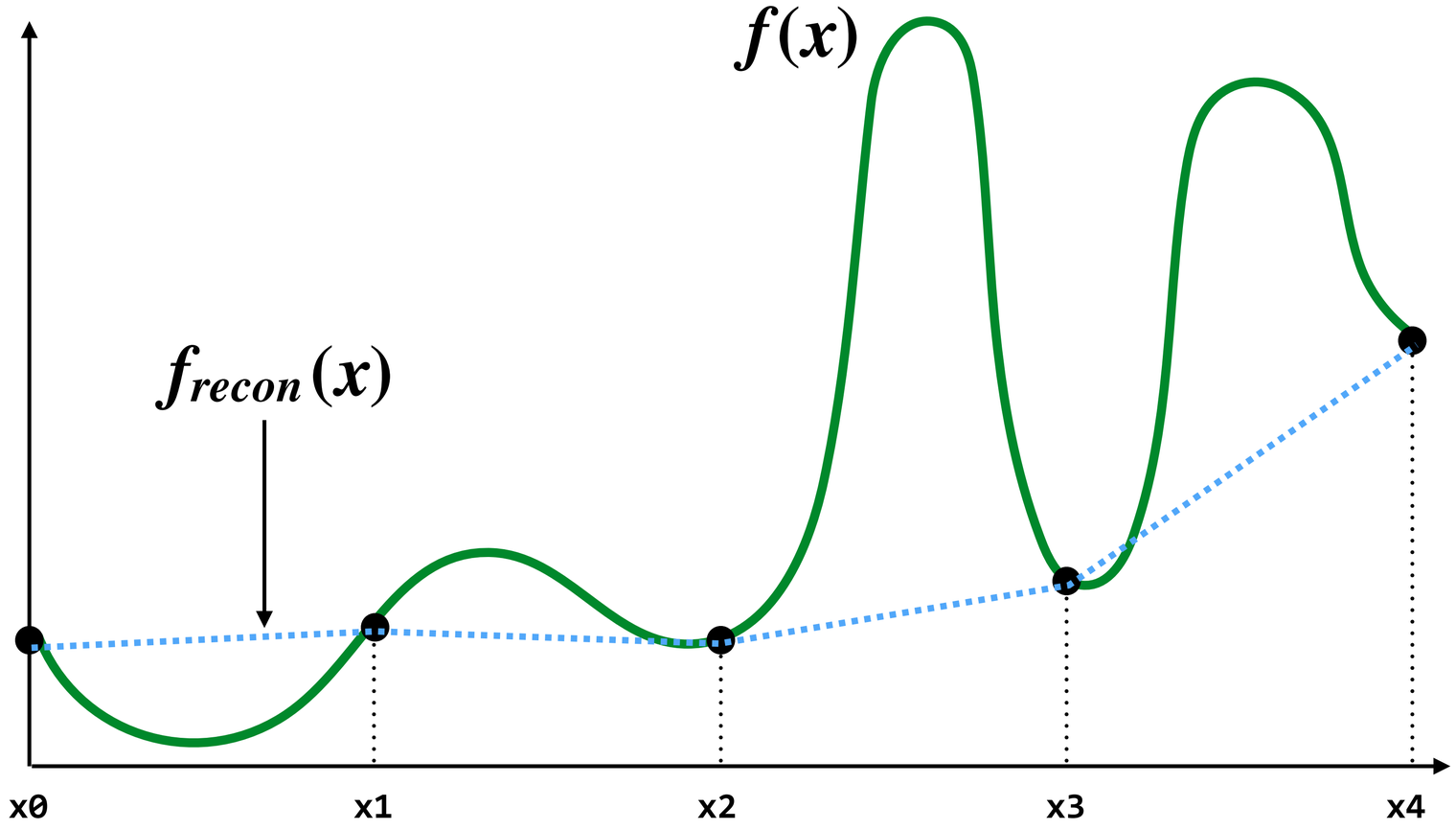
$f_{recon}(x)$ = value of sample closest to x

$f_{recon}(x)$ approximates $f(x)$

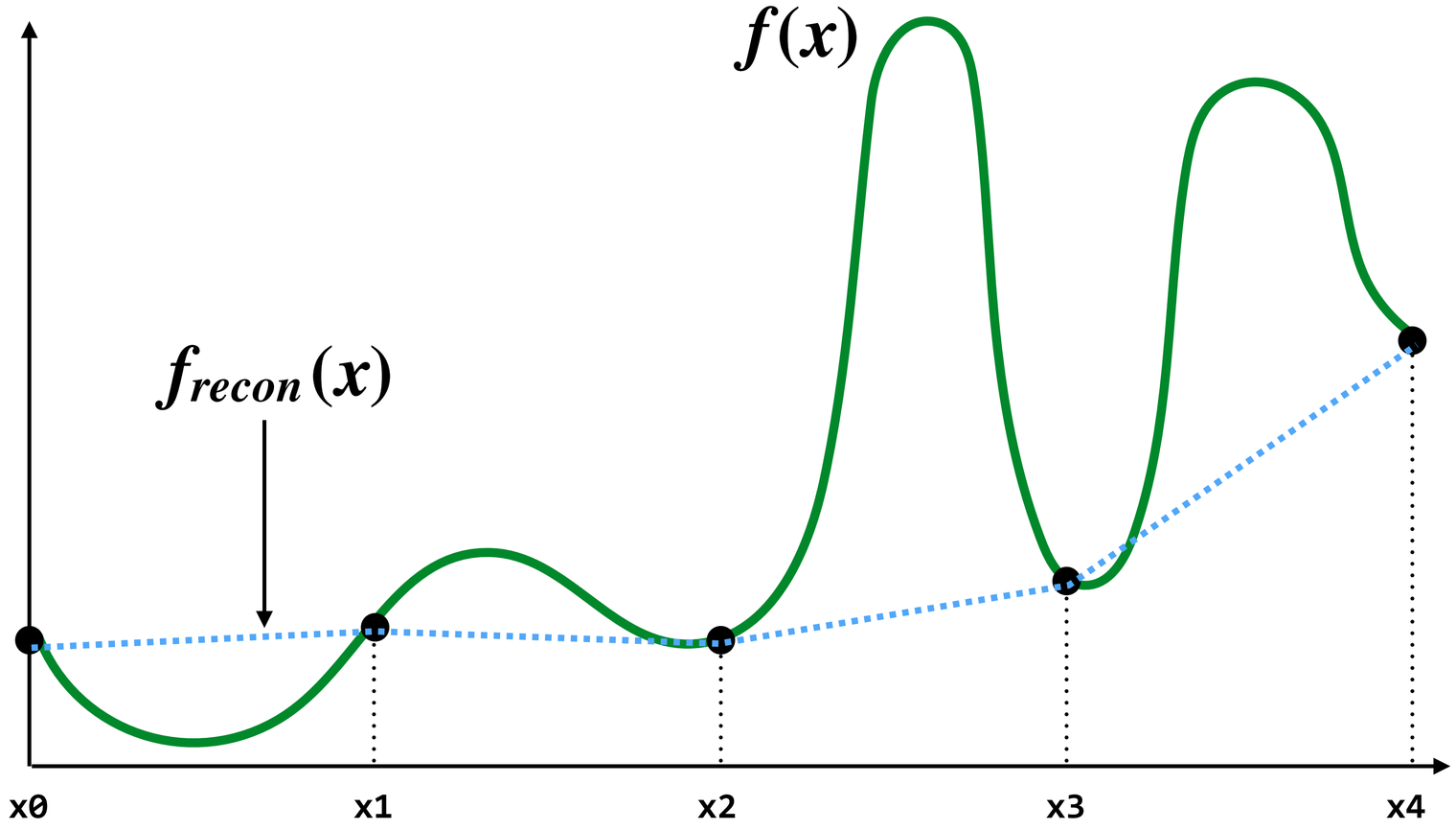


Piecewise linear approximation

$f_{recon}(x)$ = linear interpolation between values of two closest samples to x



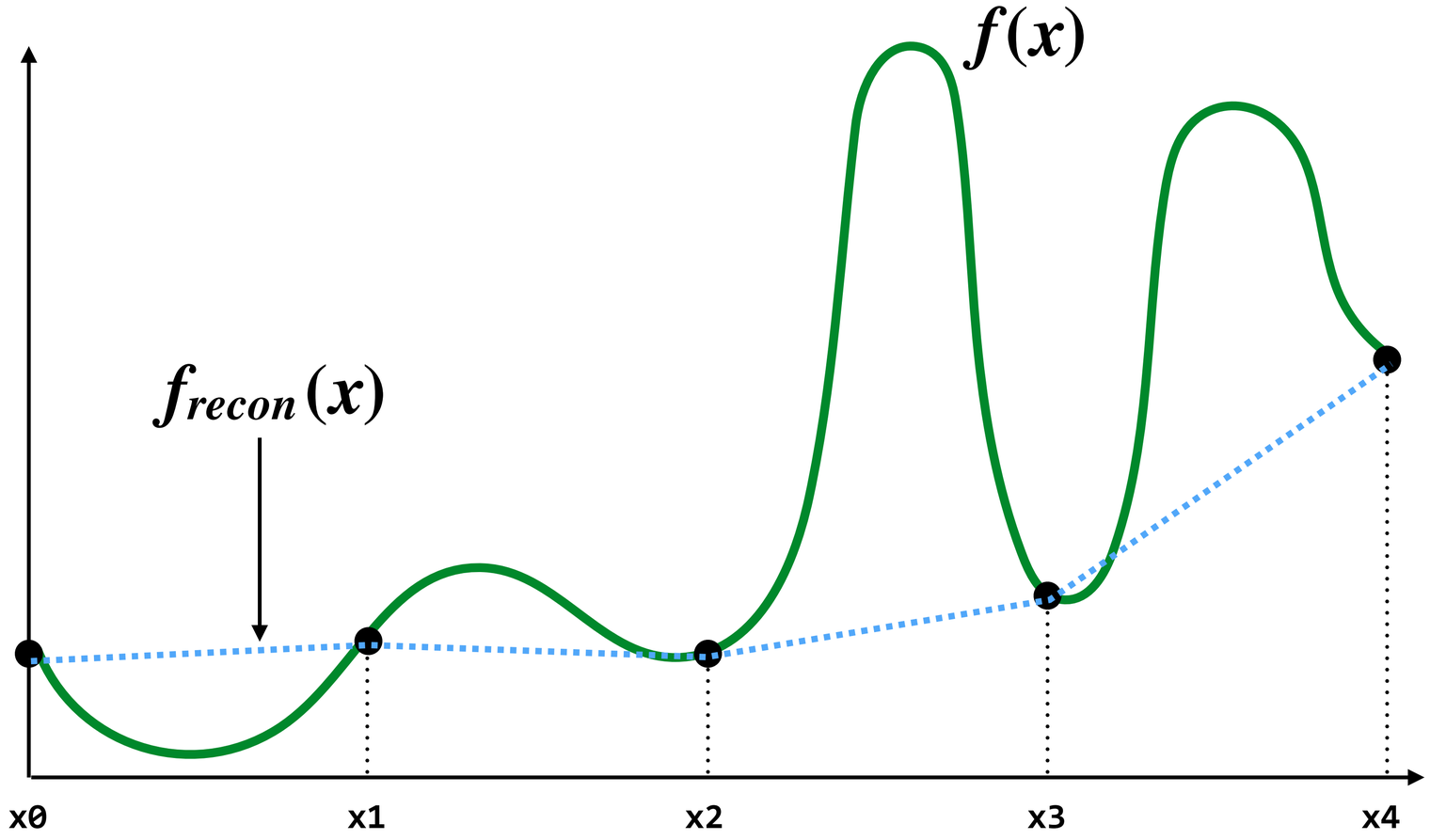
How can we represent the signal more accurately?



Answer: sample signal more densely (increase sampling rate)

Reconstruction from sparse sampling

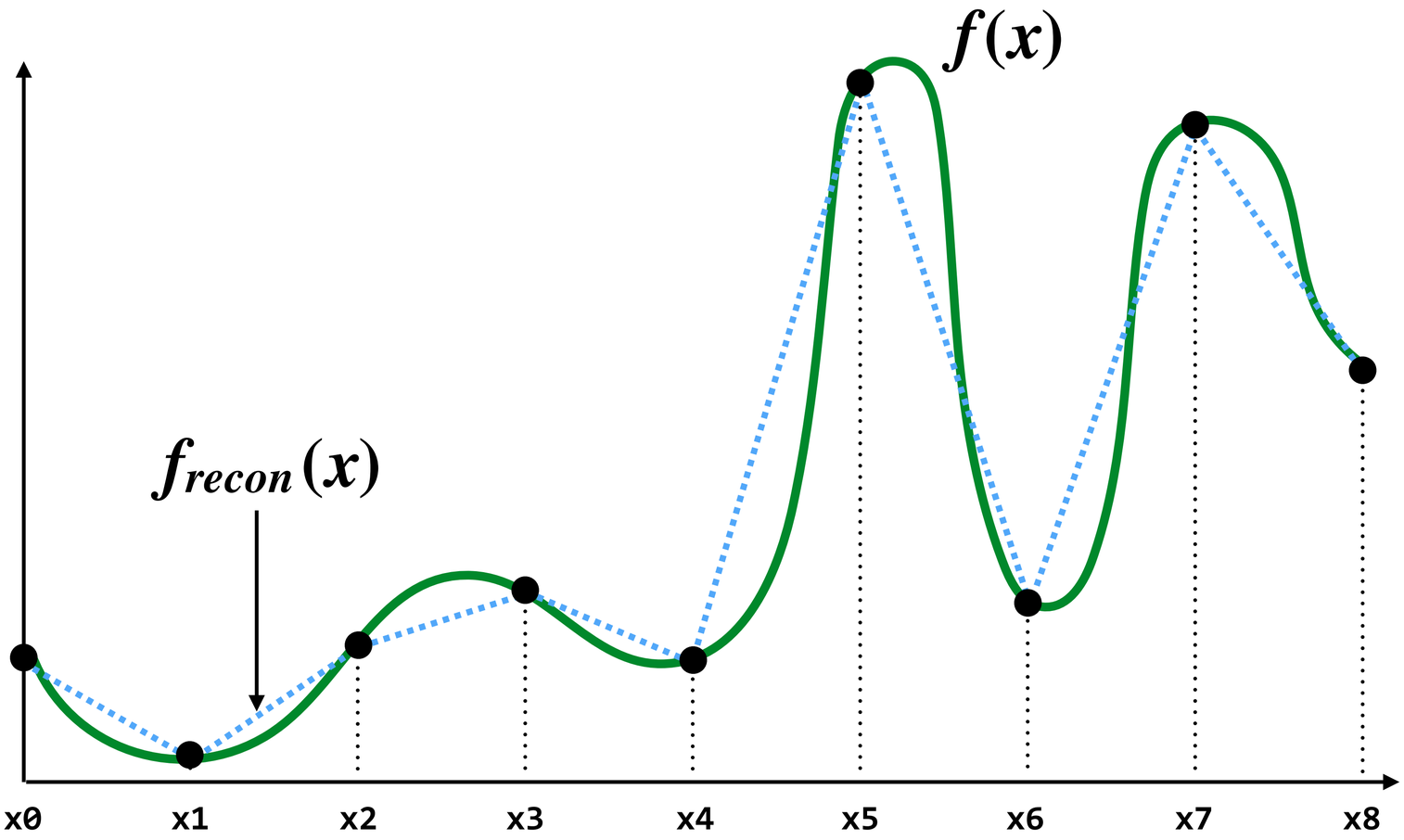
(5 samples)



= reconstruction via linear interpolation

More accurate reconstructions result from denser sampling

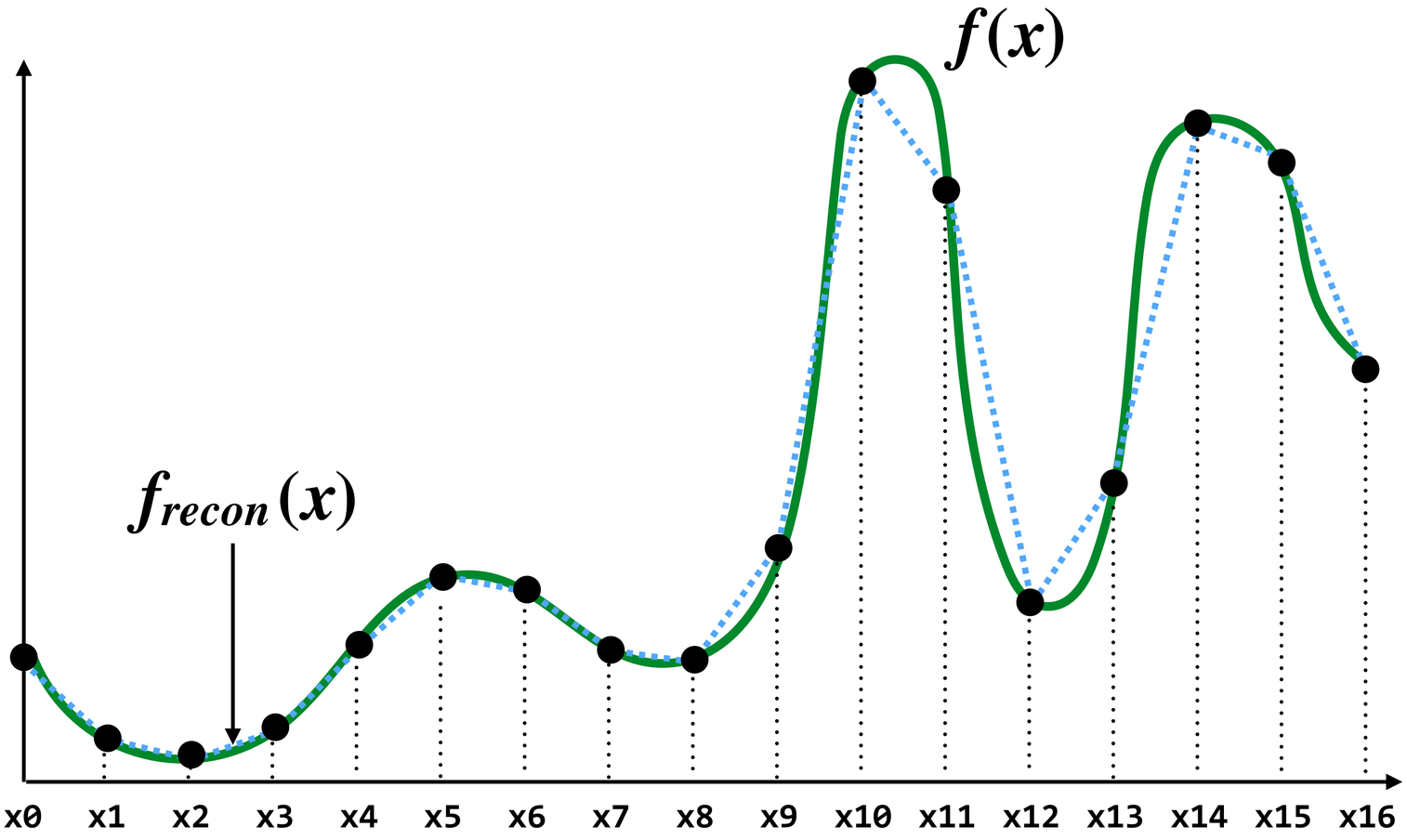
(9 samples)



= reconstruction via linear interpolation

More accurate reconstructions result from denser sampling

(17 samples)



= reconstruction via linear interpolation

Drawing a triangle by 2D sampling

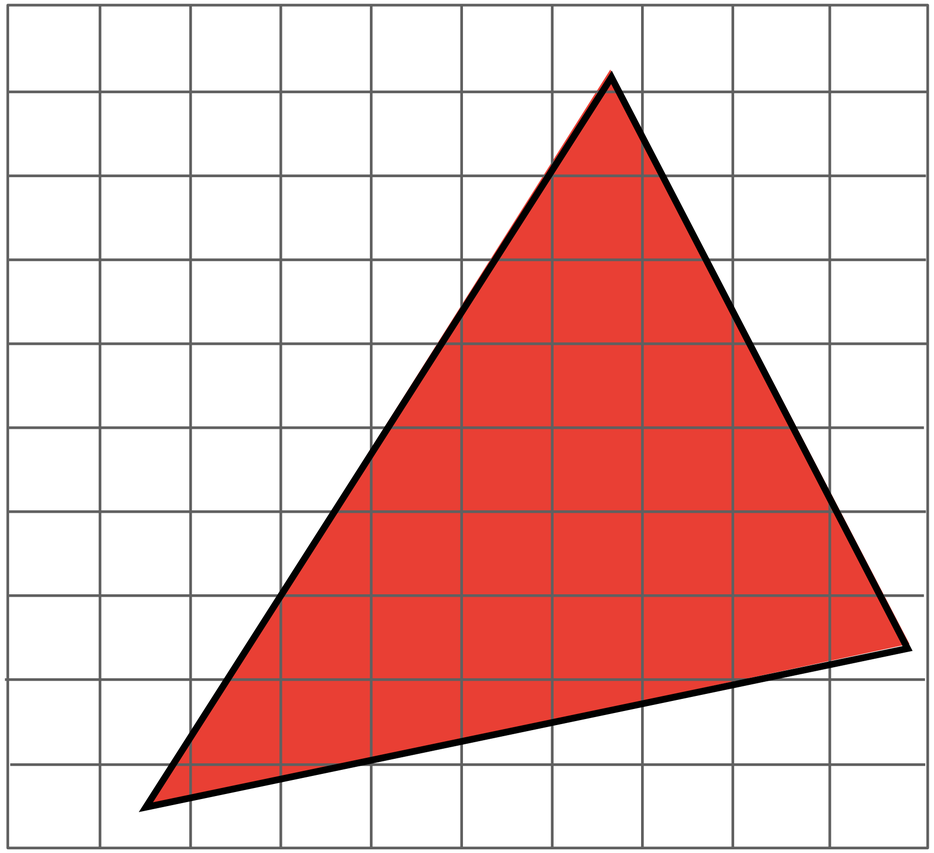


Image as a 2D matrix of pixels

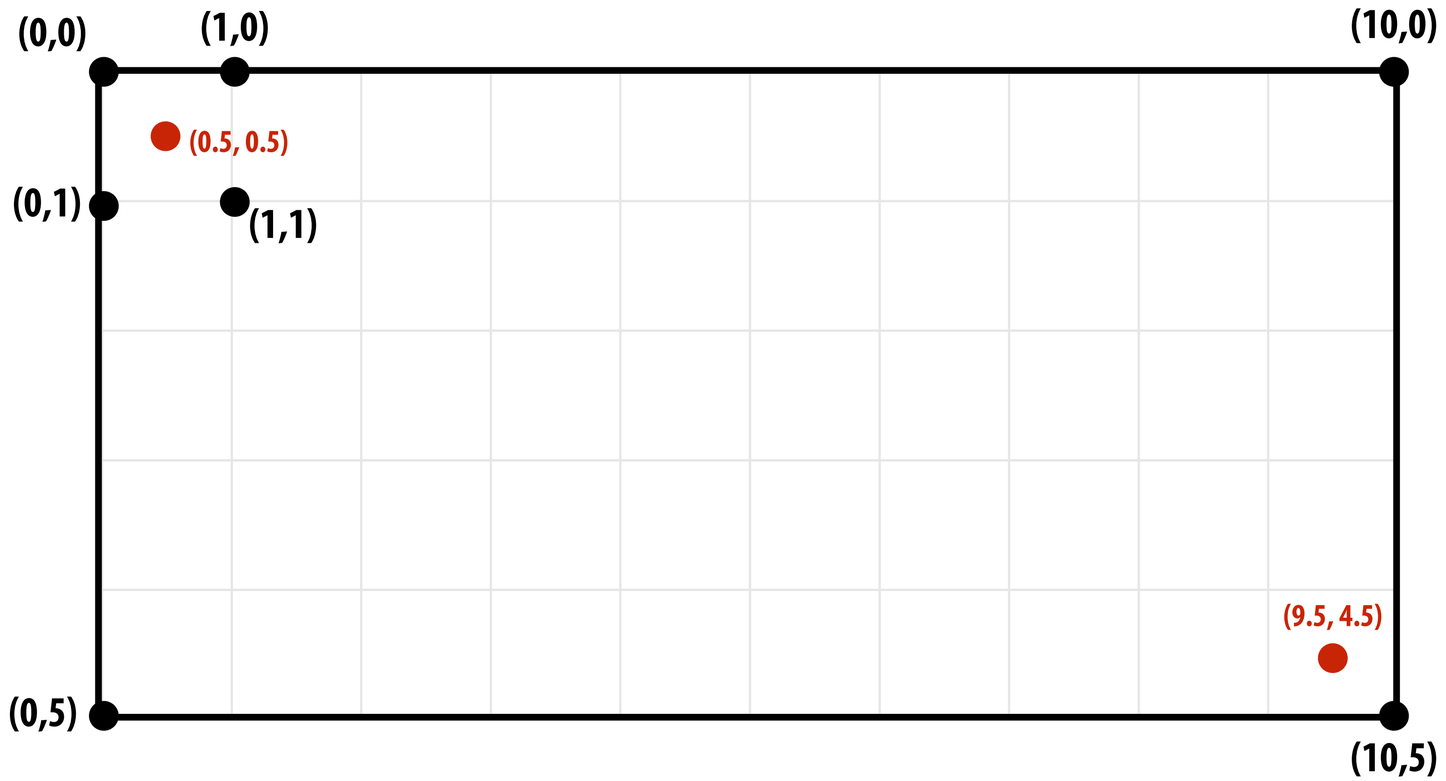
Here I'm showing a 10 x 5 image

Identify pixel by its integer (x,y) coordinates

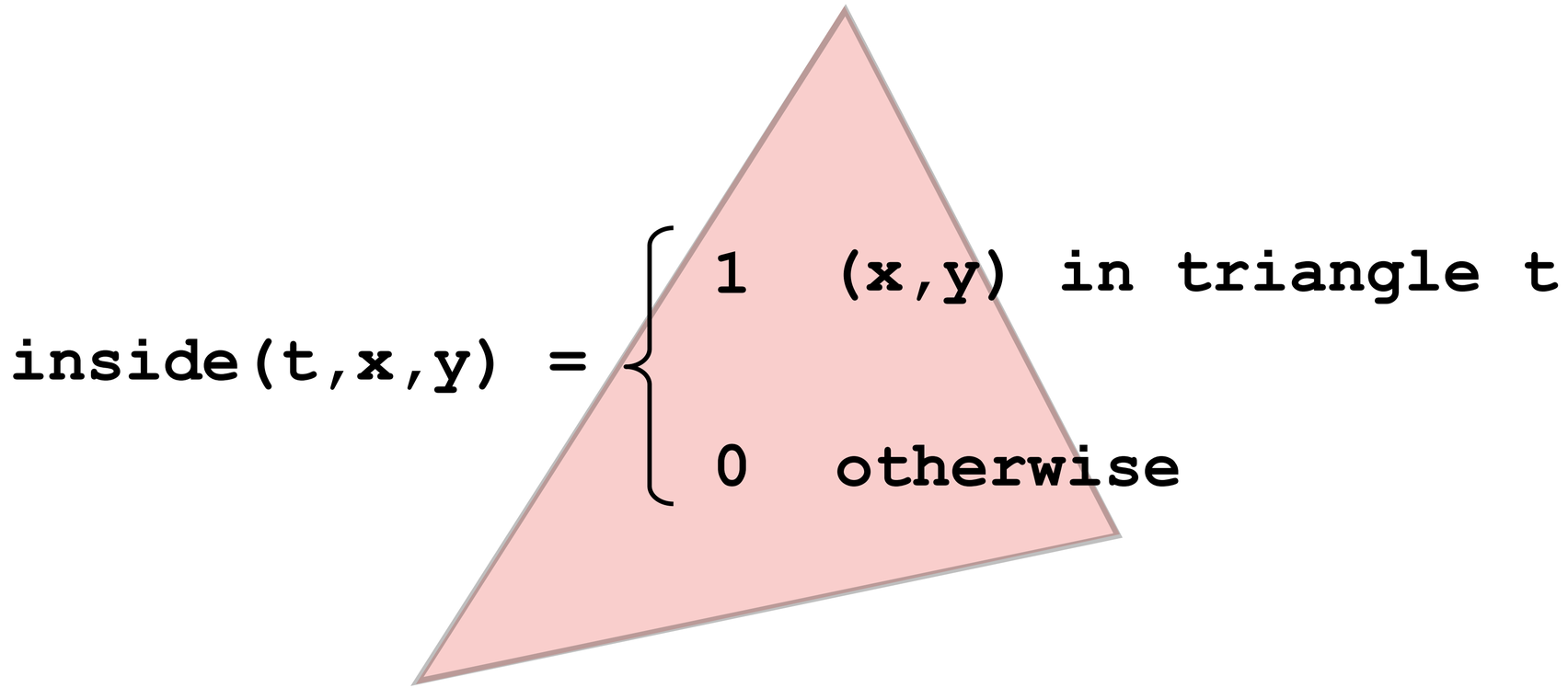
(0,0)	(1,0)								(9,0)
(0,1)	(1,1)								
(0,4)									(9,4)

Continuous coordinate space over image

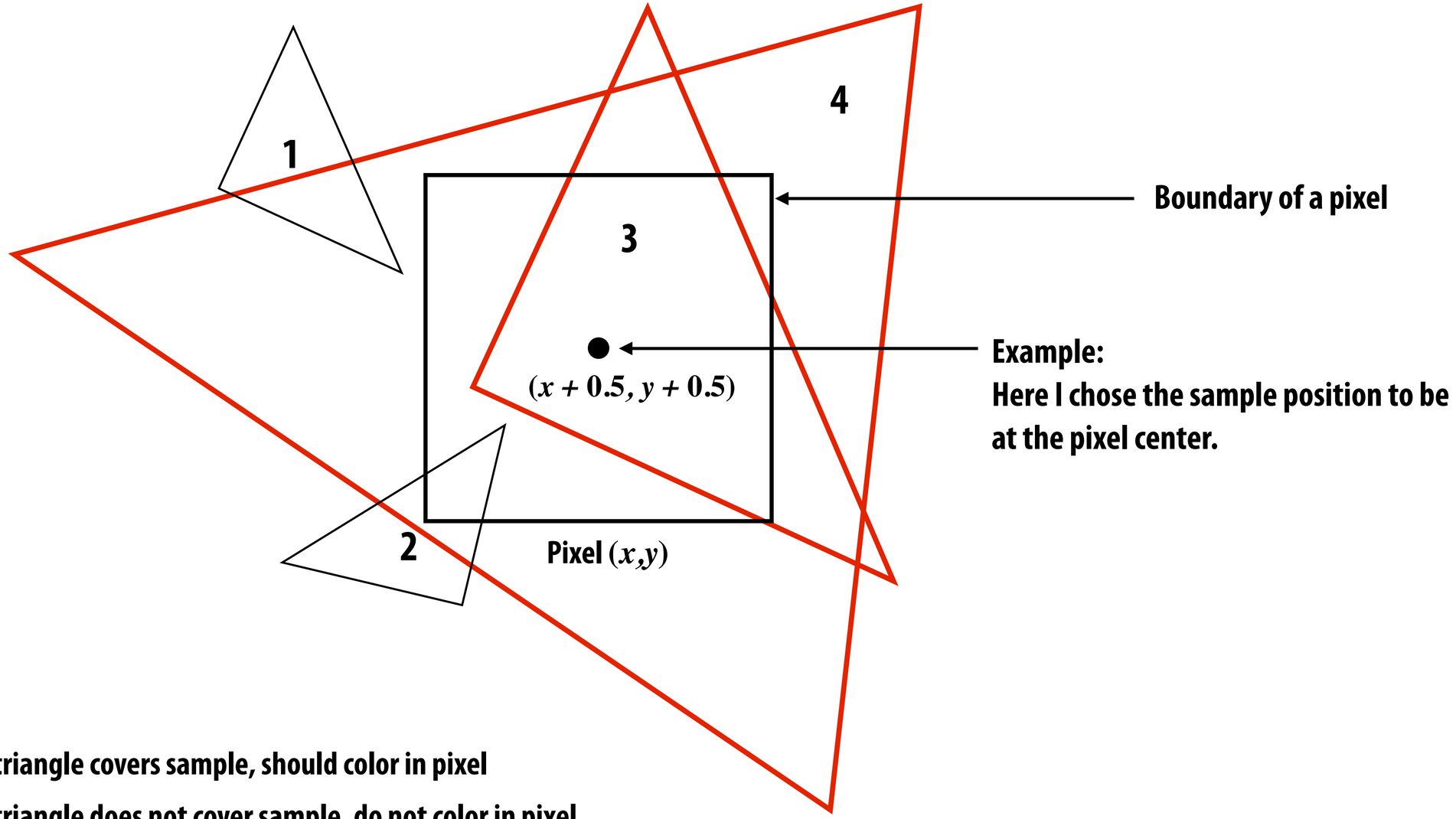
Ok, now forget about pixels!



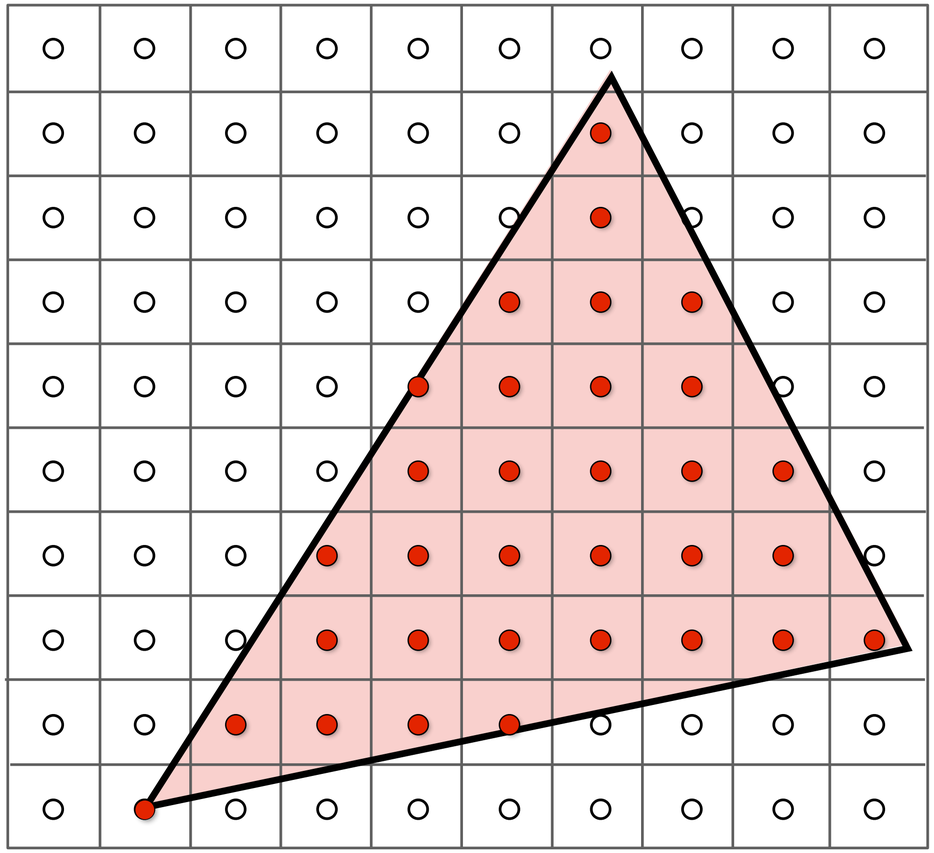
Define binary function: `inside(tri,x,y)`



Sampling the binary function: `inside(tri,x,y)`

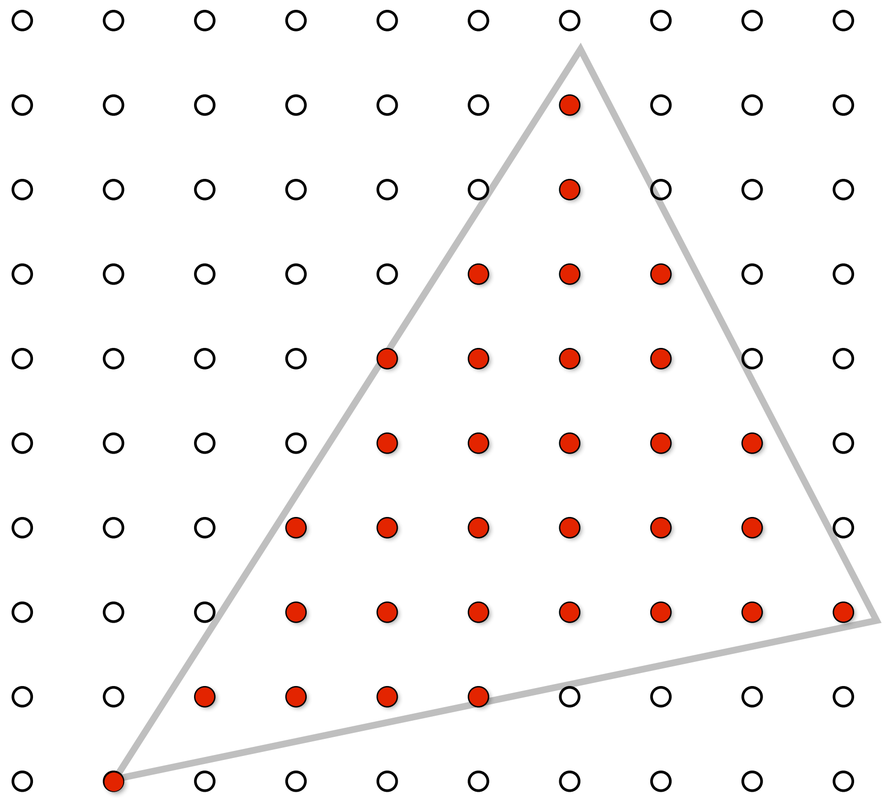


Sample coverage at pixel centers



Sample coverage at pixel centers

I only want you to think about evaluating triangle-point coverage!



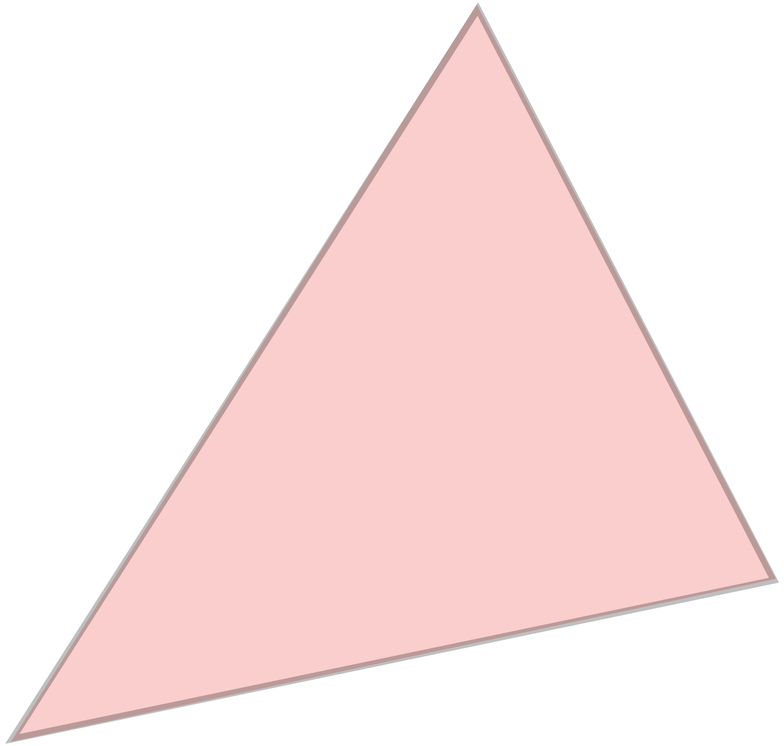
Rasterization = sampling a 2D binary function

- Rasterize triangle `tri` by sampling the function

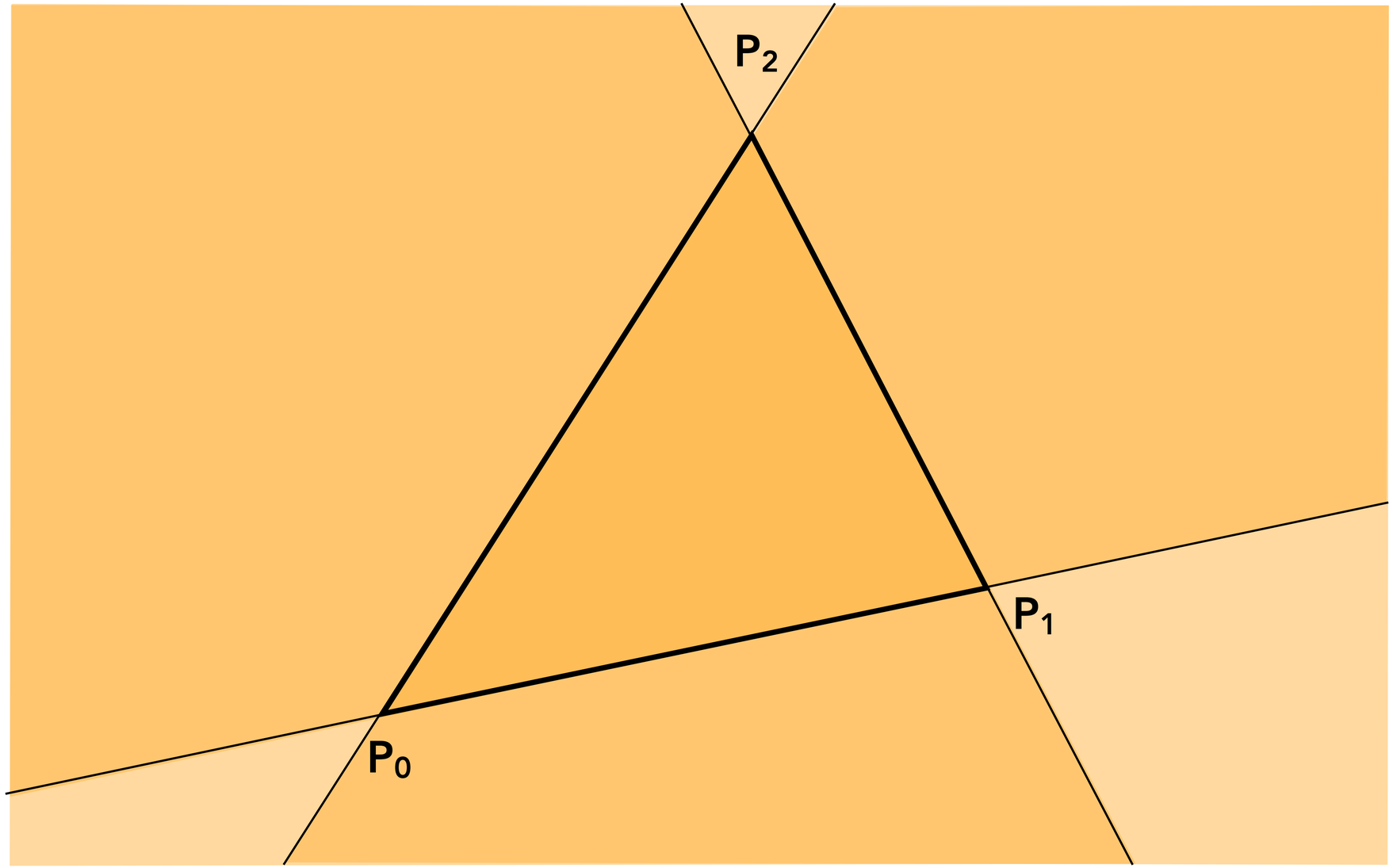
```
f(x,y) = inside(tri,x,y)
```

```
for (int x = 0; x < xmax; x++)
  for (int y = 0; y < ymax; y++)
    image[x][y] = f(x + 0.5, y + 0.5);
```

Evaluating `inside(tri,x,y)`



Triangle = intersection of three half planes

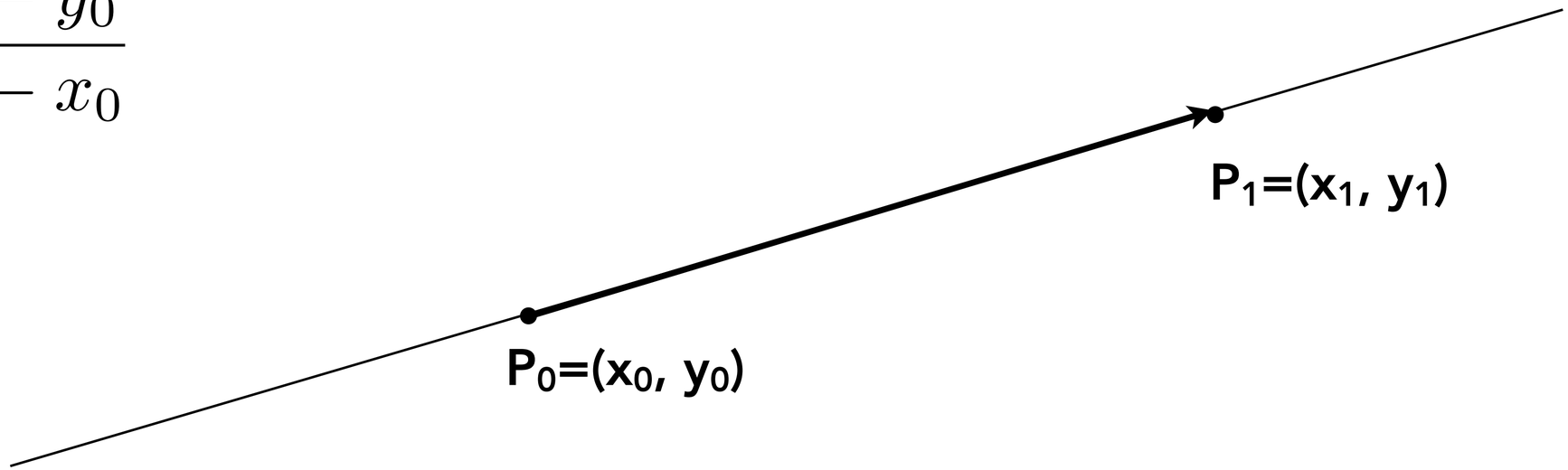


Point-slope form of a line

(You might have seen this in high school)

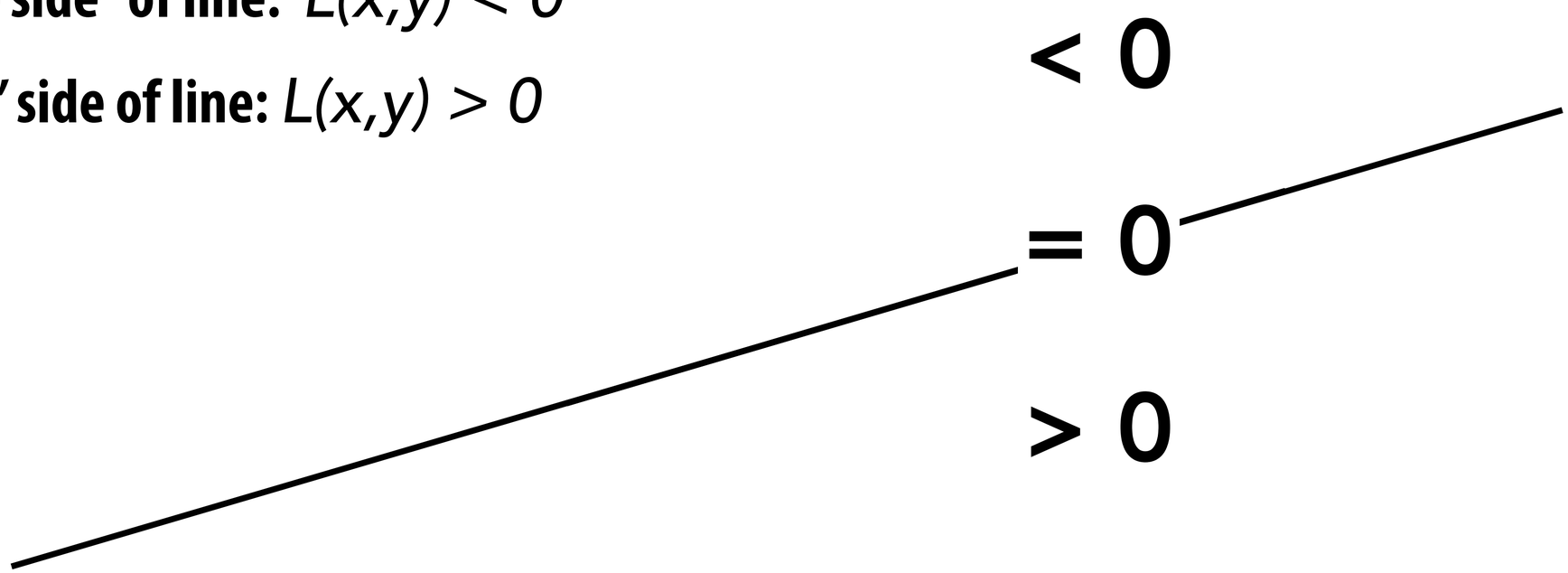
$$y - y_0 = m(x - x_0)$$

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

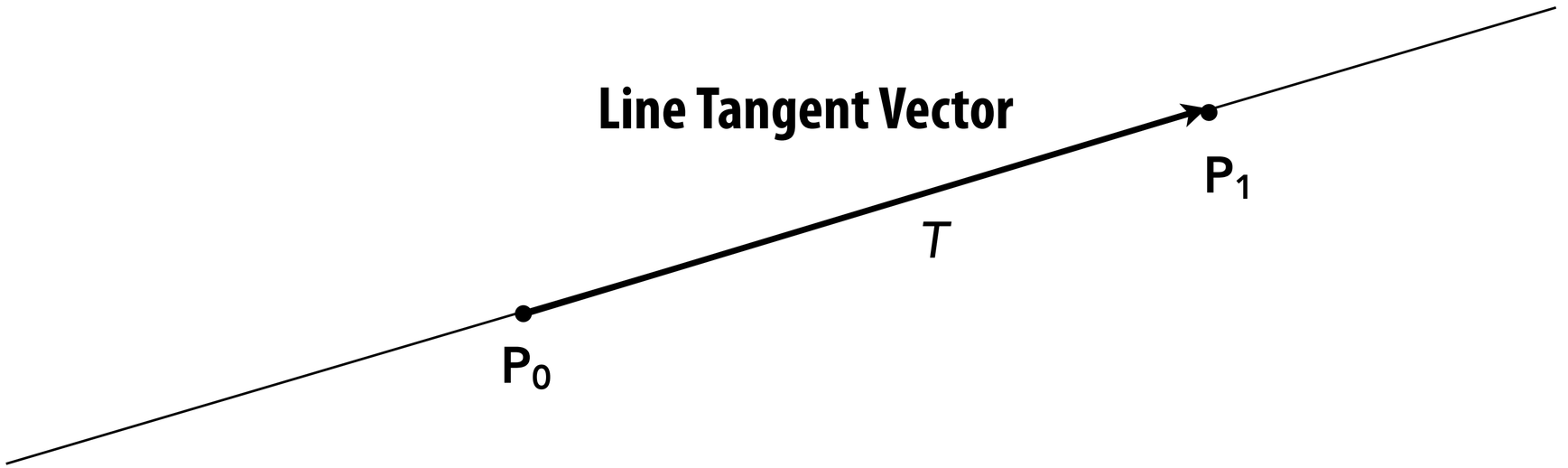


Each line defines two half-planes

- **Implicit line equation**
  - $L(x,y) = Ax + By + C$
  - **On the line:** $L(x,y) = 0$
  - **"Negative side" of line:** $L(x,y) < 0$
  - **"Positive" side of line:** $L(x,y) > 0$

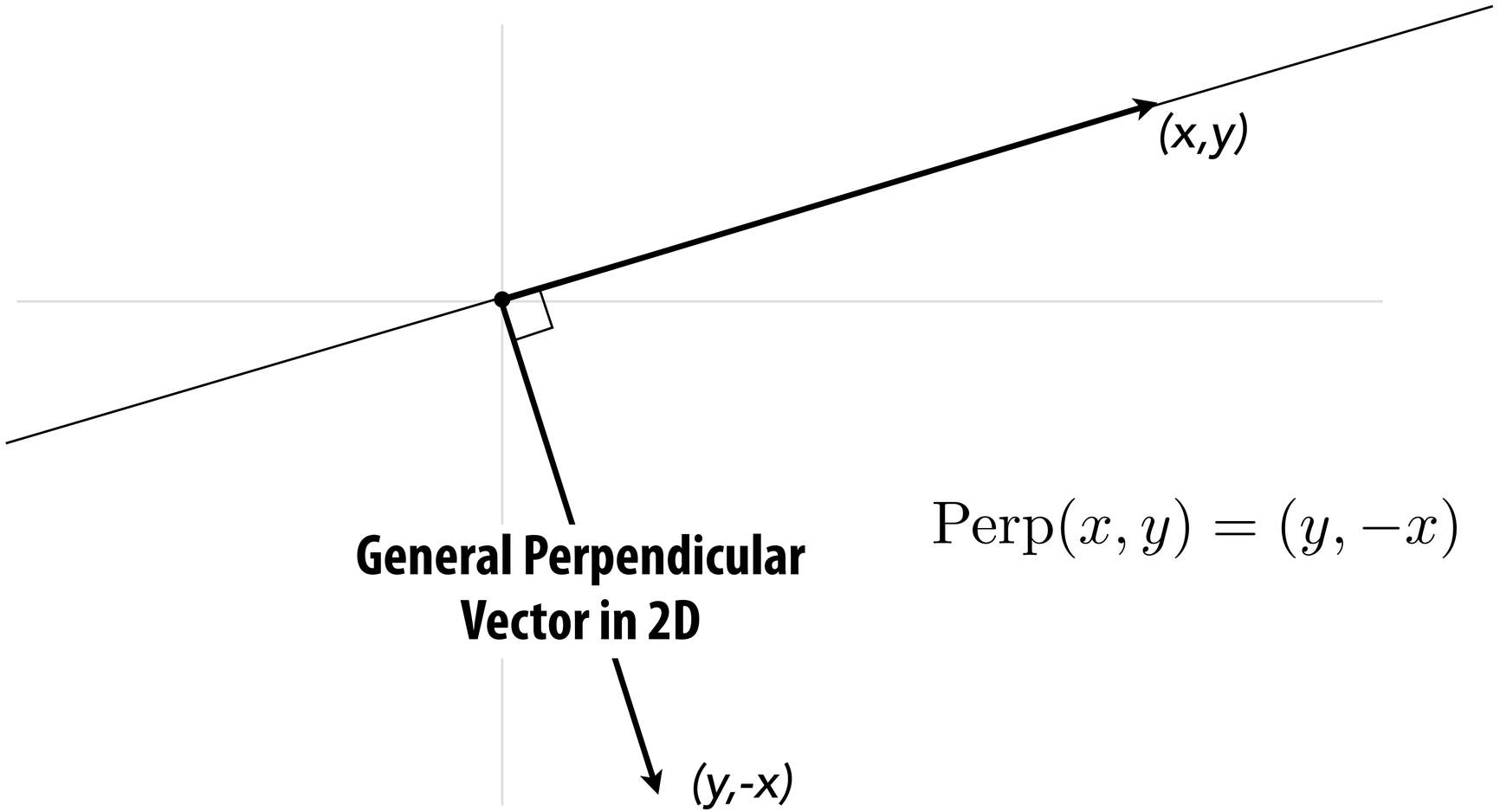


Line equation derivation



$$T = P_1 - P_0 = (x_1 - x_0, y_1 - y_0)$$

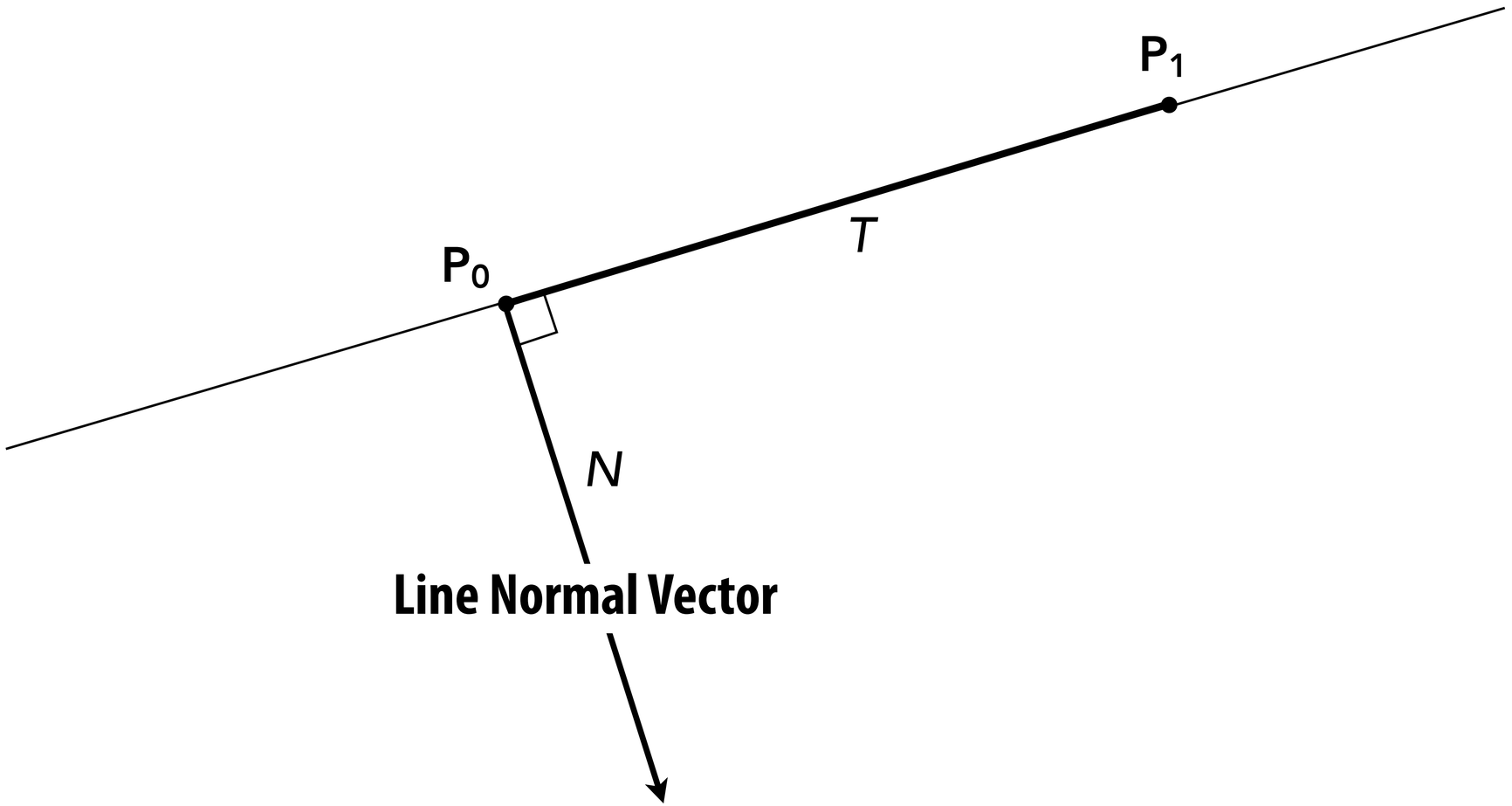
Line equation derivation



$$\mathrm{Perp}(x, y) = (y, -x)$$

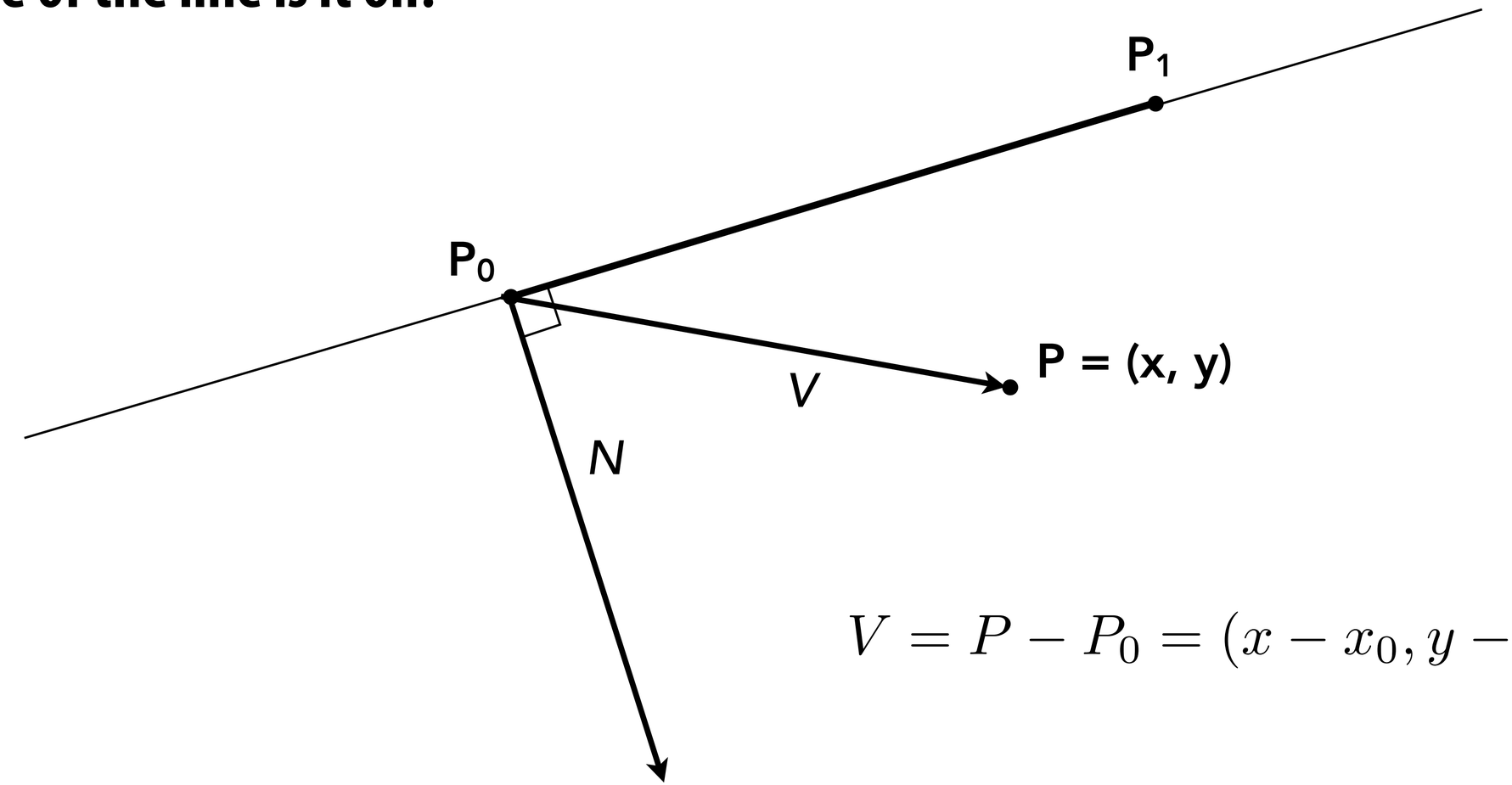
Line equation derivation

$$N = \mathrm{Perp}(T) = (y_1 - y_0, -(x_1 - x_0))$$



Line equation derivation

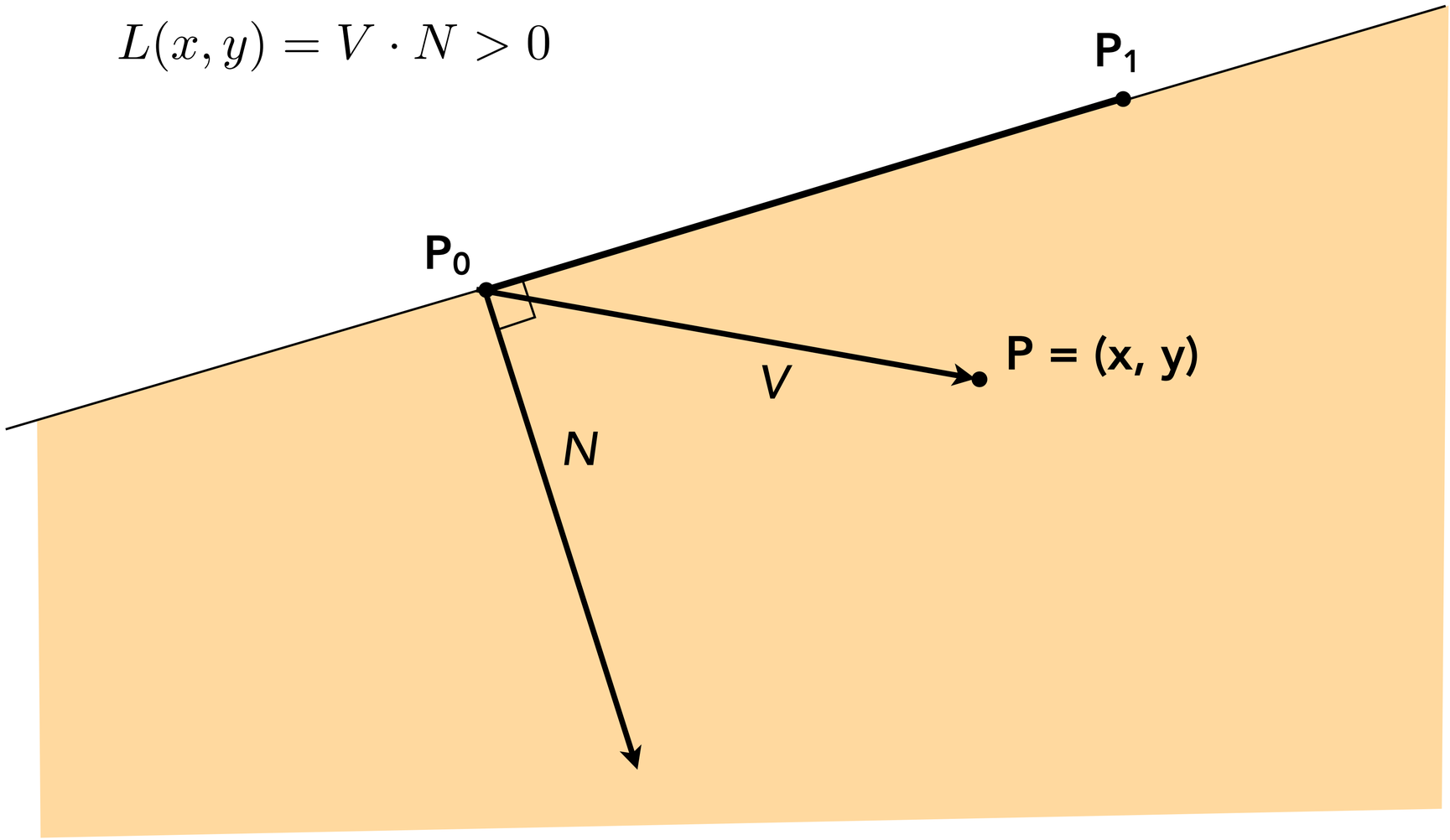
Now consider a point *P*=(x,y).
Which side of the line is it on?



$$V = P - P_0 = (x - x_0, y - y_0)$$

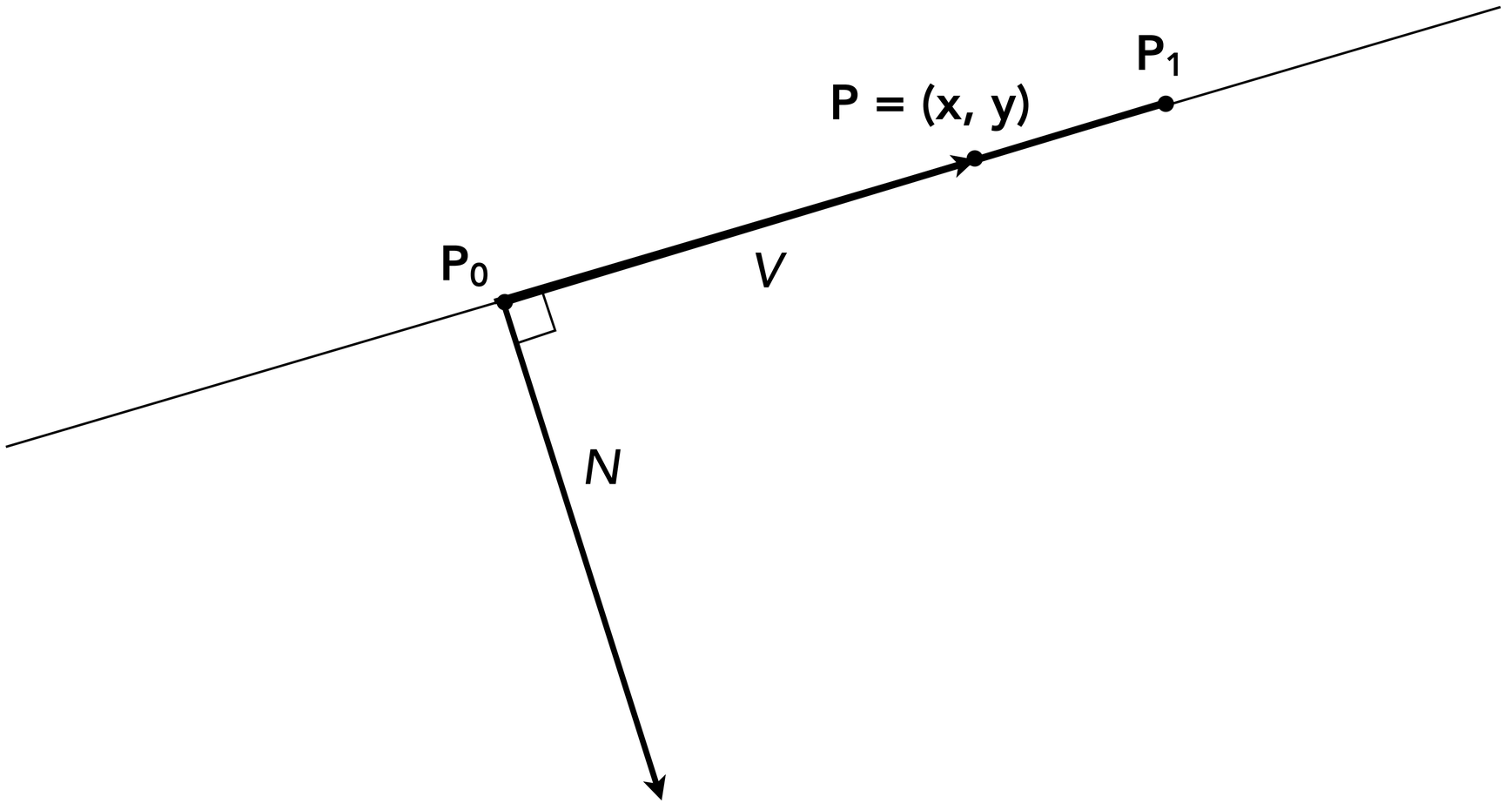
Line equation tests

$$L(x, y) = V \cdot N > 0$$

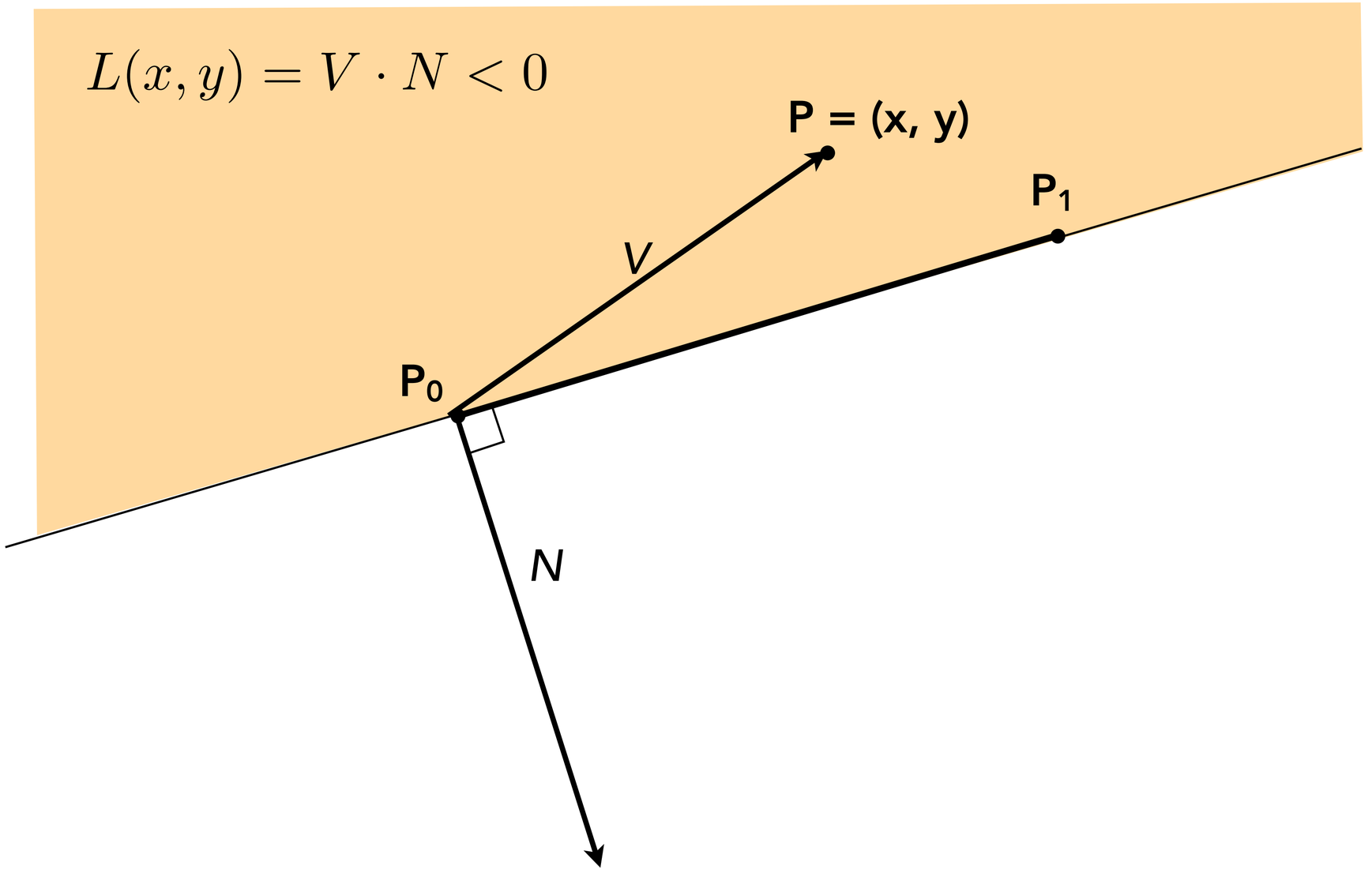


Line equation tests

$$L(x, y) = V \cdot N = 0$$

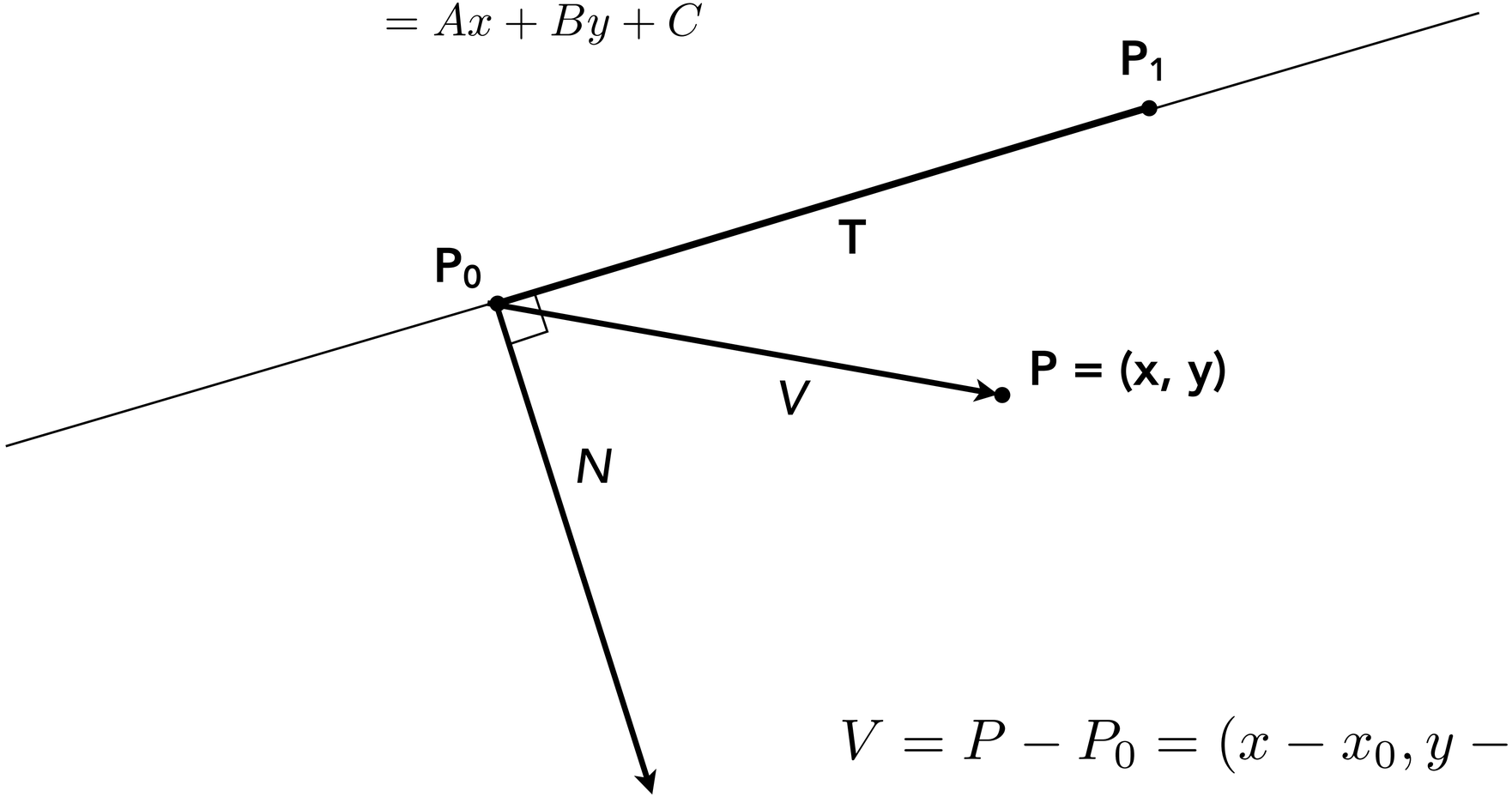


Line equation tests



Line equation derivation

$$\begin{aligned} L(x,y) = V \cdot N &= -(y-y_0)(x_1-x_0) + (x-x_0)(y_1-y_0) \\ &= (y_1-y_0)x - (x_1-x_0)y + y_0(x_1-x_0) - x_0(y_1-y_0) \\ &= Ax + By + C \end{aligned}$$



$$V = P - P_0 = (x - x_0, y - y_0)$$

$$N = \mathrm{Perp}(T) = (y_1 - y_0, -(x_1 - x_0))$$

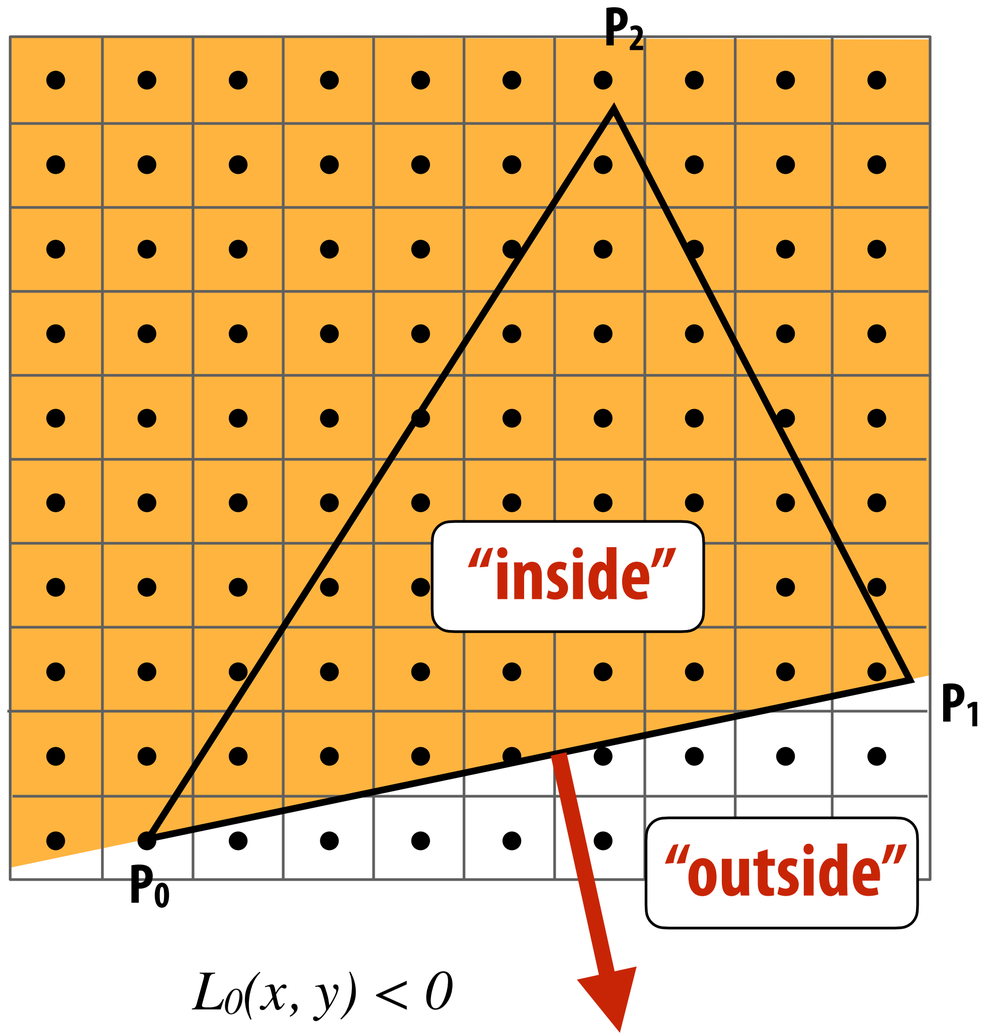
Point-in-triangle test

$$P_i = (X_i, Y_i)$$

$$A_i = dY_i = Y_{i+1} - Y_i$$
$$B_i = dX_i = X_{i+1} - X_i$$
$$C_i = Y_i(X_{i+1} - X_i) - X_i(Y_{i+1} - Y_i)$$

$$L_i(x, y) = dY_i\,x - dX_i\,y + C_i$$

$$L_i(x, y) = 0 \text{ : point on edge}$$
$$> 0 \text{ : outside edge}$$
$$< 0 \text{ : inside edge}$$



Point-in-triangle test

$$P_i = (X_i, Y_i)$$

$$A_i = dY_i = Y_{i+1} - Y_i$$
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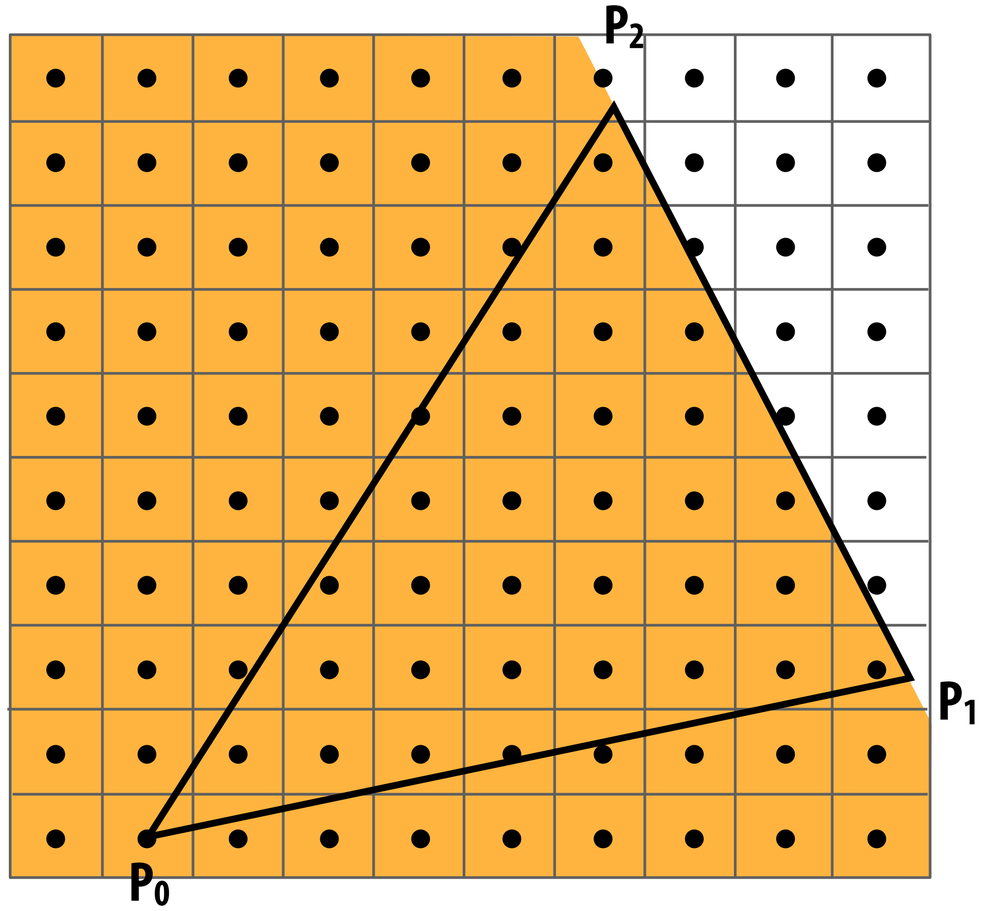
$$C_i = Y_i(X_{i+1} - X_i) - X_i(Y_{i+1} - Y_i)$$

$$L_i(x, y) = dY_i\, x - dX_i\, y + C_i$$

$$L_i(x, y) = 0 \text{ : point on edge}$$
$$> 0 \text{ : outside edge}$$
$$< 0 \text{ : inside edge}$$



$$L_1(x, y) < 0$$

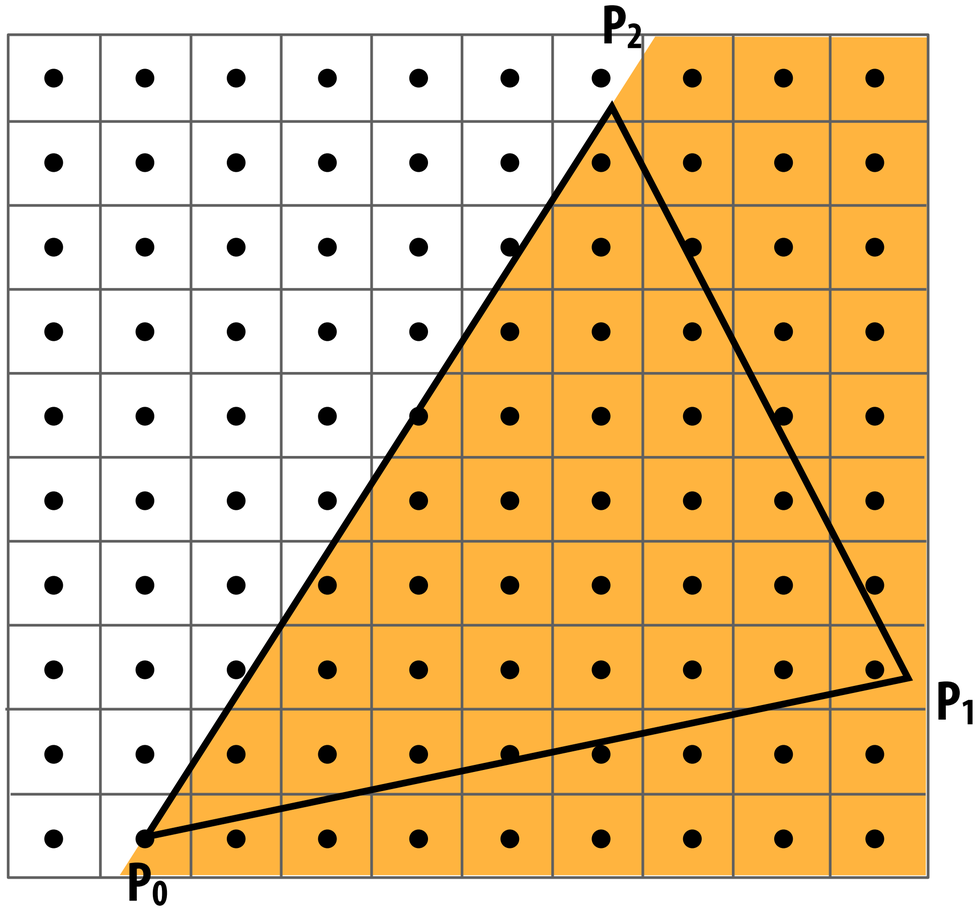
Point-in-triangle test

$$P_i = (X_i, Y_i)$$

$$A_i = dY_i = Y_{i+1} - Y_i$$
$$B_i = dX_i = X_{i+1} - X_i$$
$$C_i = Y_i(X_{i+1} - X_i) - X_i(Y_{i+1} - Y_i)$$

$$L_i(x, y) = dY_i\, x - dX_i\, y + C_i$$

$$L_i(x, y) = 0 \text{ : point on edge}$$
$$> 0 \text{ : outside edge}$$
$$< 0 \text{ : inside edge}$$



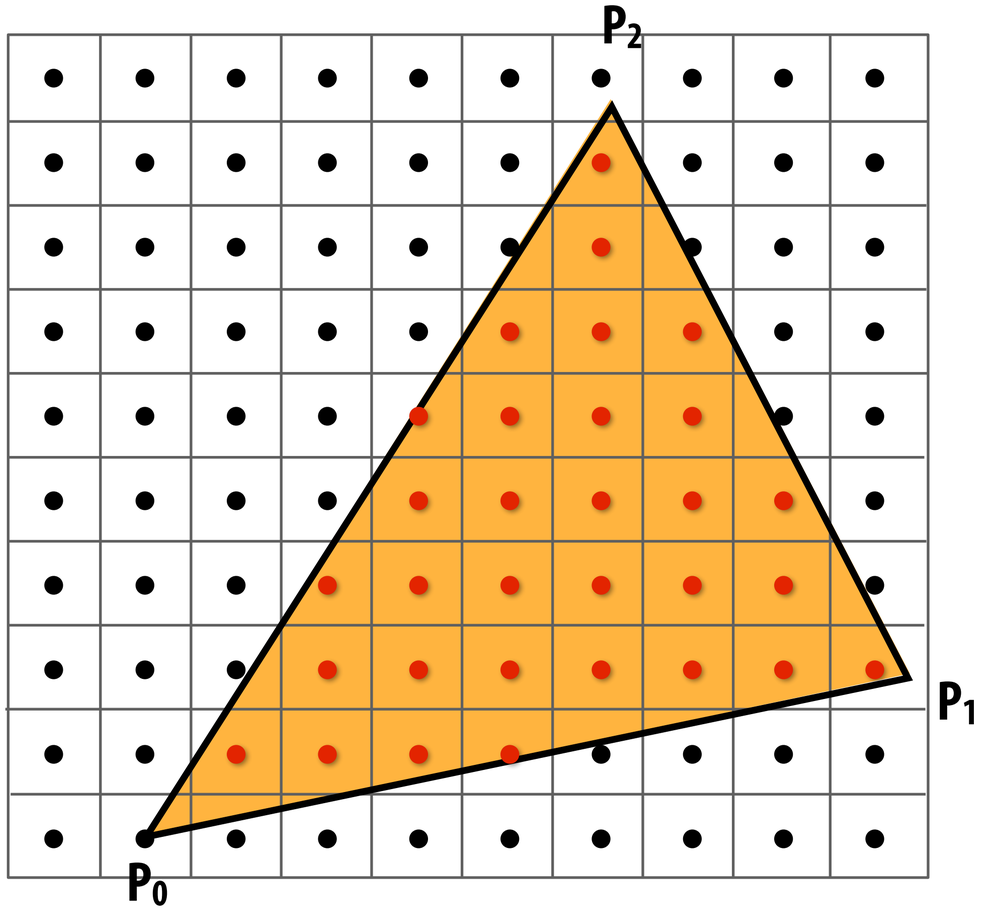
$$L_2(x, y) < 0$$

Point-in-triangle test

Sample point $s = (sx, sy)$ is inside the triangle if it is inside all three edges.

$$inside(sx, sy) =$$
$$L_0(sx, sy) < 0 \;\&\&$$
$$L_1(sx, sy) < 0 \;\&\&$$
$$L_2(sx, sy) < 0;$$

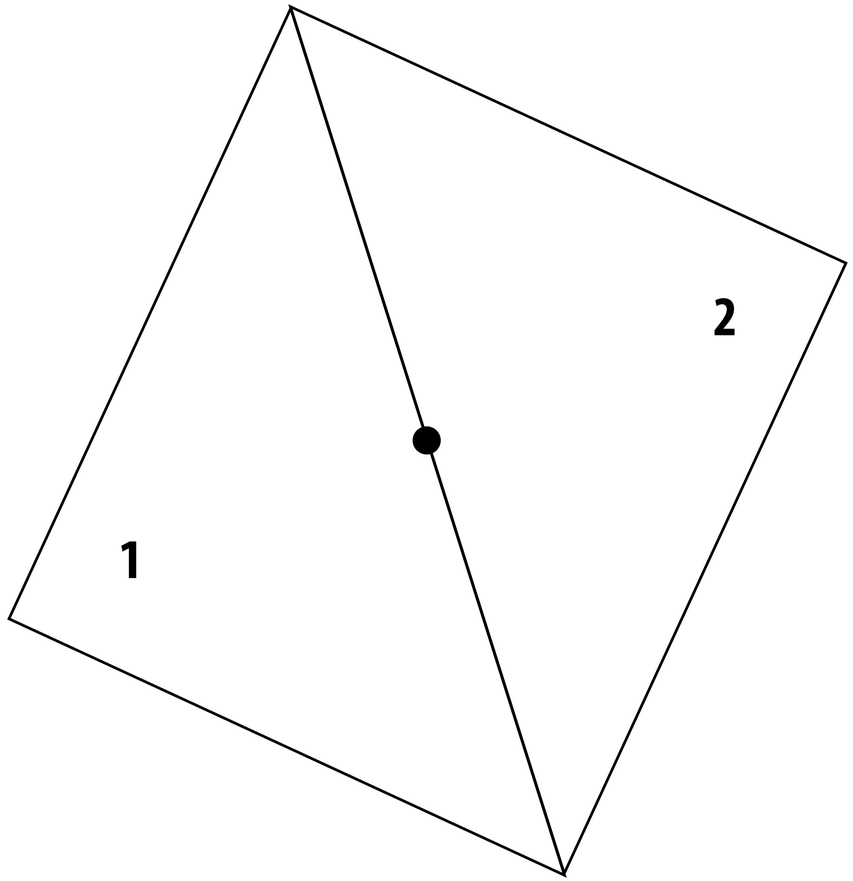
Note: actual implementation of $inside(sx,sy)$ involves ≤ checks based on the triangle coverage edge rules (see next slide)



Sample points inside triangle are highlighted red.

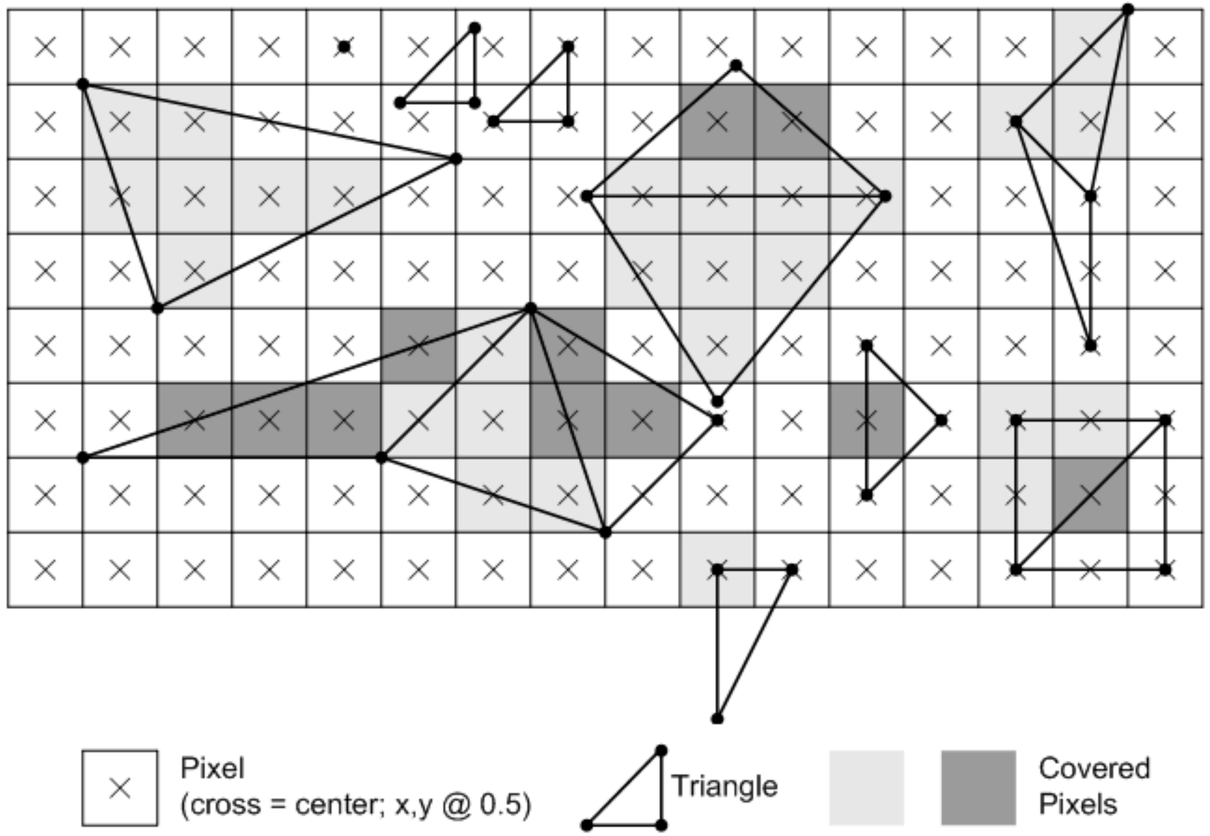
Edge cases (literally)

Is this sample point covered by triangle 1? or triangle 2? or both?



OpenGL/Direct3D edge rules

- When edge falls directly on a screen sample point, the sample is classified as within triangle if the edge is a "top edge" or "left edge"
  - Top edge: horizontal edge that is above all other edges
  - Left edge: an edge that is not exactly horizontal and is on the left side of the triangle. (triangle can have one or two left edges)



Source: Direct3D Programming Guide, Microsoft

Finding covered samples: incremental triangle traversal

$$P_i = (X_i, Y_i)$$

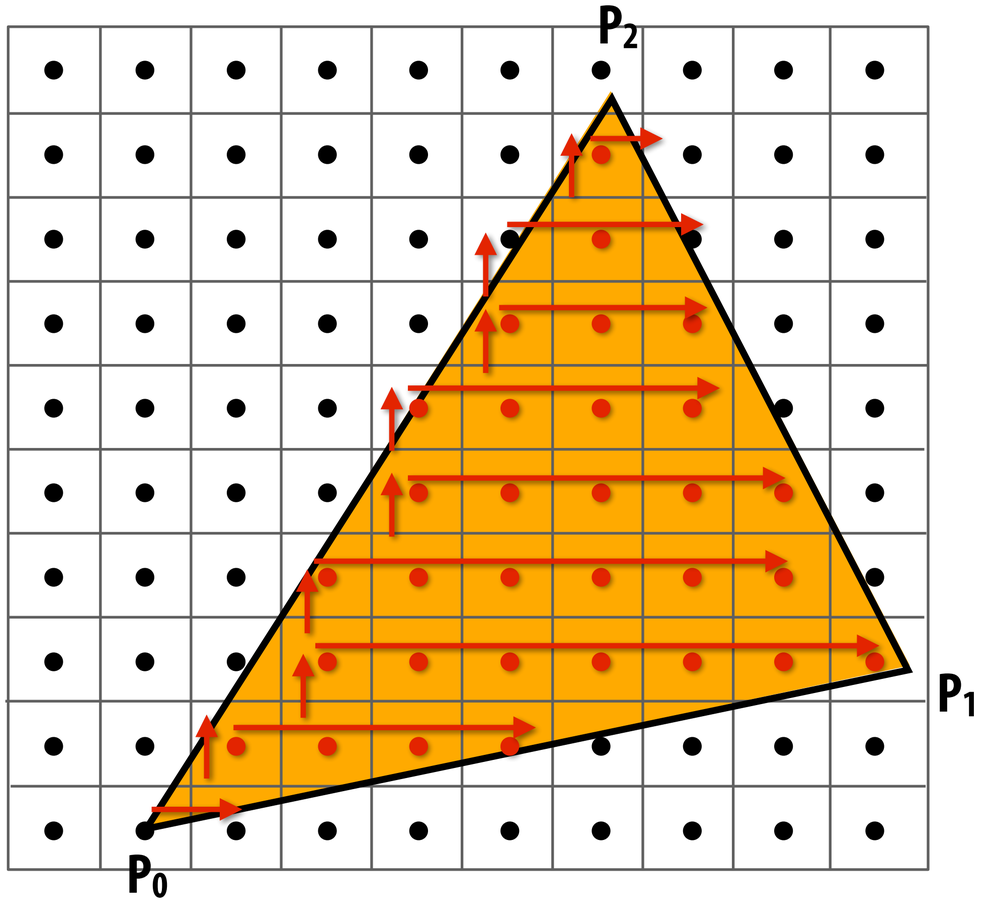
$$A_i = dY_i = Y_{i+1} - Y_i$$
$$B_i = dX_i = X_{i+1} - X_i$$
$$C_i = Y_i(X_{i+1} - X_i) - X_i(Y_{i+1} - Y_i)$$

$$L_i(x, y) = dY_i x - dX_i y + C_i$$

$$L_i(x, y) = 0 \text{ : point on edge}$$
$$> 0 \text{ : outside edge}$$
$$< 0 \text{ : inside edge}$$

Efficient incremental update:

$$L_i(x+1,y) = L_i(x,y) + dY_i = L_i(x,y) + A_i$$
$$L_i(x,y+1) = L_i(x,y) - dX_i = L_i(x,y) + B_i$$



Incremental update saves computation:
Only one addition per edge, per sample test

Many traversal orders are possible: backtrack, zig-zag, Hilbert/Morton curves

Modern approach: tiled triangle traversal

Traverse triangle in blocks

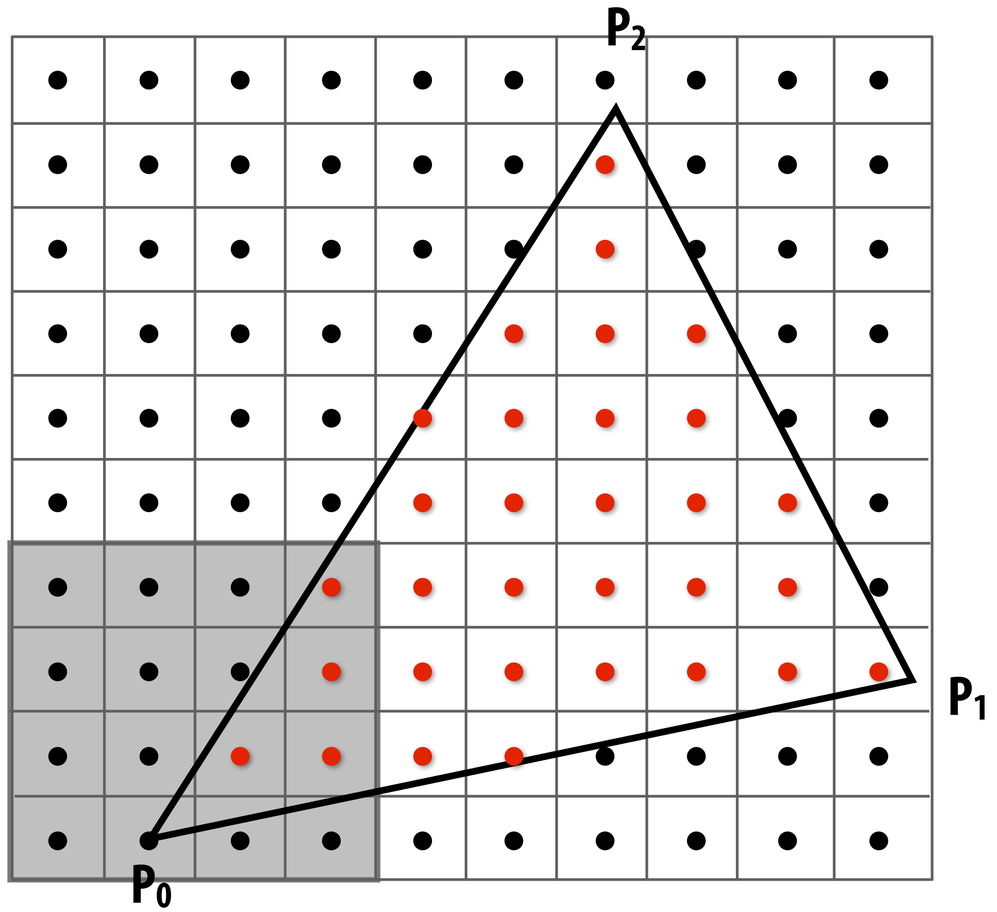
Test all samples in block against triangle in parallel

Advantages:

- Simplicity of parallel execution overcomes cost of extra point-in-triangle tests (most triangles are big enough to cover many samples)

- Can skip sample testing work: entire block not in triangle ("early out"), entire block entirely within triangle ("early in")

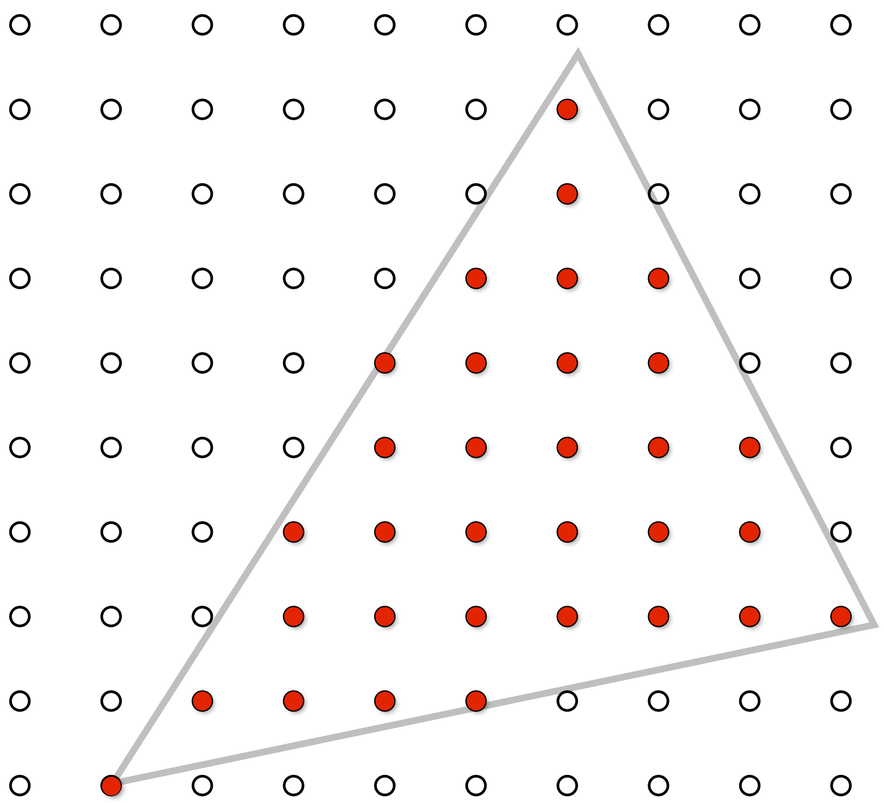
- Additional advantages related to accelerating occlusion computations (not discussed today)



All modern graphics processors (GPUs) have special-purpose hardware for efficiently performing point-in-triangle tests

Where are we now

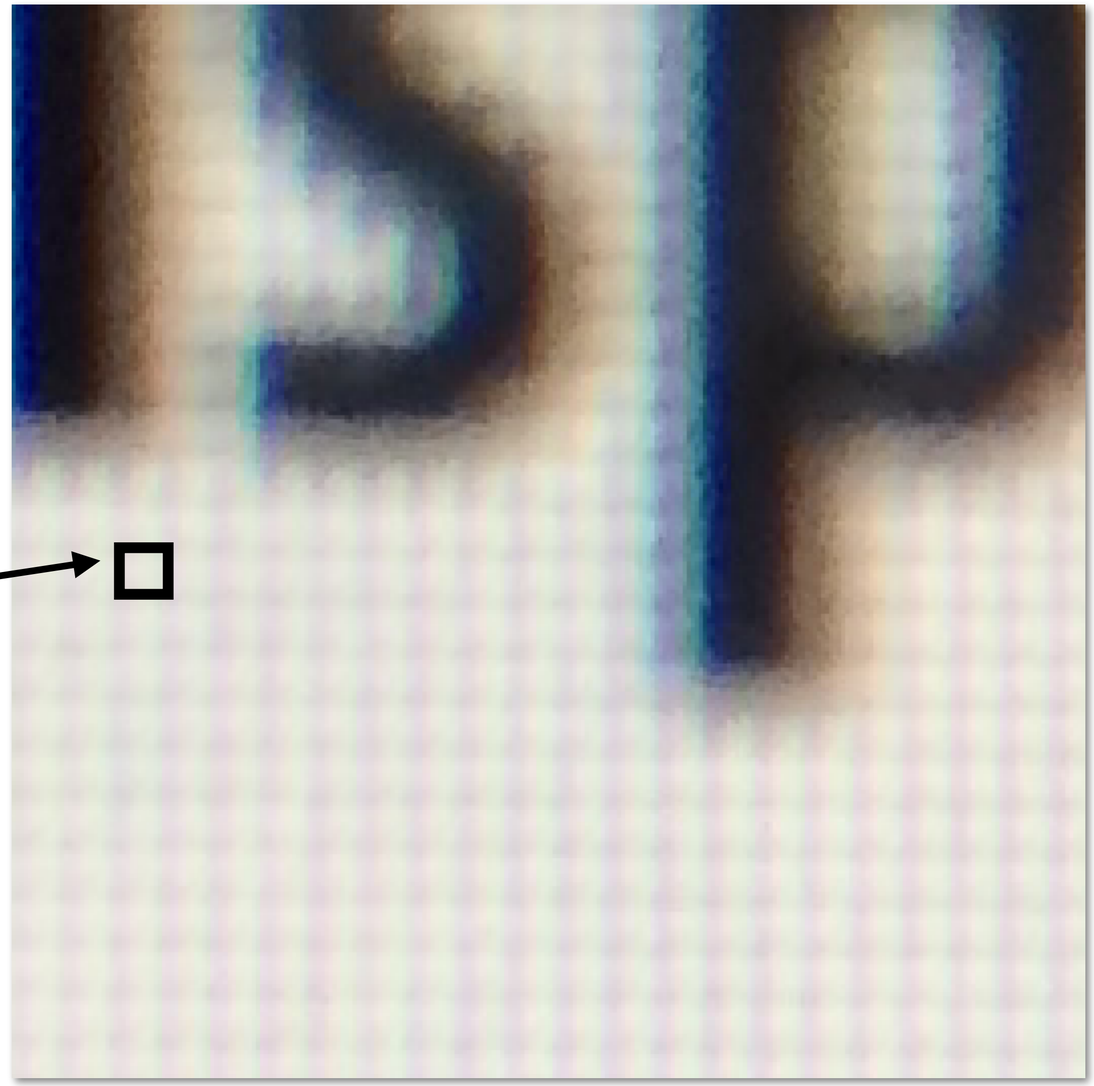
- We have the ability to determine if any point in the image is inside or outside the triangle



- How to we interpret these results as an image to display? (Recall, there's no pixels above)

Recall: pixels on a screen

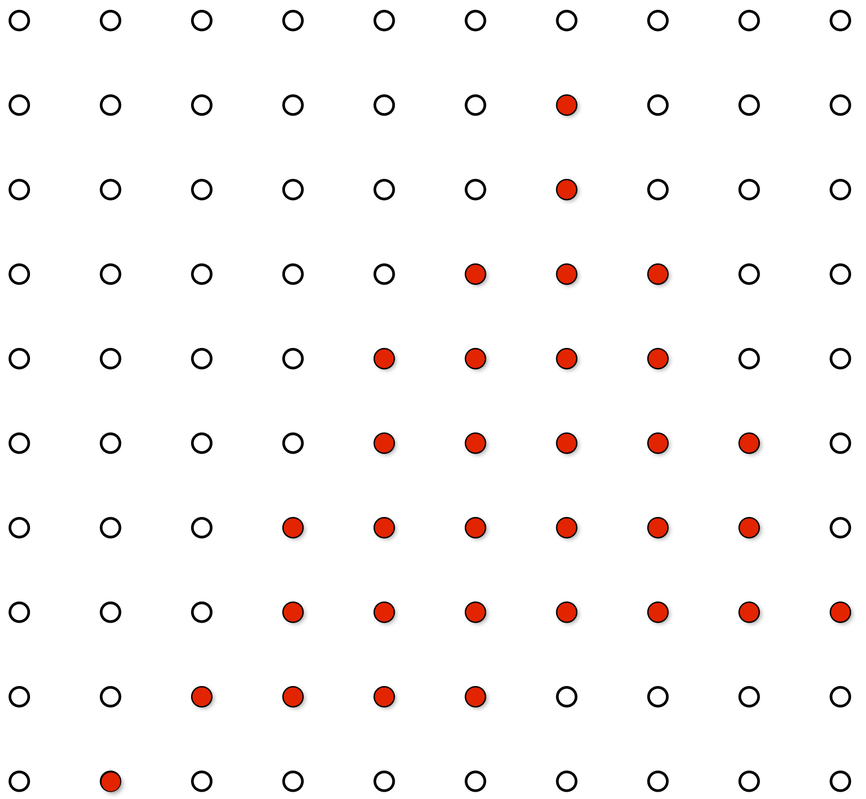
Each image sample sent to the display is converted into a little square of light of the appropriate color:
(a pixel = picture element)



LCD display pixel on my laptop

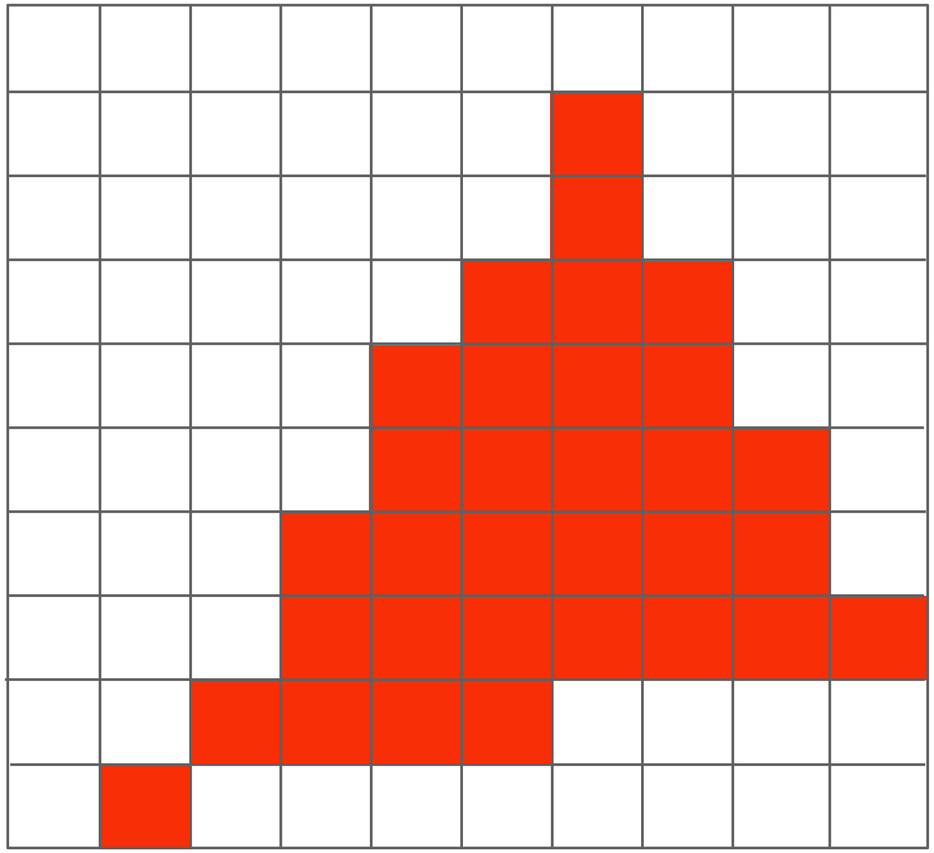
* Thinking of each LCD pixel as emitting a square of uniform intensity light of a single color is a bit of an approximation to how real displays work, but it will do for now.

So, if we send the display this sampled signal…



…and each value determines the light emitted from a pixel…

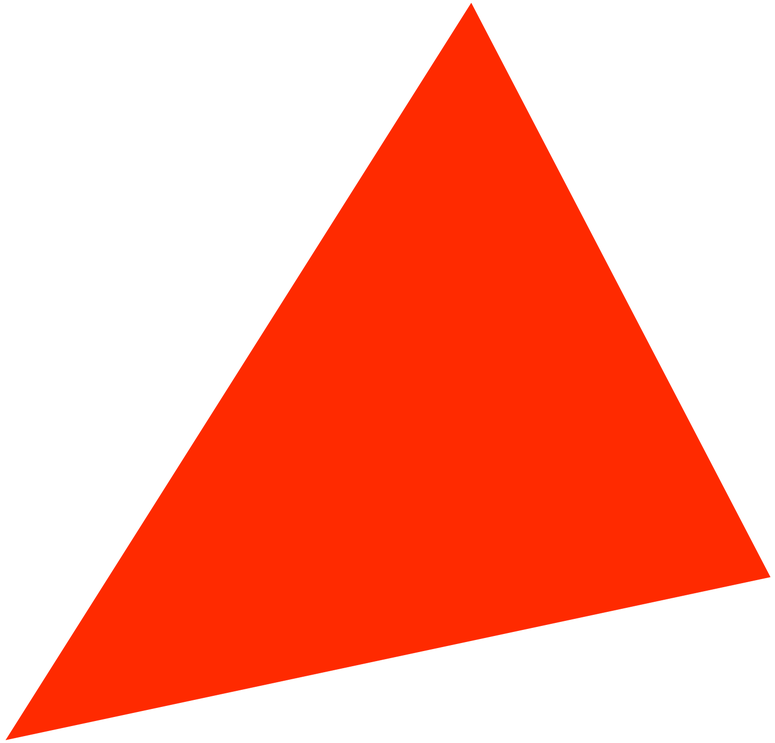
The display physically emits this signal



Given our simplified "square pixel" display assumption, the emitted light is a piecewise constant reconstruction of the samples

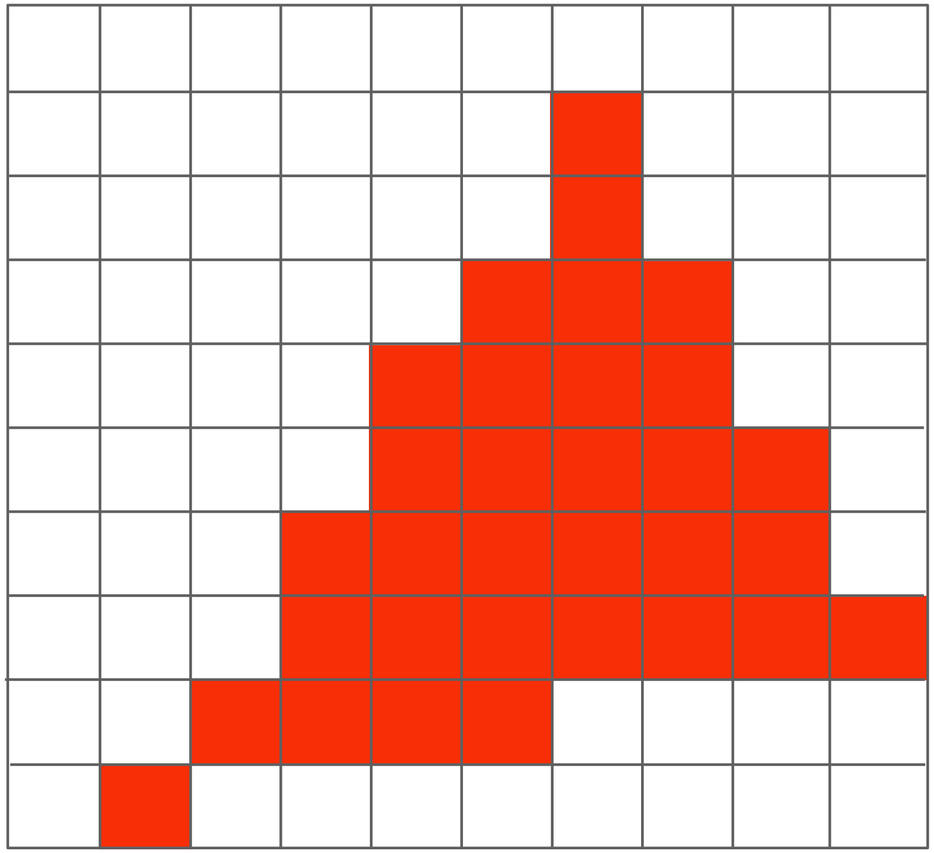
Compare: the continuous triangle function

(This is the function we sampled)



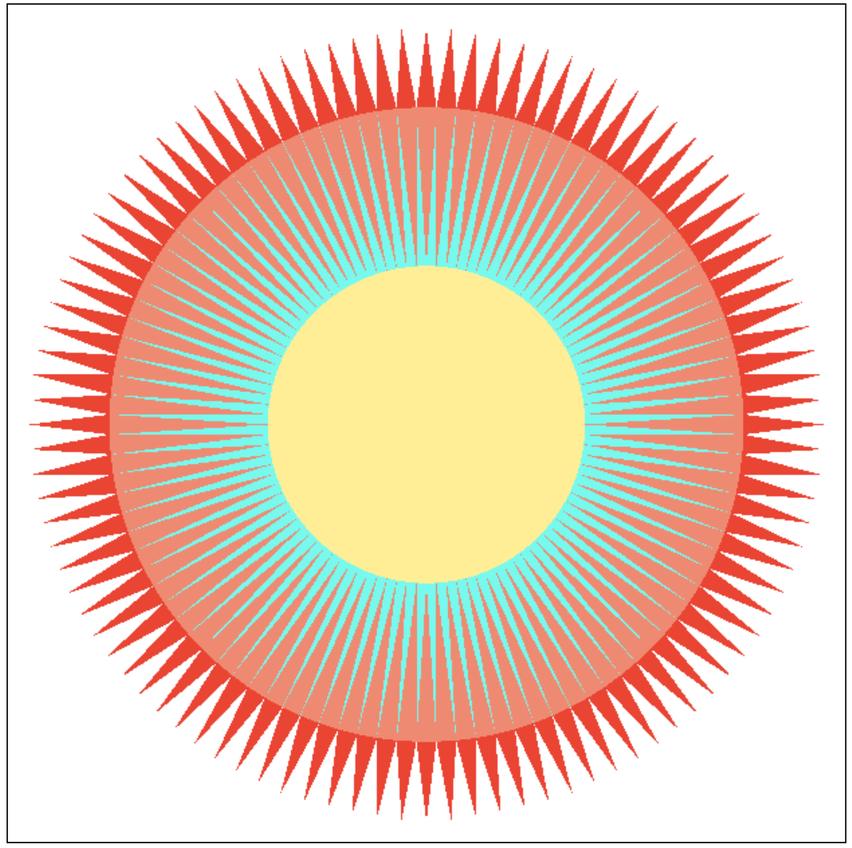
What's wrong with this picture?

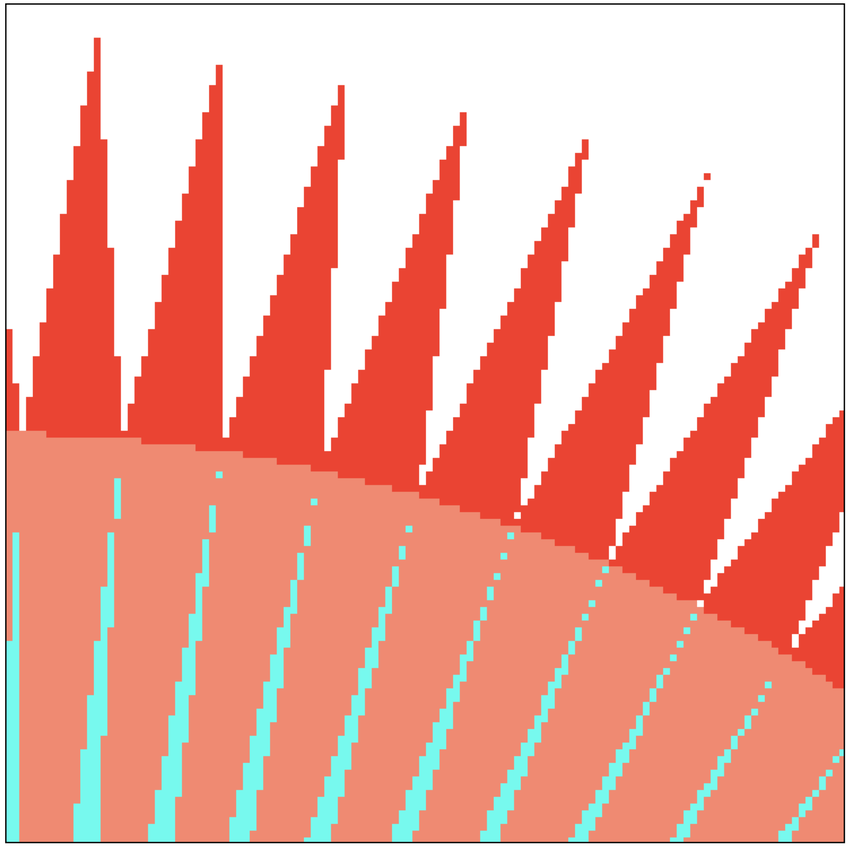
(This is the reconstruction emitted by the display)



Jaggies!

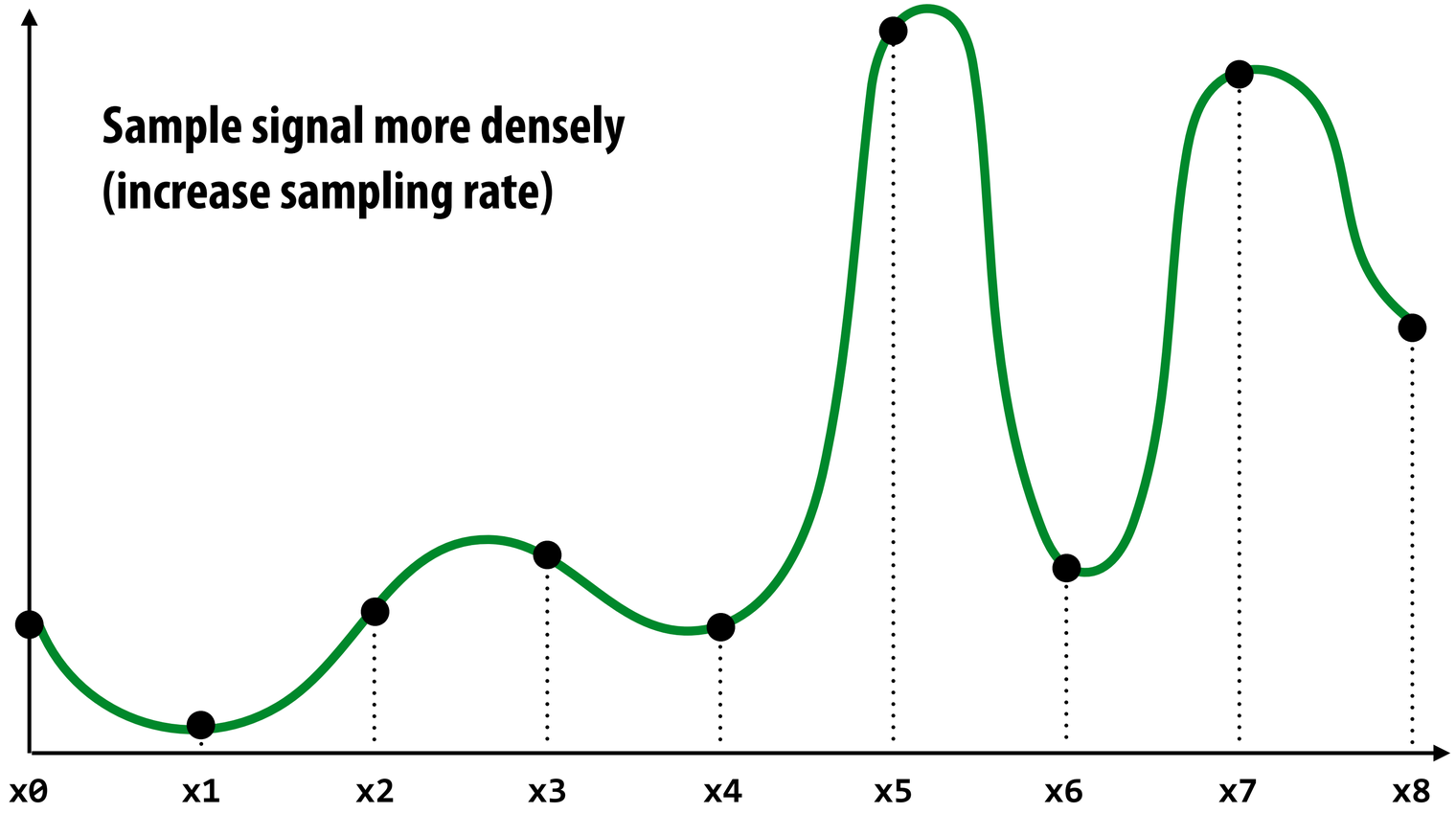
Jaggies (staircase pattern)



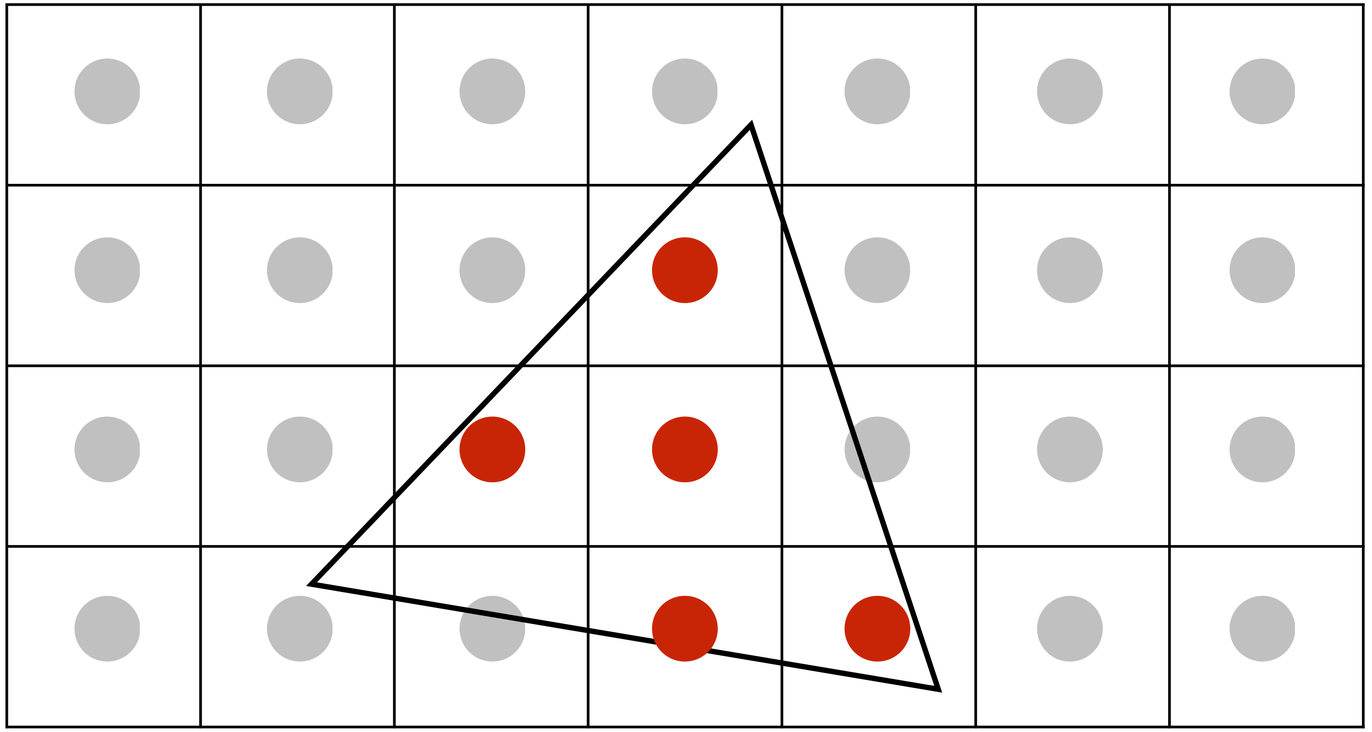


Is this the best we can do?

Reminder: how can we represent a sampled signal more accurately?



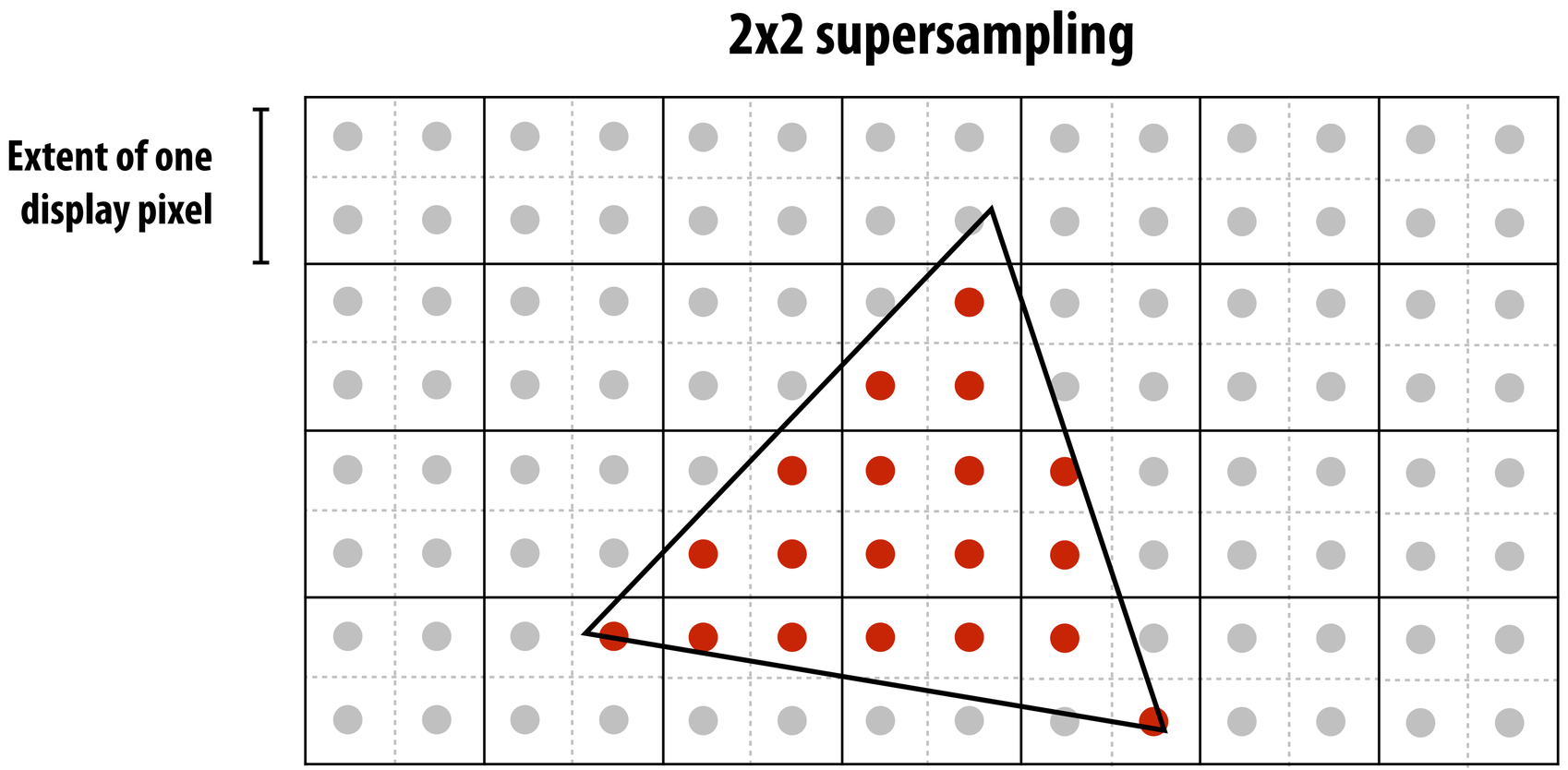
Sampling using one sample per pixel



Supersampling: step 1

Sample the input signal more densely in the image plane

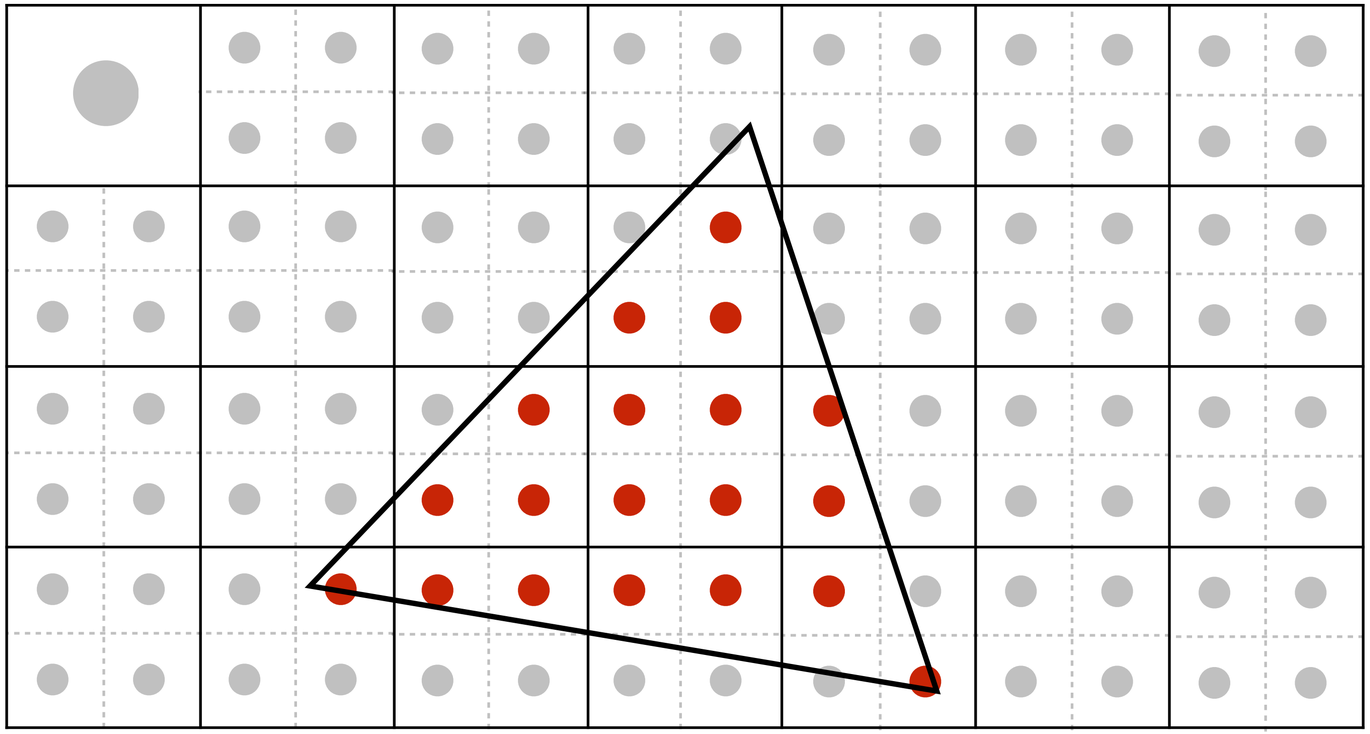
In this example: take 2 x 2 samples in the area spanned by a pixel



(but how do we use these samples to drive a display, since there are four times more samples than display pixels!). 🤔

Supersampling: step 2

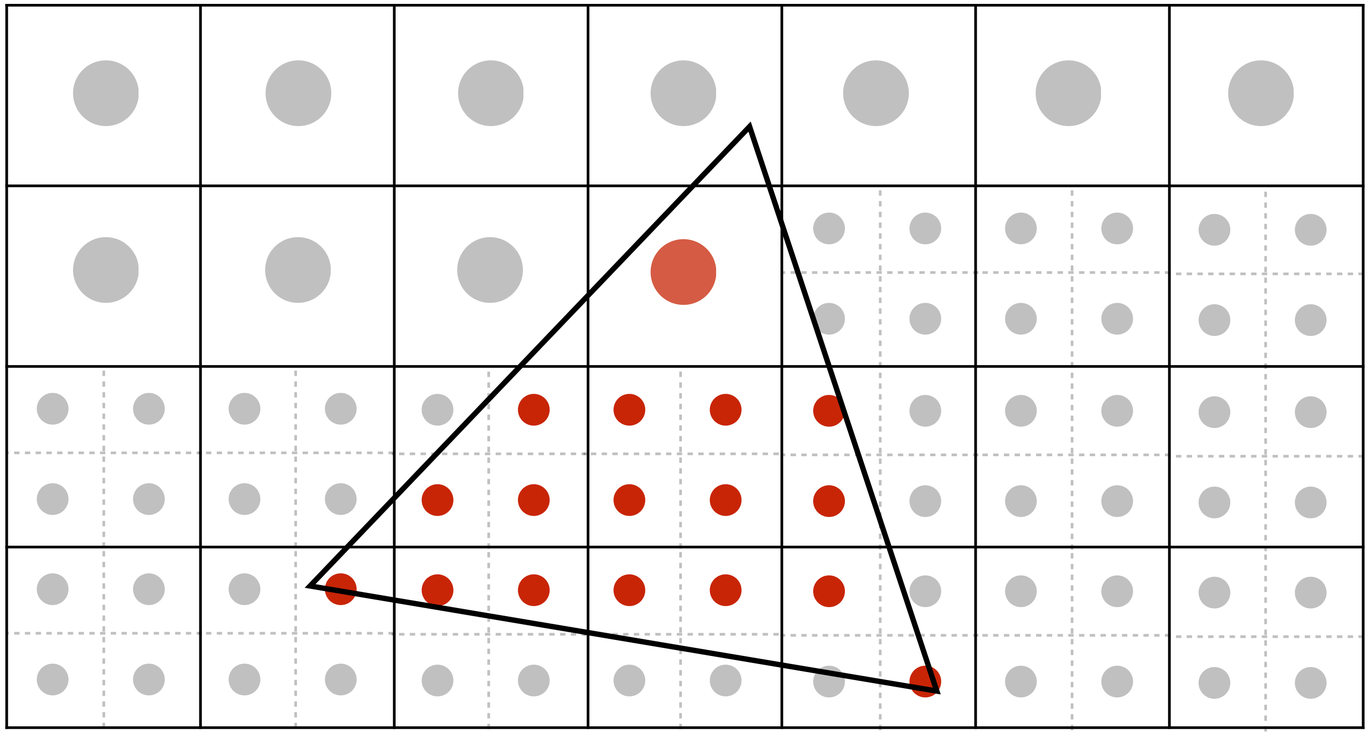
Average the N x N samples "inside" each pixel



Averaging down

Supersampling: step 2

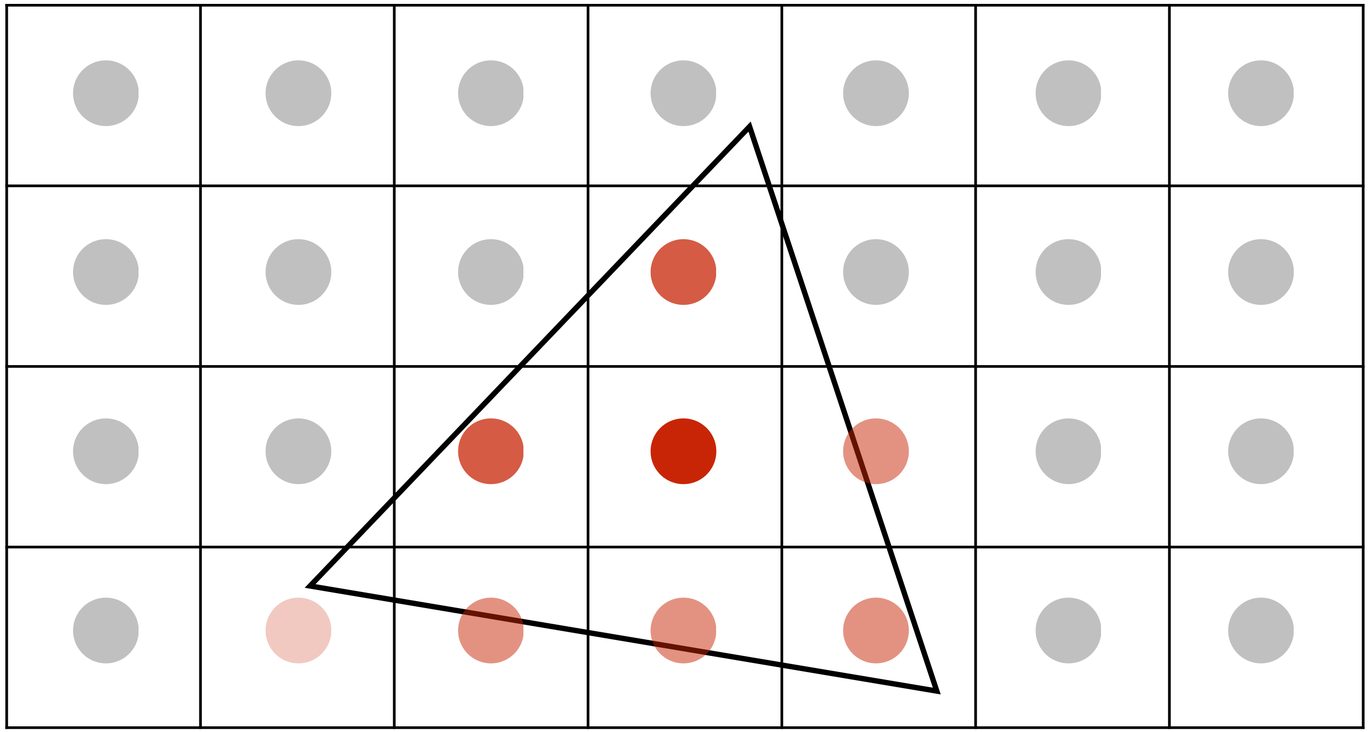
Average the N x N samples "inside" each pixel



Averaging down

Supersampling: step 2

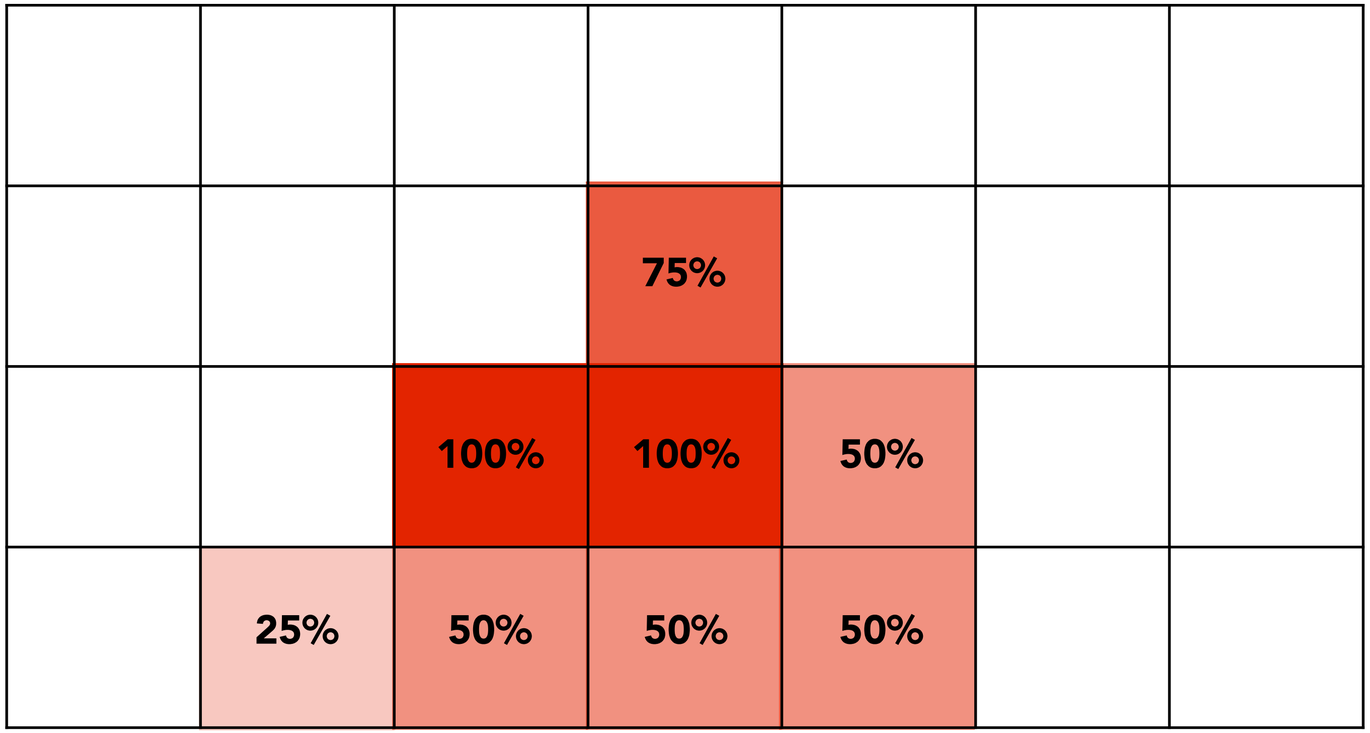
Average the N x N samples "inside" each pixel



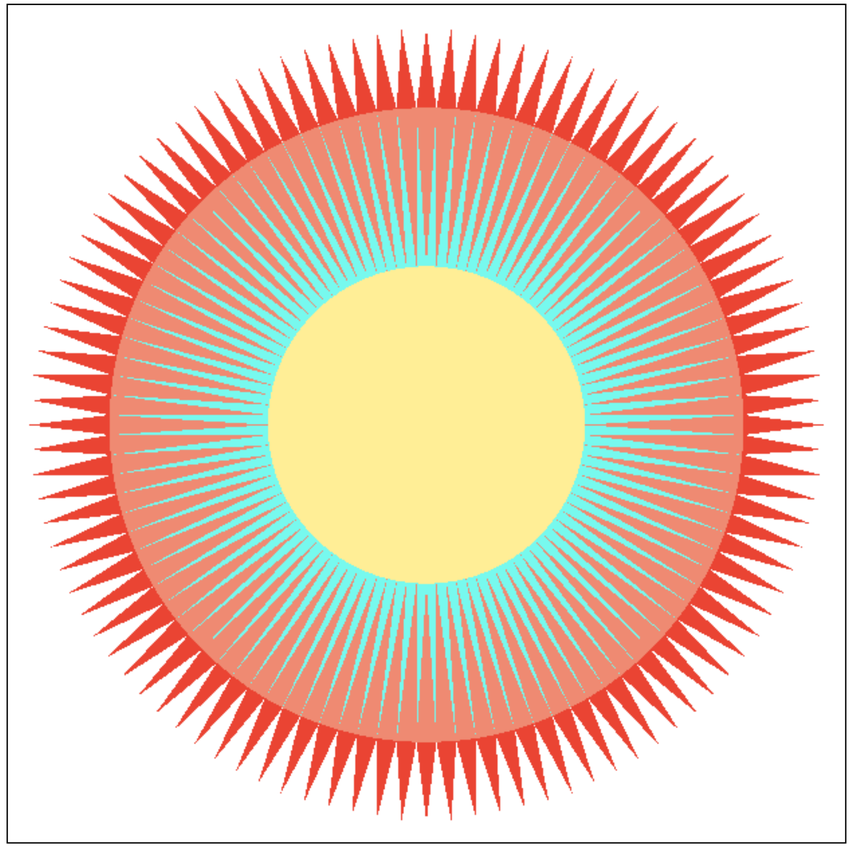
Averaging down

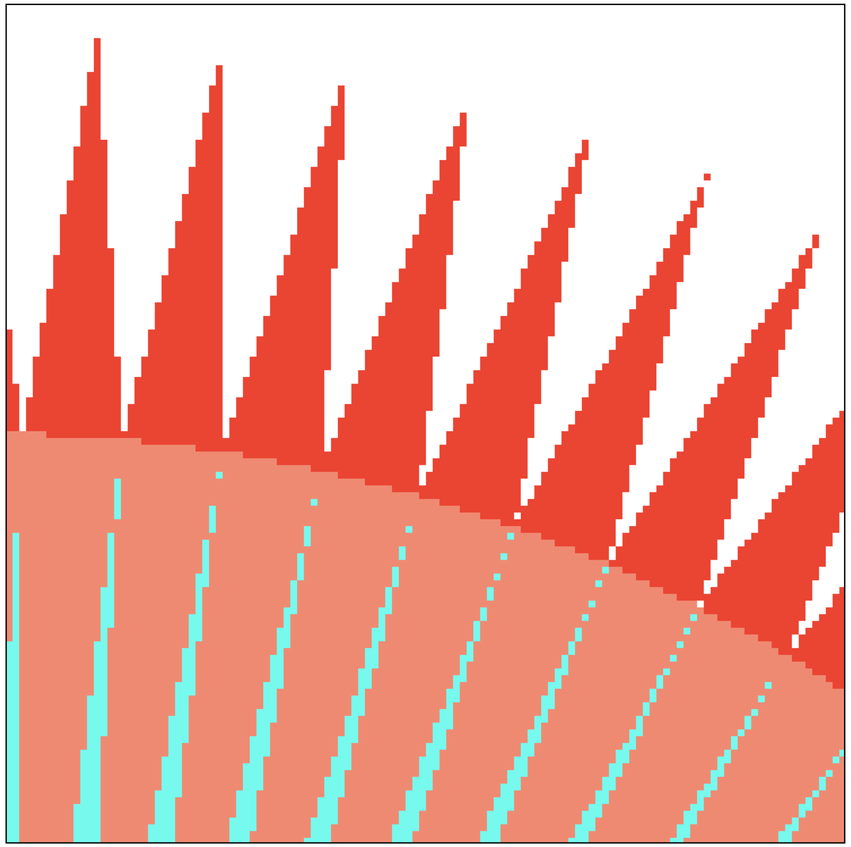
Displayed result

This is the corresponding signal emitted by the display
(value provided to each display pixel is the average of the values sampled in that region)

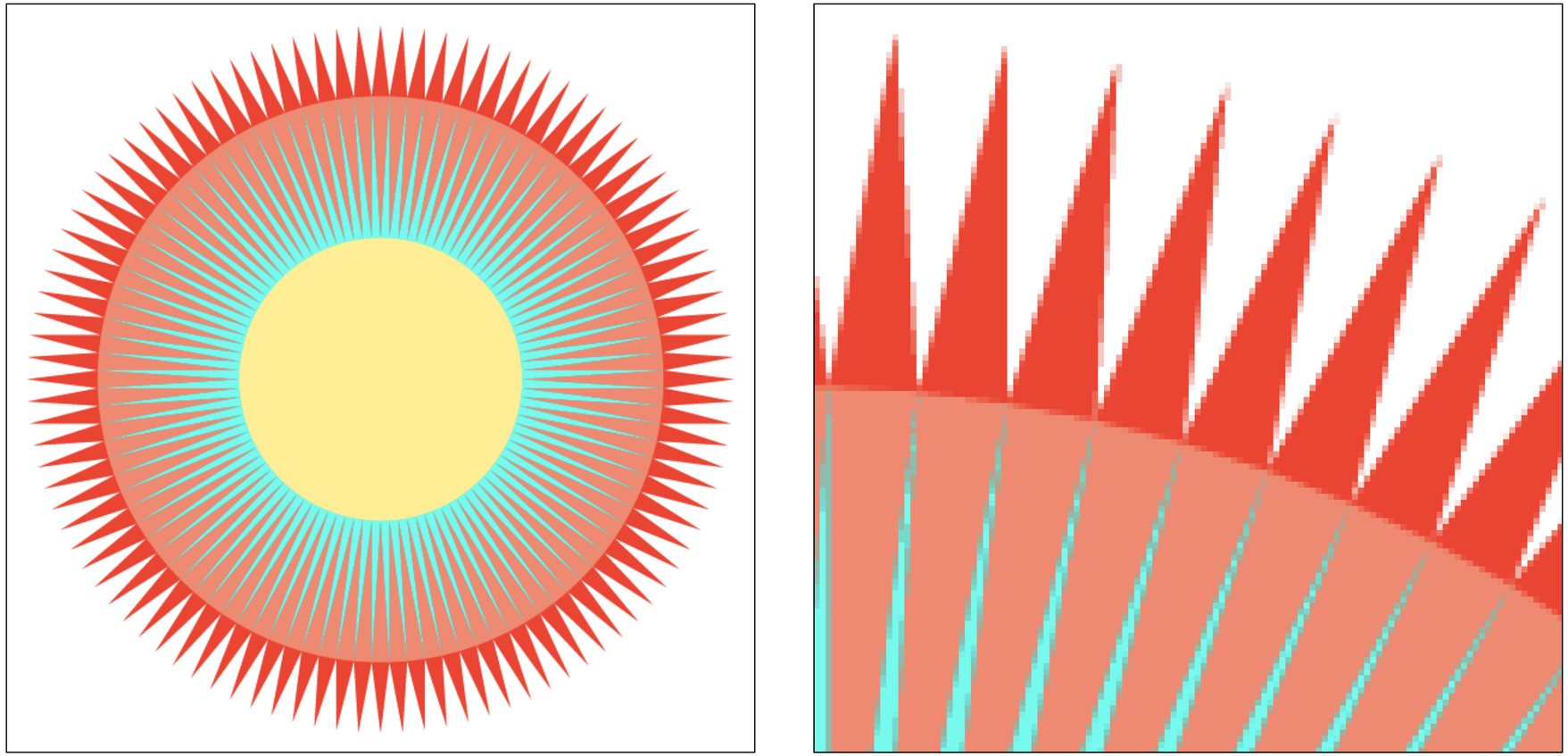


Images rendered using one sample per pixel





4x4 supersampling + downsampling

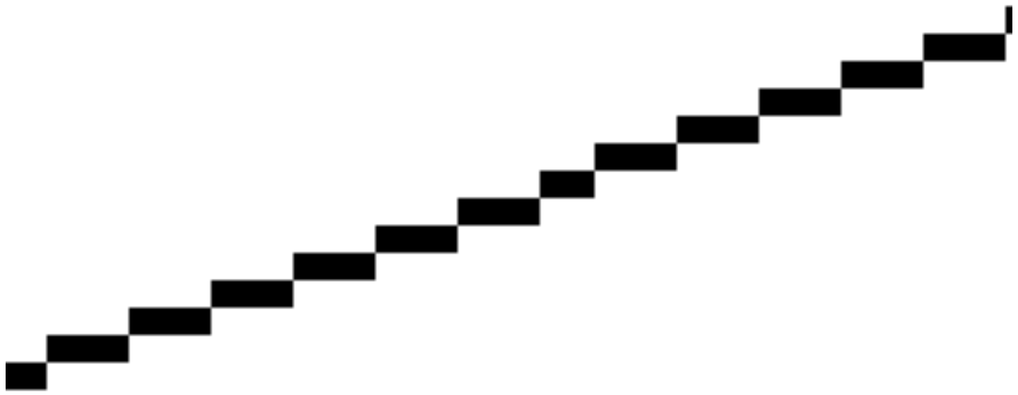


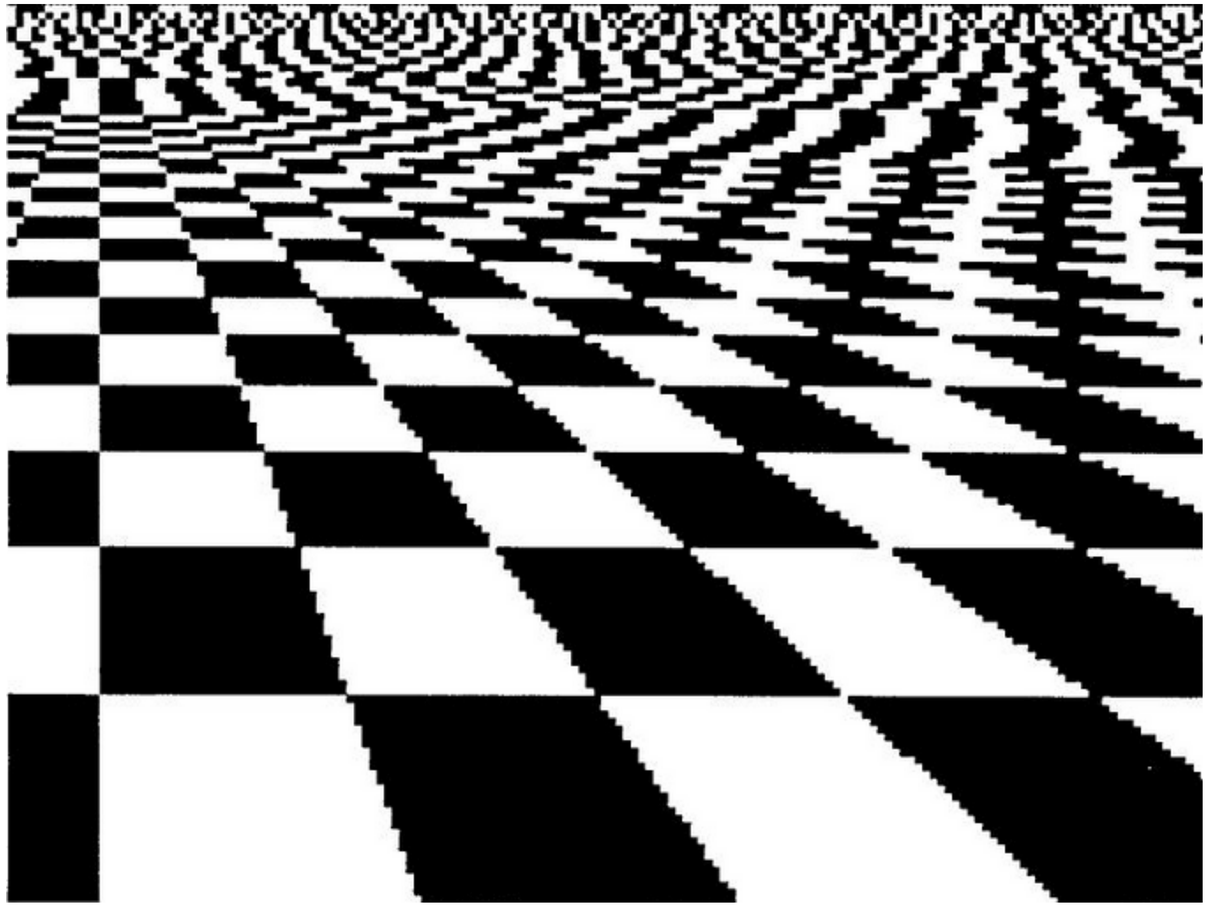
Each pixel's value is the average of the values of the 4x4 samples per pixel

Let's understand what just happened in a more principled way

More examples of sampling artifacts in computer graphics

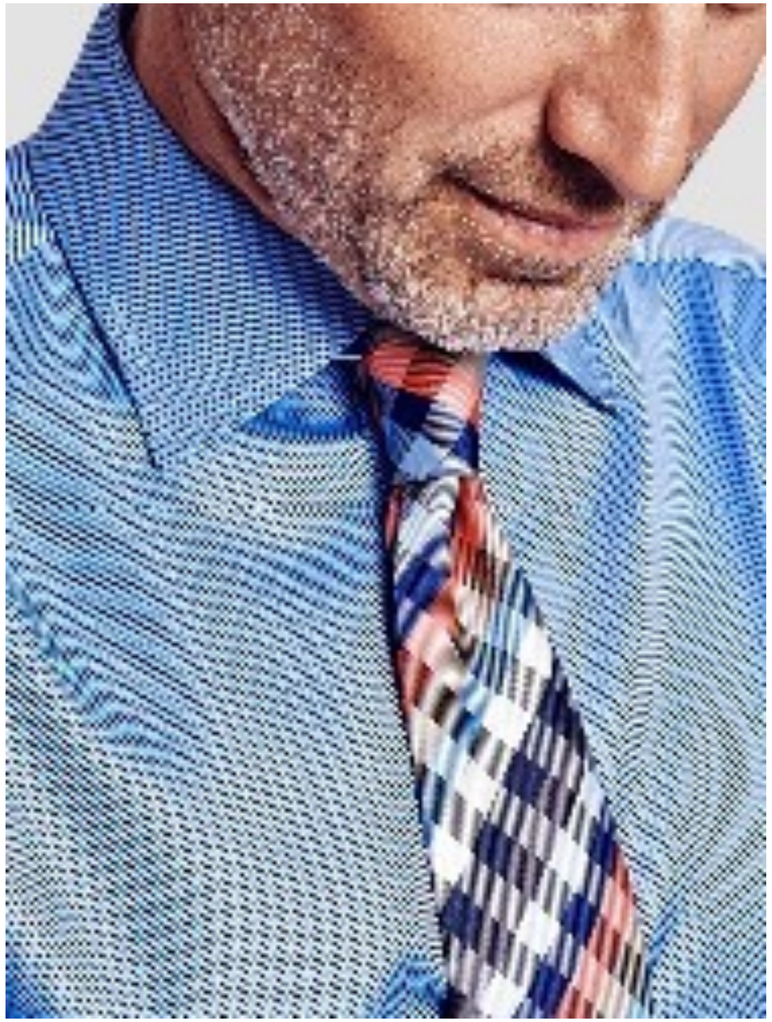
Jaggies (staircase pattern)





Moiré patterns in imaging

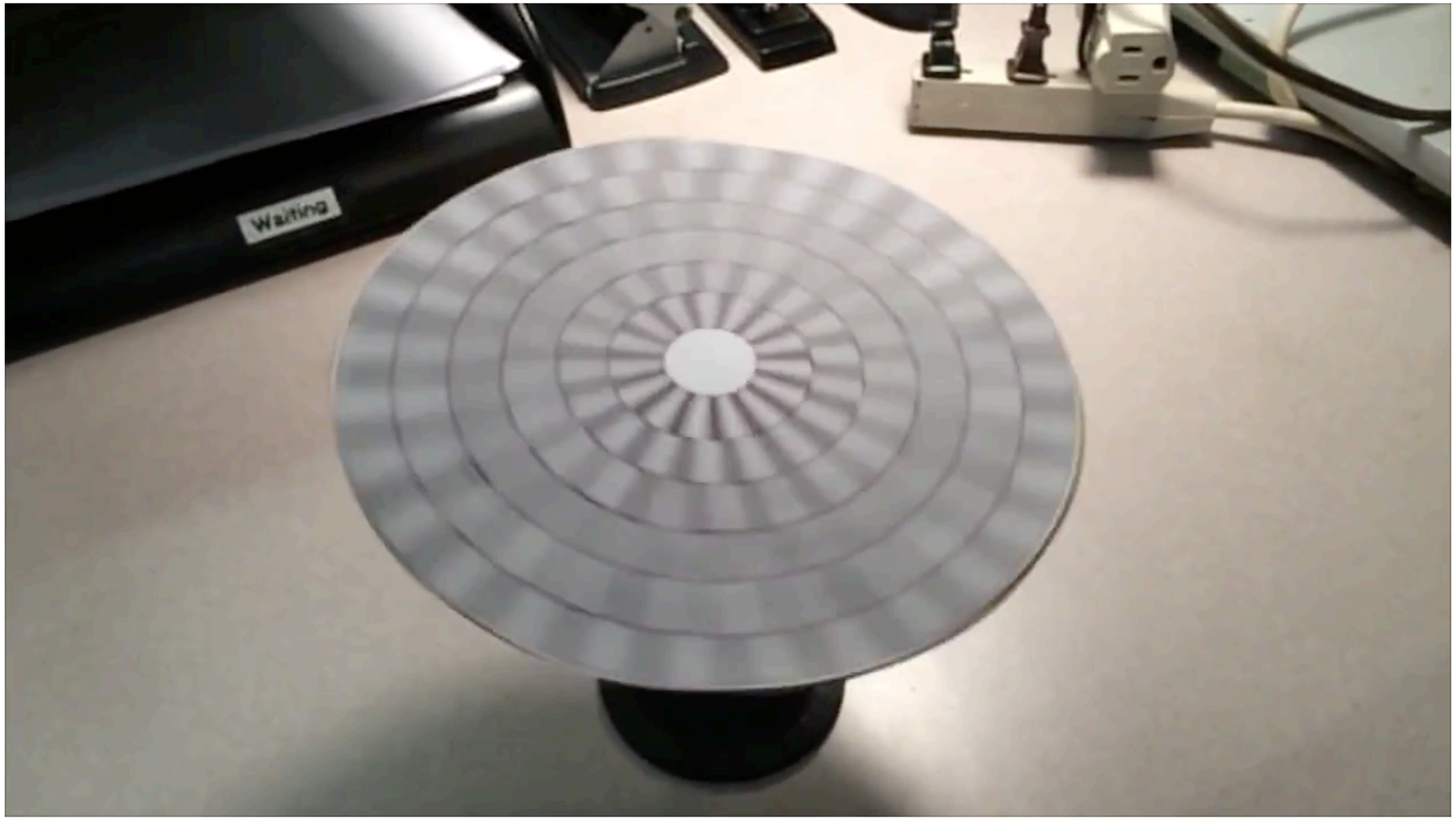
Full resolution image



1/2 resolution image:
skip pixel odd rows and columns

lystit.com

Wagon wheel illusion (false motion)



Camera's frame rate (temporal sampling rate) is too low for rapidly spinning wheel.

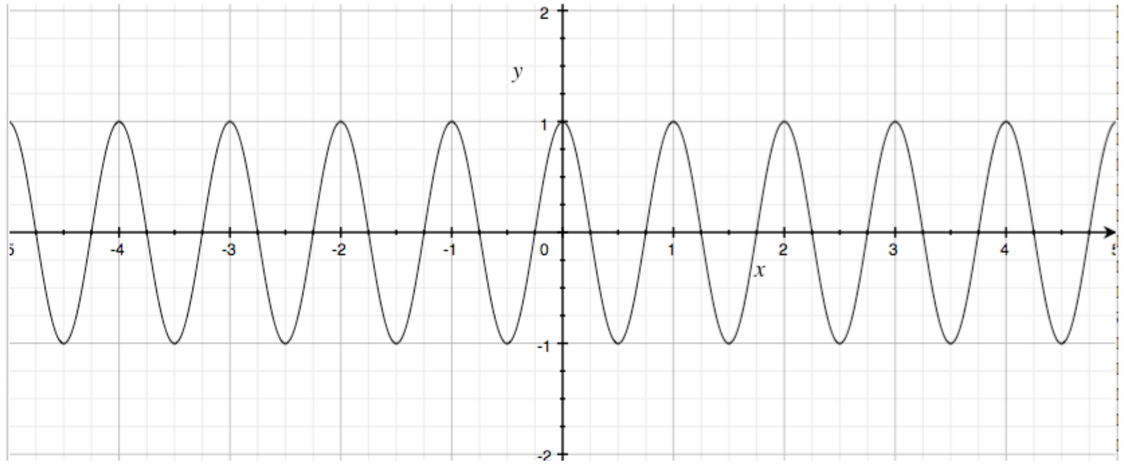
Created by Jesse Mason, https://www.youtube.com/watch?v=QOwzkND_ooU

Sampling artifacts in computer graphics

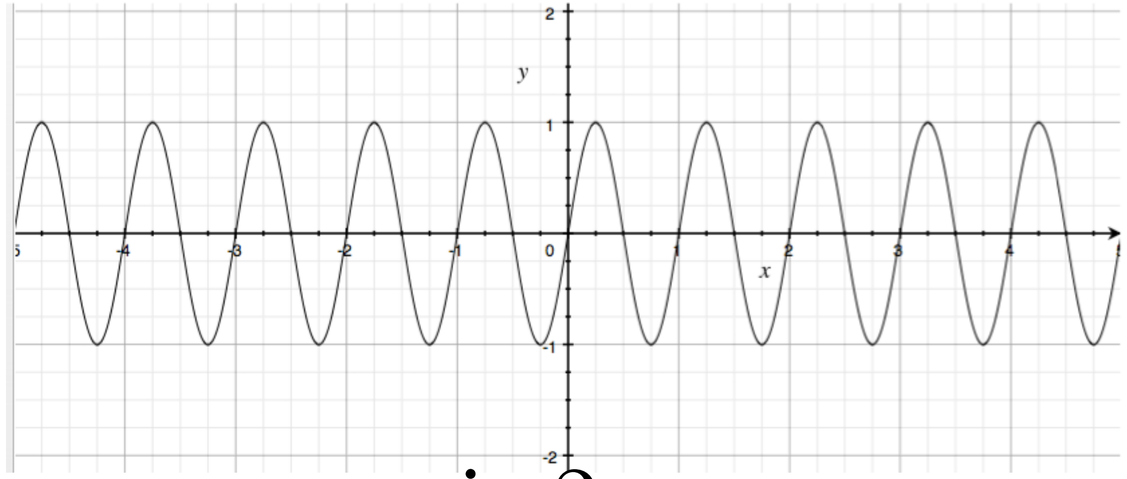
- Artifacts due to sampling - "Aliasing"
  - Jaggies – sampling in space
  - Wagon wheel effect – sampling in time
  - Moire – undersampling images (and texture maps)
  - [Many more] . . .

- We notice this in fast-changing signals, when we sample the signal too sparsely

Sines and cosines



$$\cos 2\pi x$$

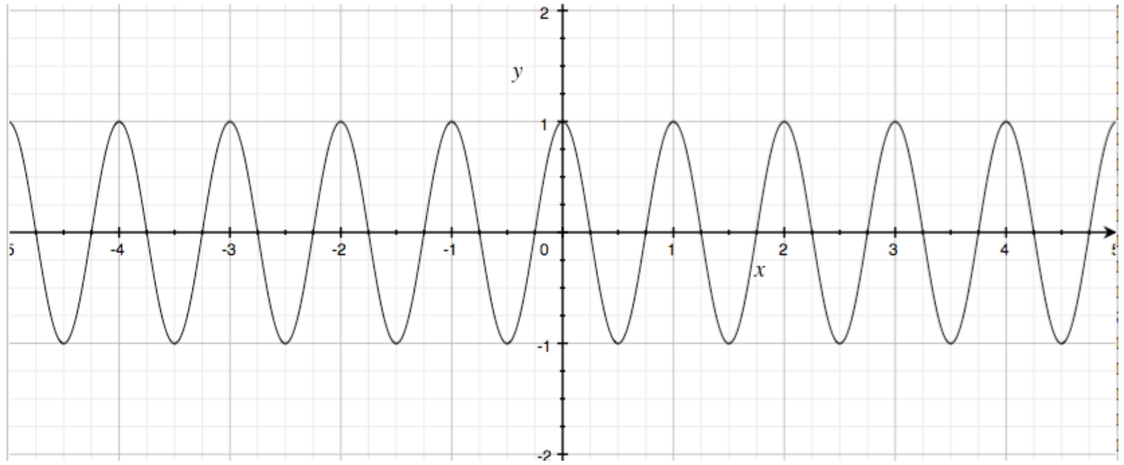


$$\sin 2\pi x$$

Frequencies

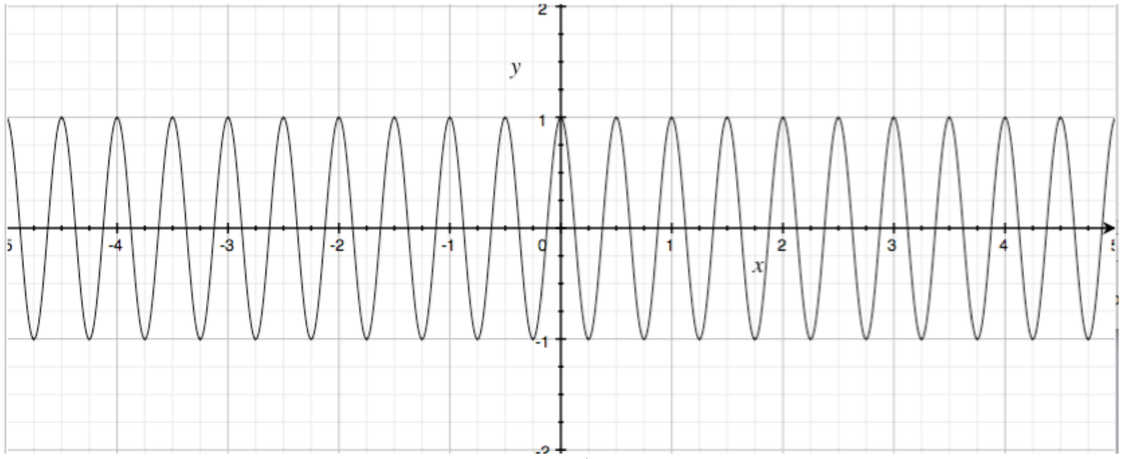
$$\cos 2\pi f x$$

$$f = \frac{1}{T}$$



$$f = 1$$

$$\cos 2\pi x$$

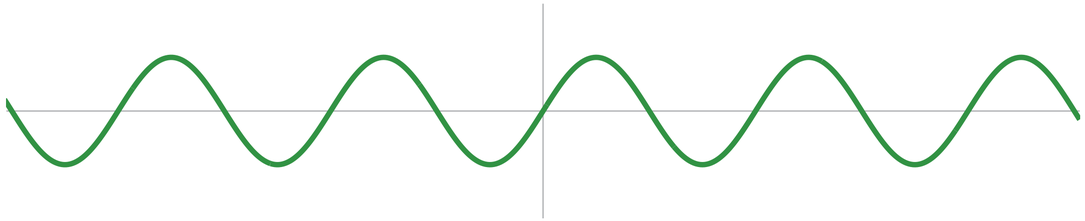


$$f = 2$$

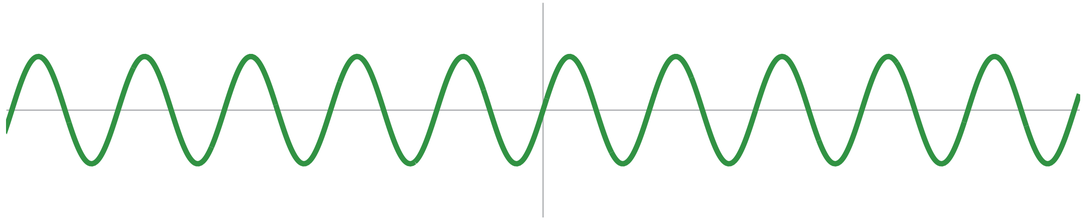
$$\cos 4\pi x$$

Representing sound wave as a superposition (linear combination) of frequencies

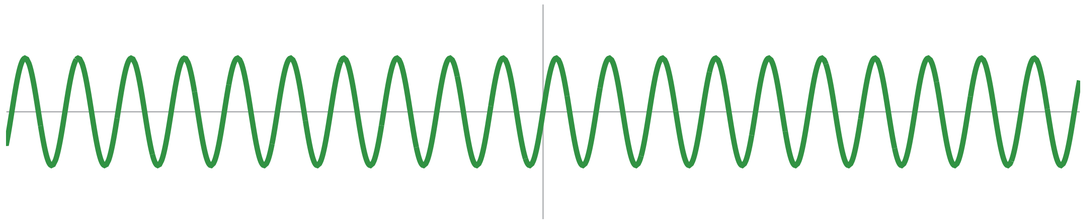
$f_1(x) = sin(\pi x)$



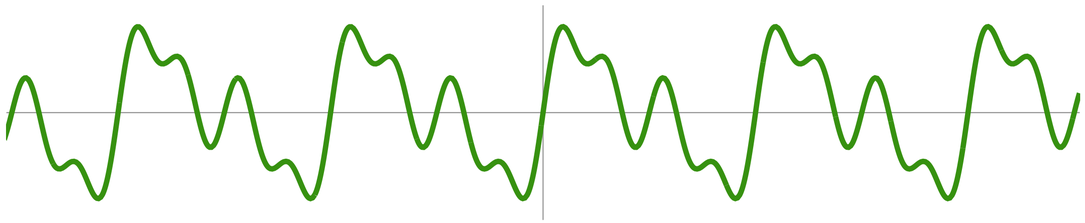
$f_2(x) = sin(2\pi x)$



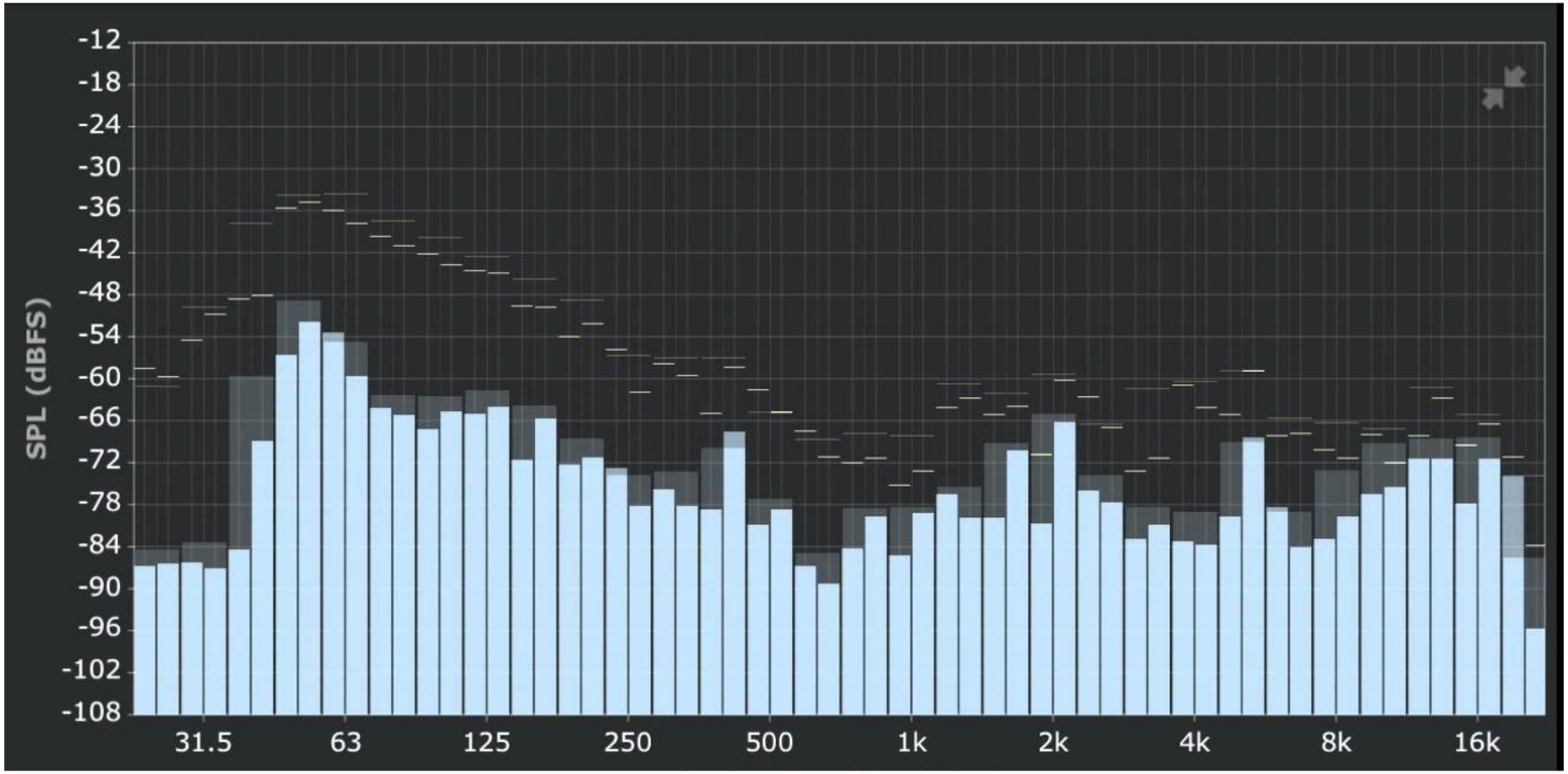
$f_4(x) = sin(4\pi x)$



$f(x) = 1.0\, f_1(x) + 0.75\, f_2(x) + 0.5\, f_4(x)$



Audio spectrum analyzer: representing sound as a sum of its constituent frequencies



Intensity of low-frequencies (bass)

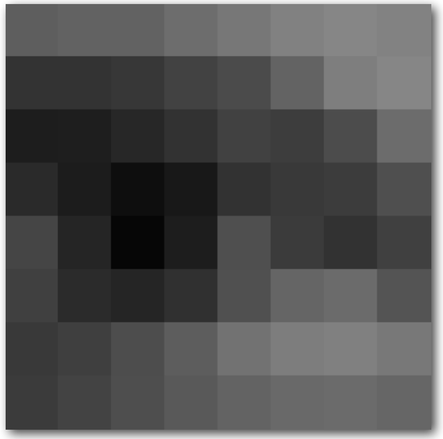
Intensity of high frequencies

Image credit: ONYX Apps

Images as a superposition of cosines

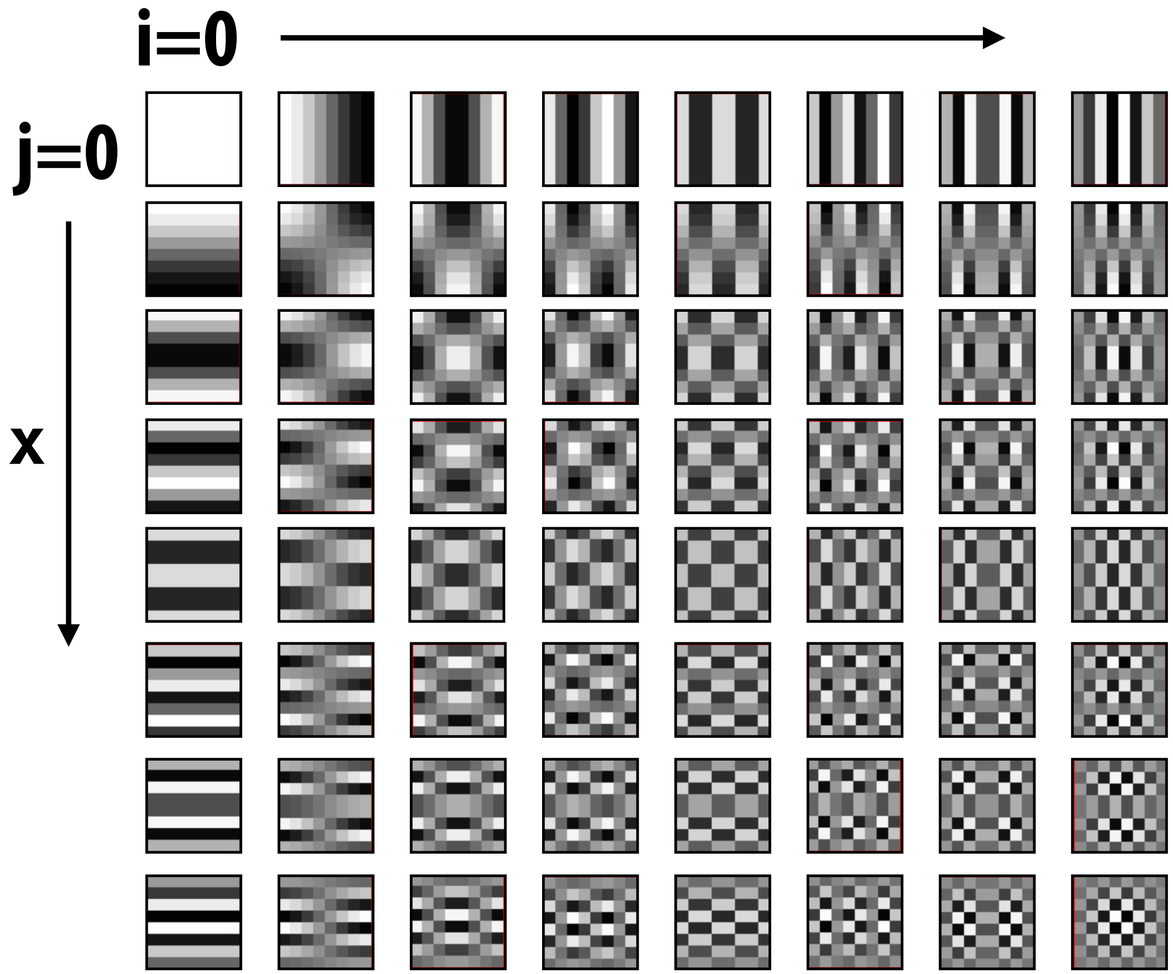
$$\cos\left[\pi\frac{i}{N}\left(x+\frac{1}{2}\right)\right]\times\cos_{j}\left[\pi\frac{j}{N}\left(y+\frac{1}{2}\right)\right]$$

8x8 images



=

$$\begin{bmatrix} -415 & -30 & -61 & 27 & 56 & -20 & -2 & 0 \\ 4 & -22 & -61 & 10 & 13 & -7 & -9 & 5 \\ -47 & 7 & 77 & -25 & -29 & 10 & 5 & -6 \\ -49 & 12 & 34 & -15 & -10 & 6 & 2 & 2 \\ 12 & -7 & -13 & -4 & -2 & 2 & -3 & 3 \\ -8 & 3 & 2 & -6 & -2 & 1 & 4 & 2 \\ -1 & 0 & 0 & -2 & -1 & -3 & 4 & -1 \\ 0 & 0 & -1 & -4 & -1 & 0 & 1 & 2 \end{bmatrix}$$

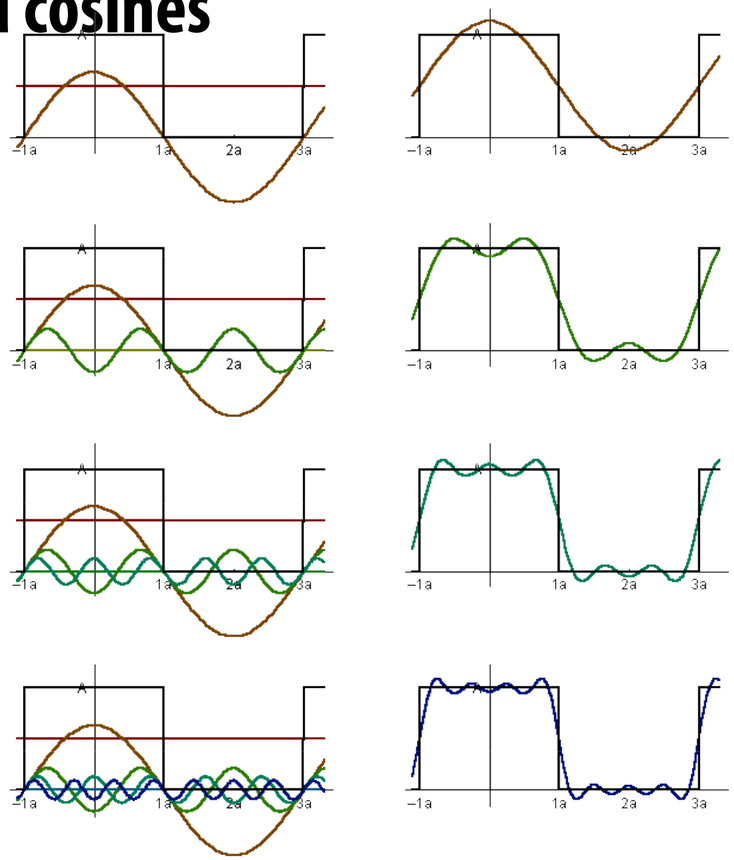


How to compute frequency-domain representation of a signal?

Fourier transform

Represent a function as a weighted sum of sines and cosines

Joseph Fourier 1768 - 1830



$$f(x) = \frac{A}{2} + \frac{2A\cos(t\omega)}{\pi} - \frac{2A\cos(3t\omega)}{3\pi} + \frac{2A\cos(5t\omega)}{5\pi} - \frac{2A\cos(7t\omega)}{7\pi} + \cdots$$

Fourier transform

- **Convert representation of signal from primal domain (spatial/temporal) to frequency domain by projecting signal into its component frequencies**

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \omega} dx$$

$$= \int_{-\infty}^{\infty} f(x)(\cos(2\pi\omega x) - i\sin(2\pi\omega x))dx$$

Recall:

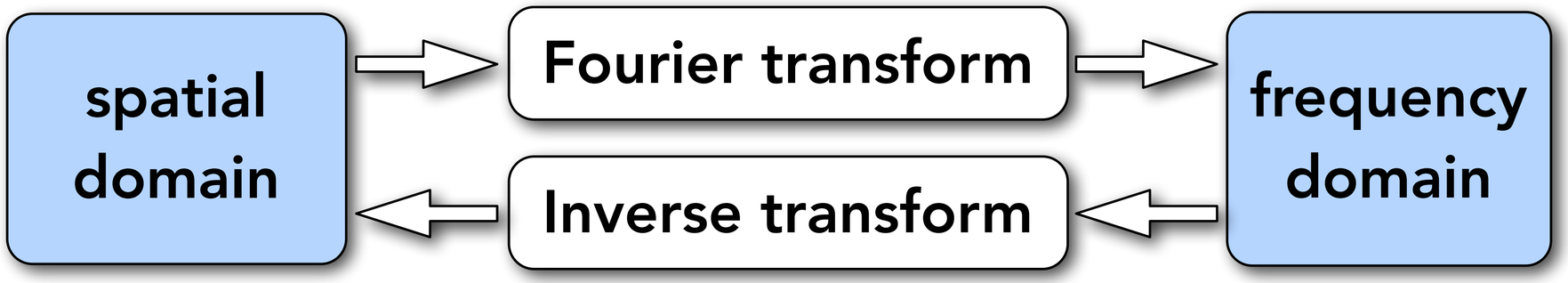
$$e^{ix} = \cos x + i \sin x$$

  - **2D form:**

$$F(u,v) = \int \int f(x,y) e^{-2\pi i(ux+vy)} dx dy$$

The Fourier transform decomposes a signal into its constituent frequencies

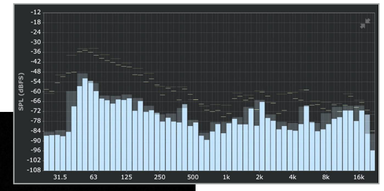
$$f(x) \qquad F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \omega x} dx \qquad F(\omega)$$



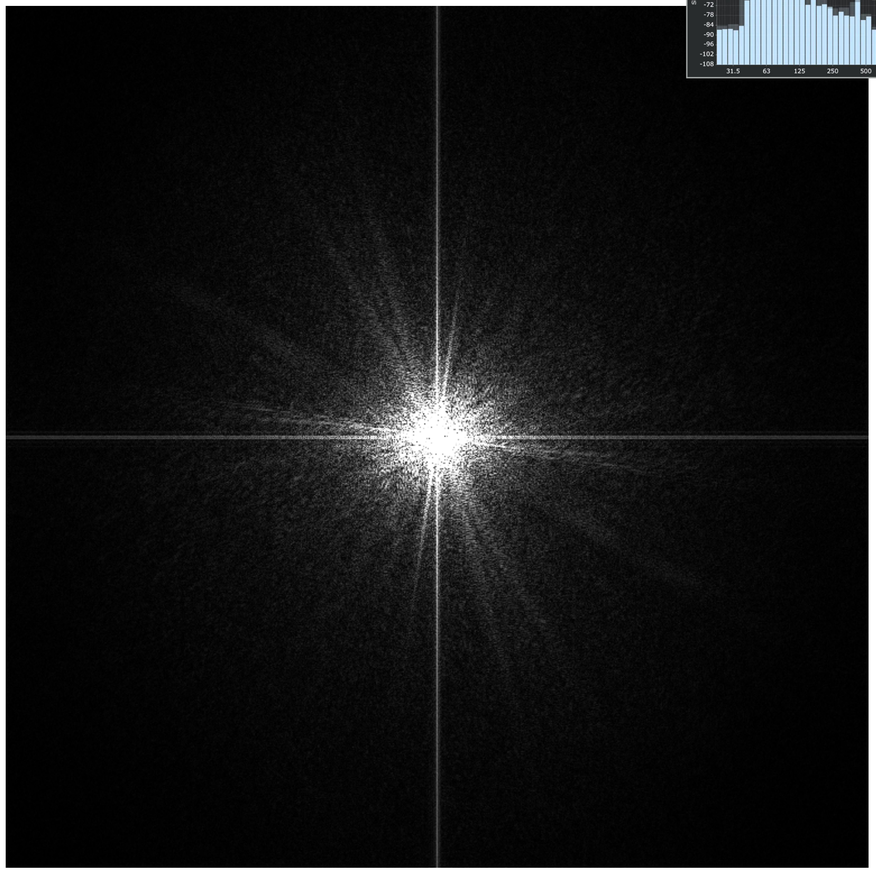
$$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{2\pi i \omega x} d\omega$$

Visualizing the frequency content of images

The visualization below is the 2D frequency domain equivalent of the 1D audio spectrum I showed you earlier *

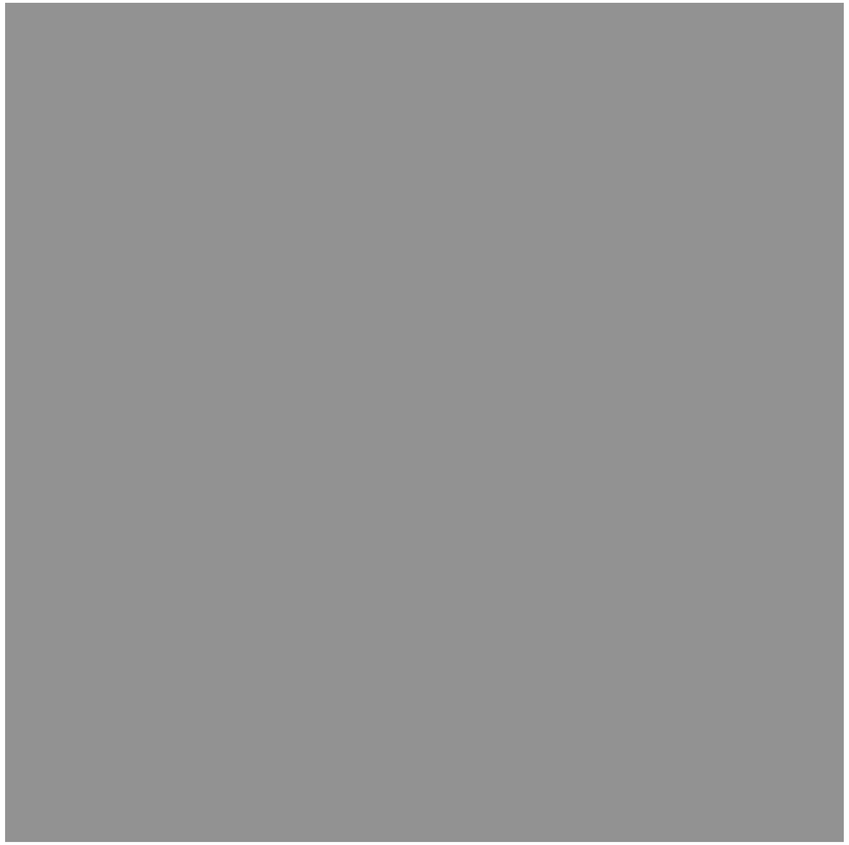


Spatial domain result

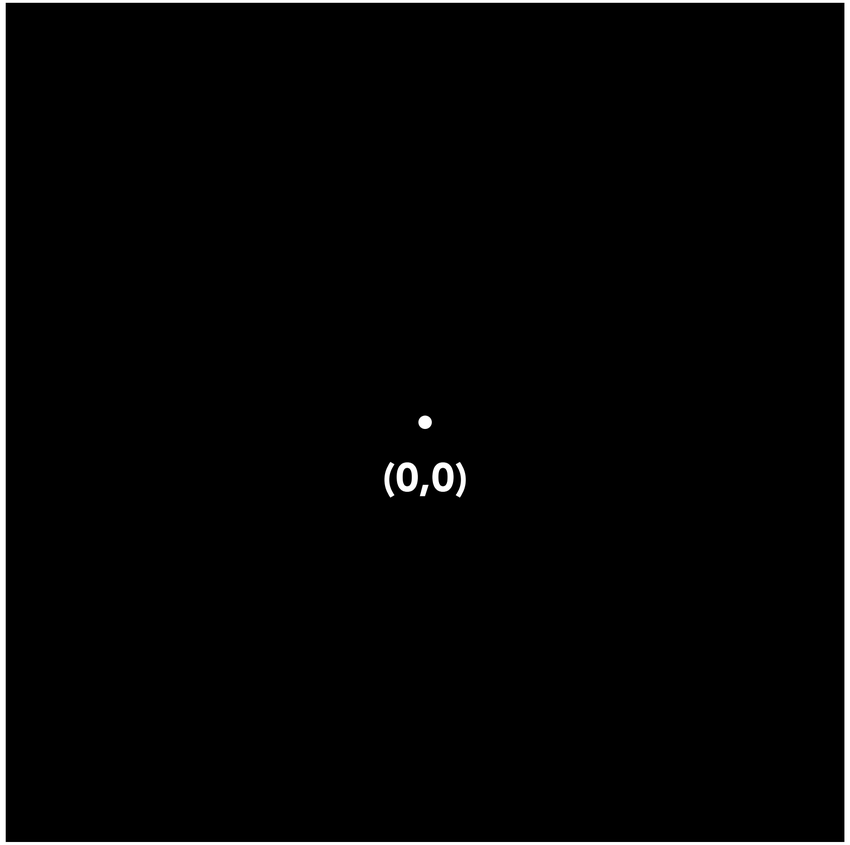


Spectrum

Constant signal (in primal domain)

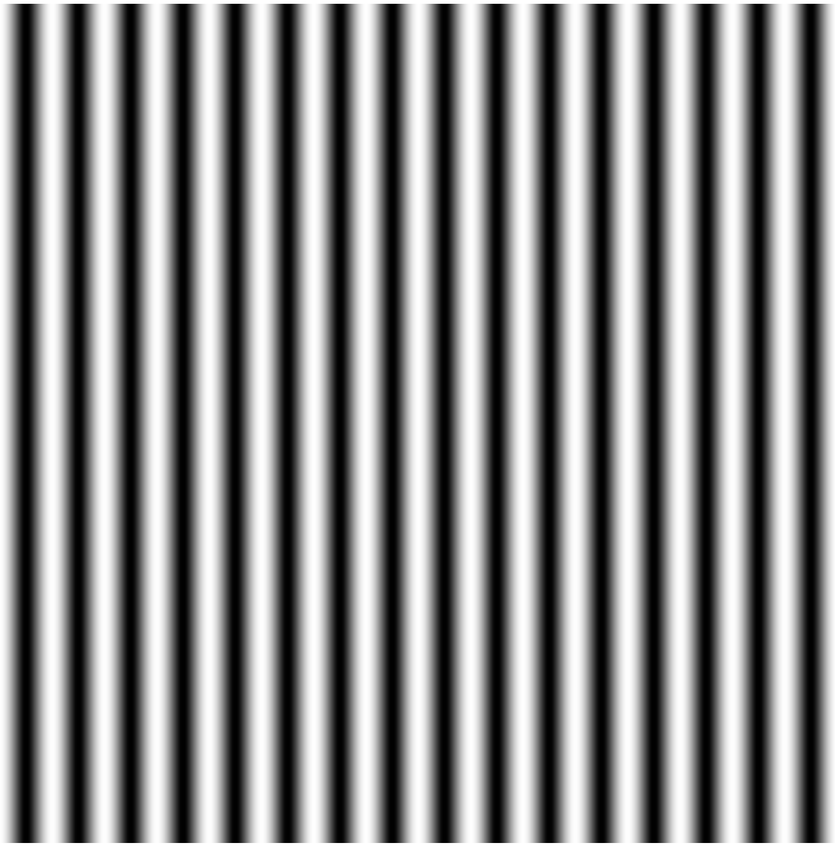


Spatial domain

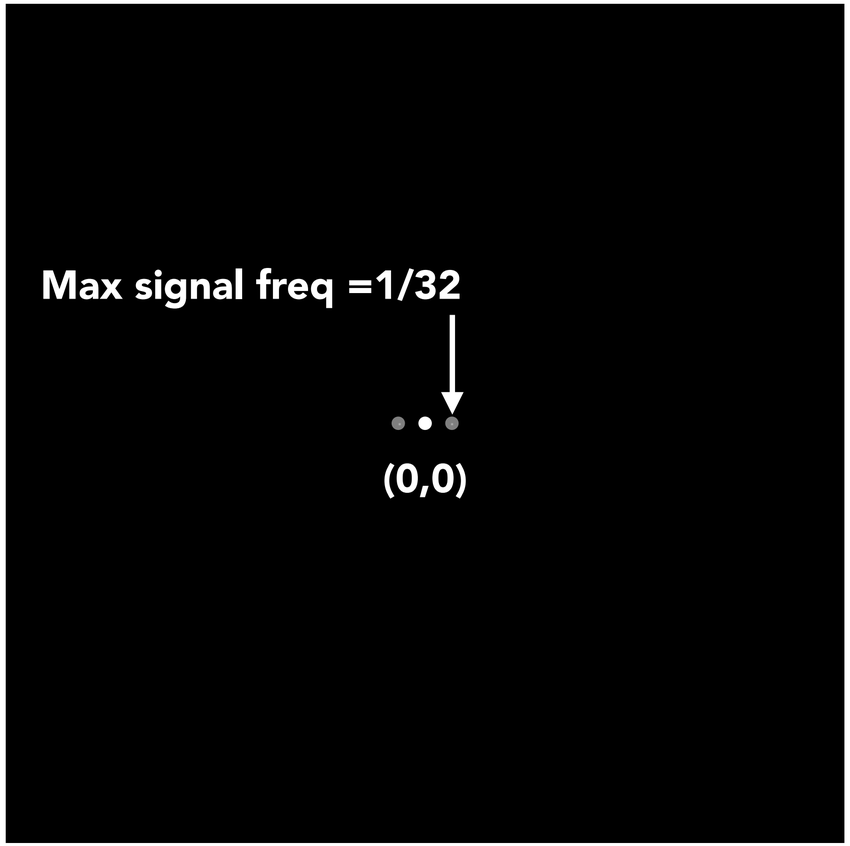


Frequency domain

$\sin(2\pi/32)x$ — frequency 1/32; 32 pixels per cycle

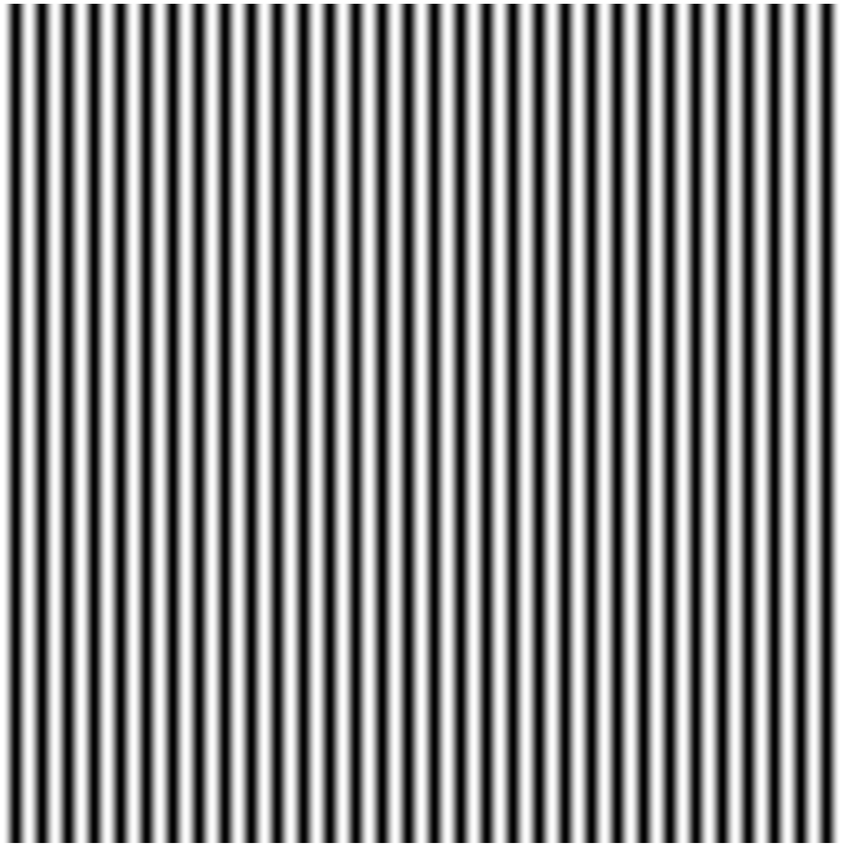


Spatial domain

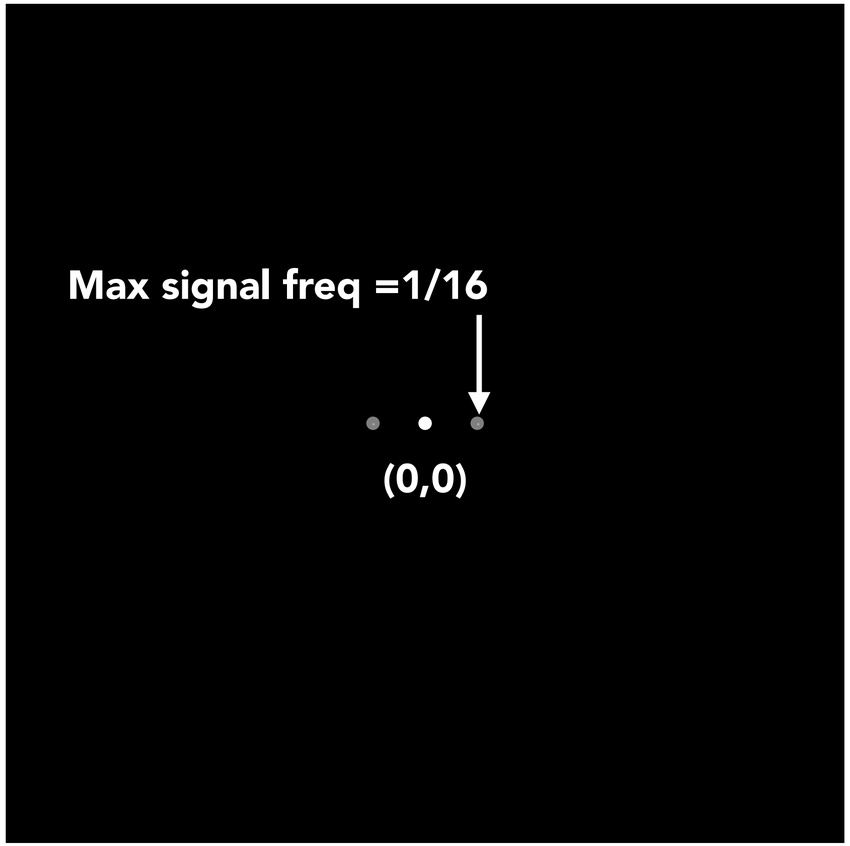


Frequency domain

$\sin(2\pi/16)x$ — frequency 1/16; 16 pixels per cycle

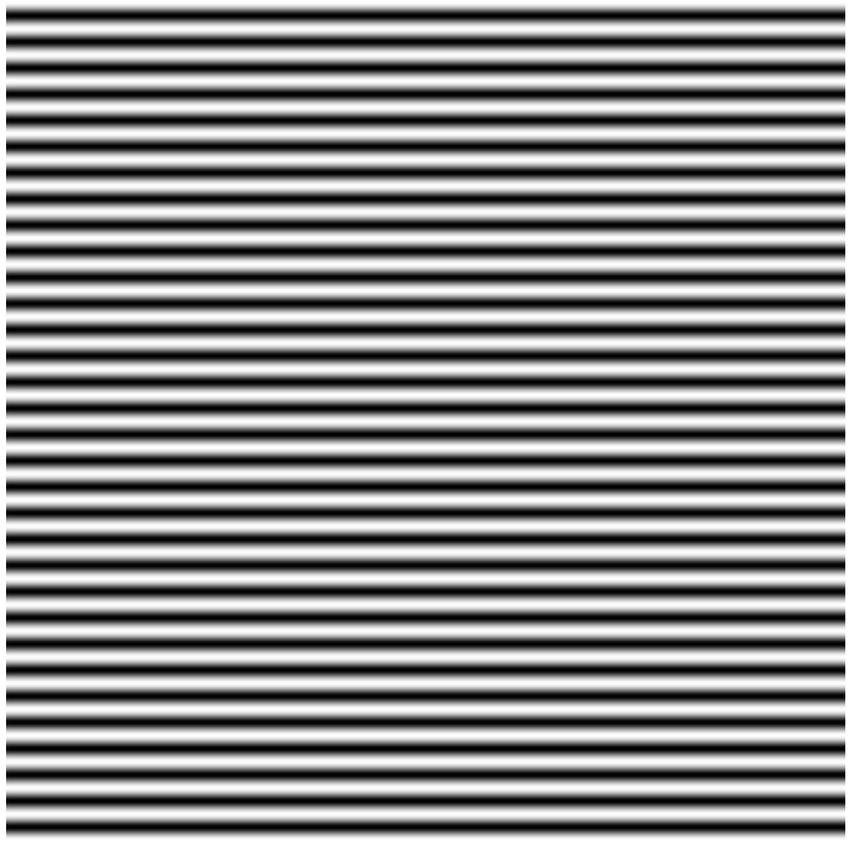


Spatial domain

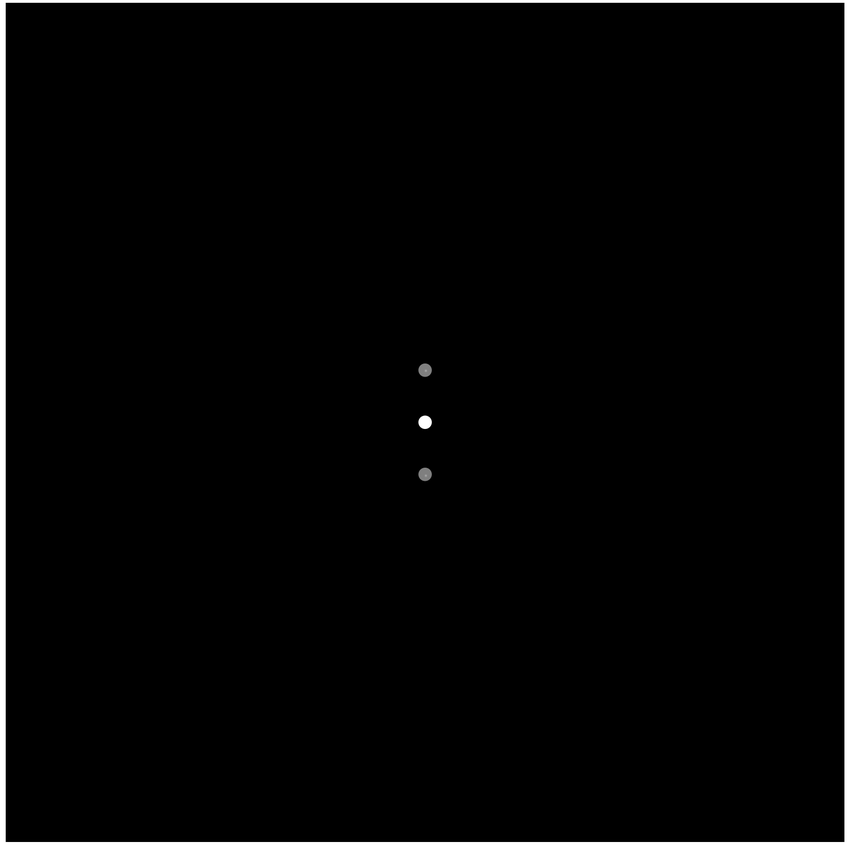


Frequency domain

$$\sin(2\pi/16)y$$

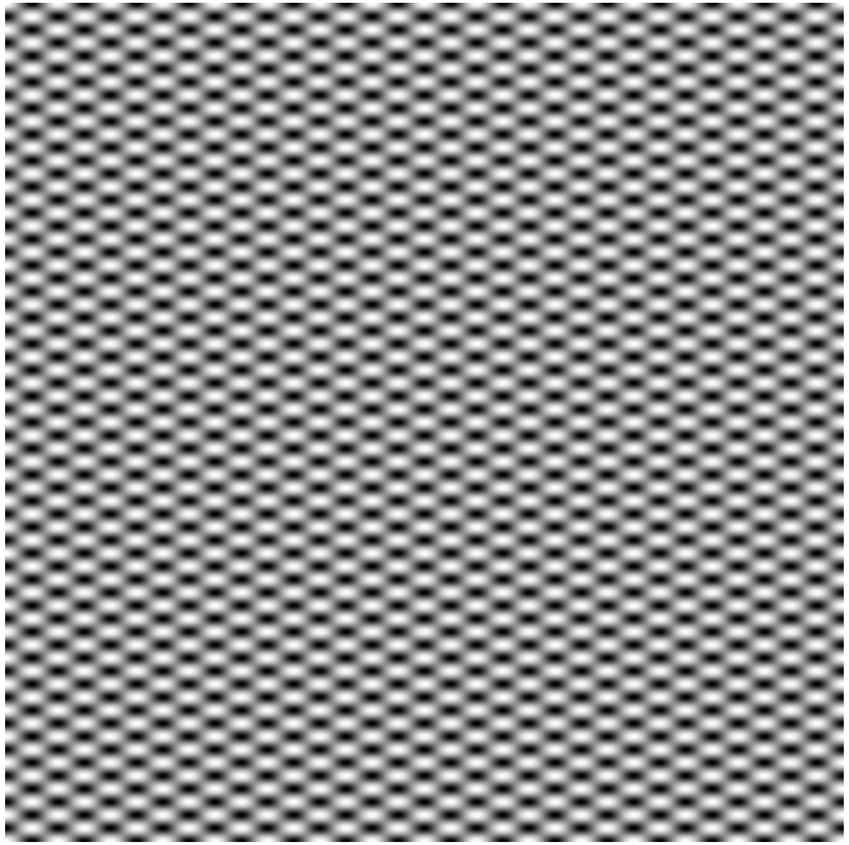


Spatial domain

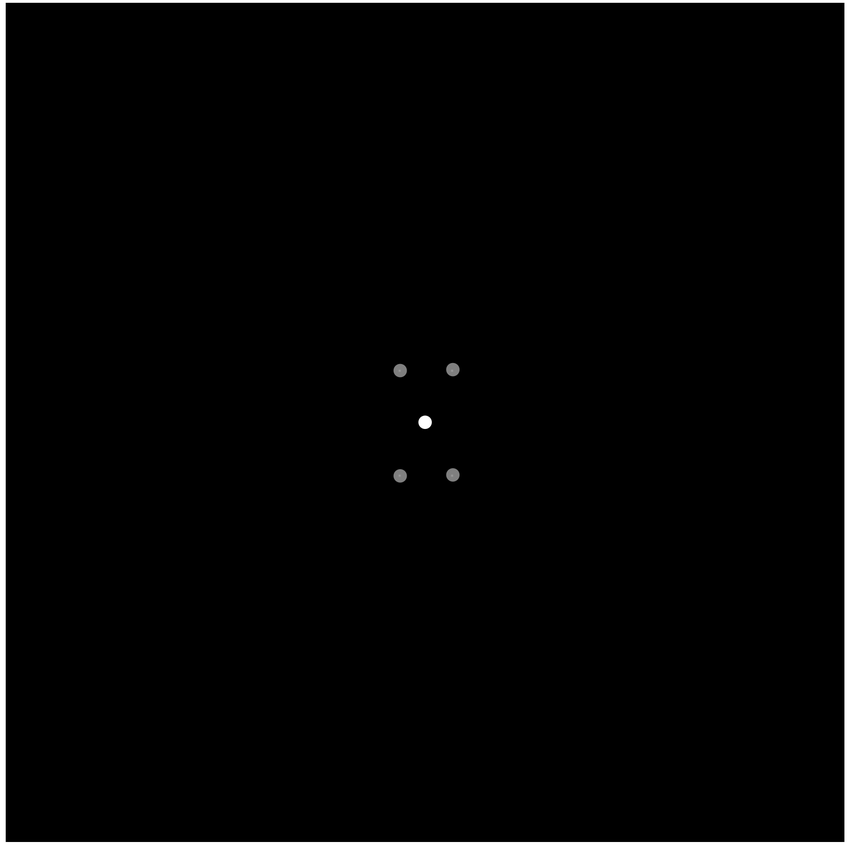


Frequency domain

$$\sin(2\pi/32)x \times \sin(2\pi/16)y$$

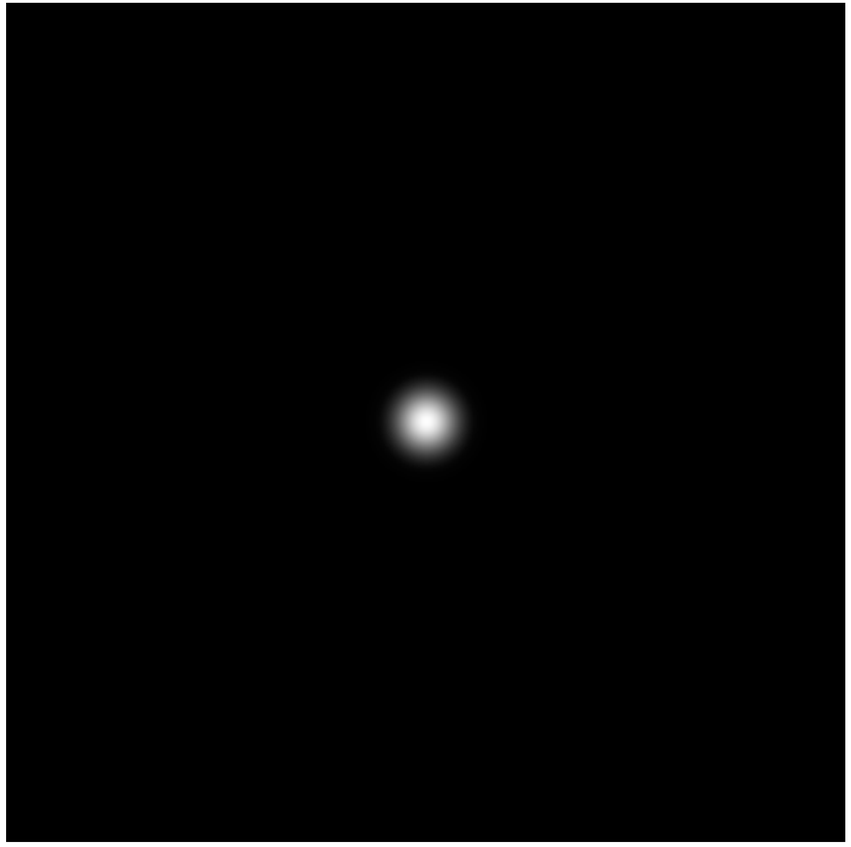


Spatial domain

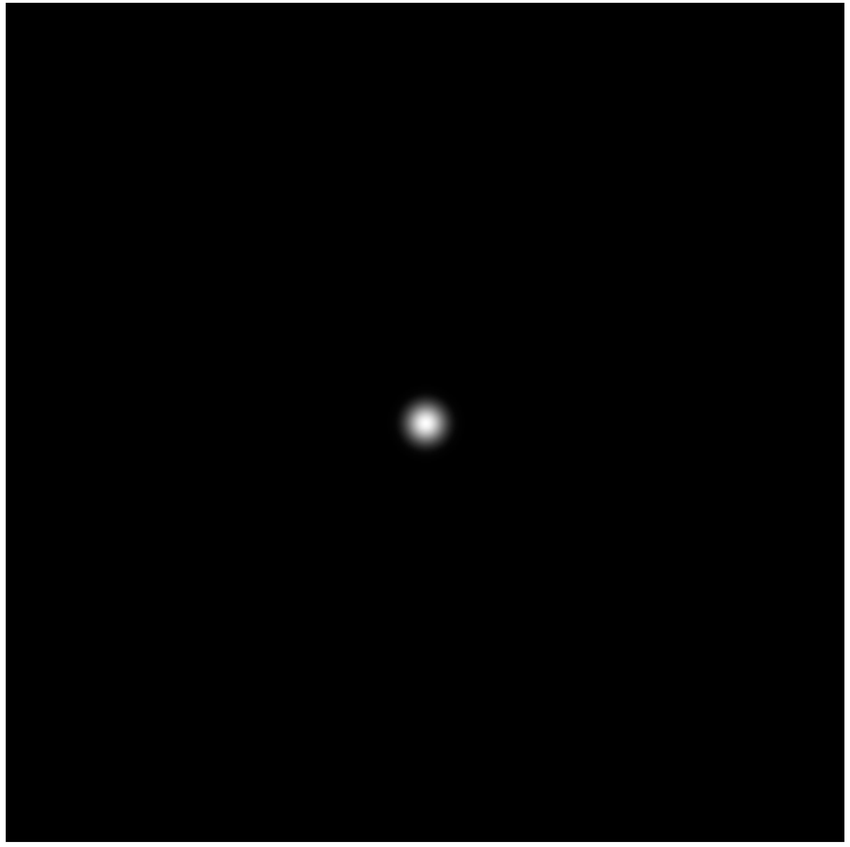


Frequency domain

$$\exp(-r^2/16^2)$$

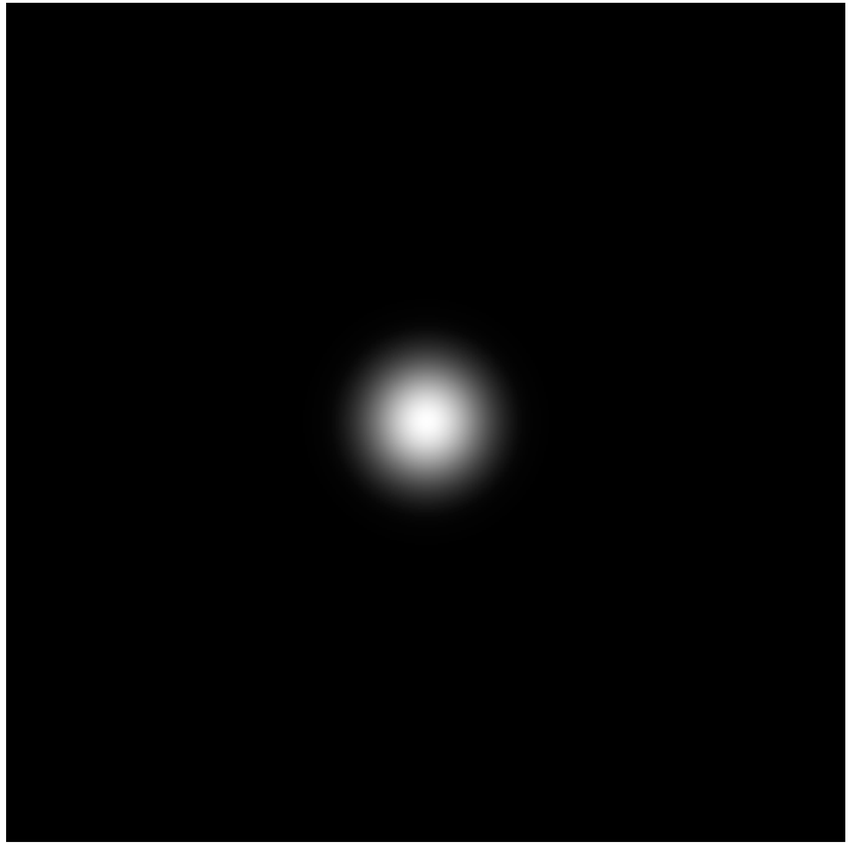


Spatial domain

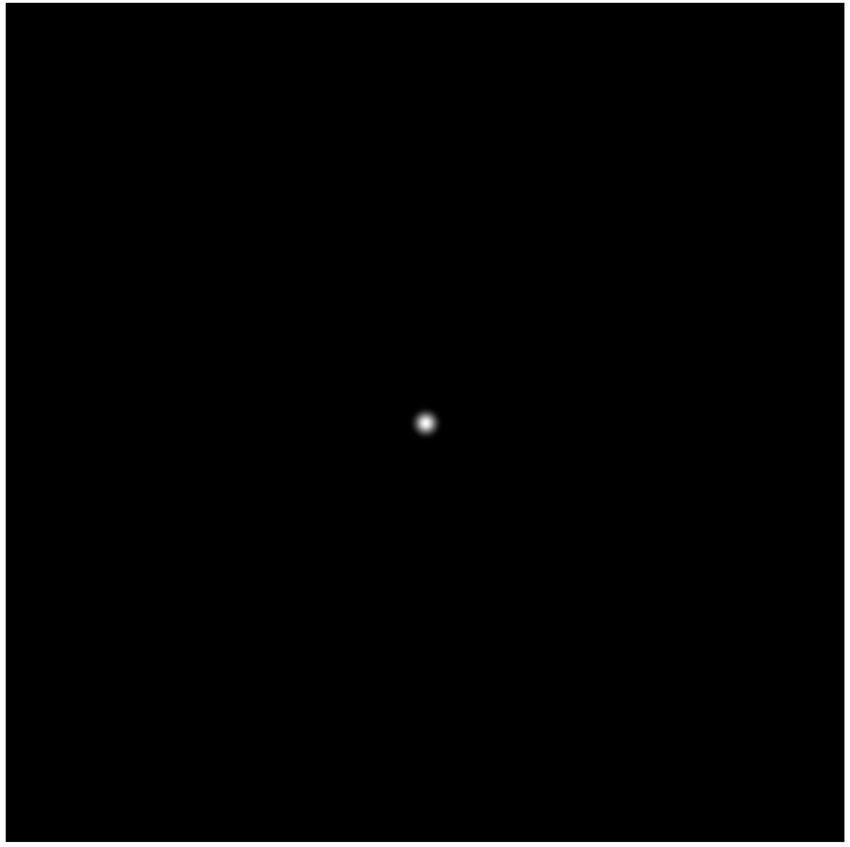


Frequency domain

$$\exp(-r^2/32^2)$$

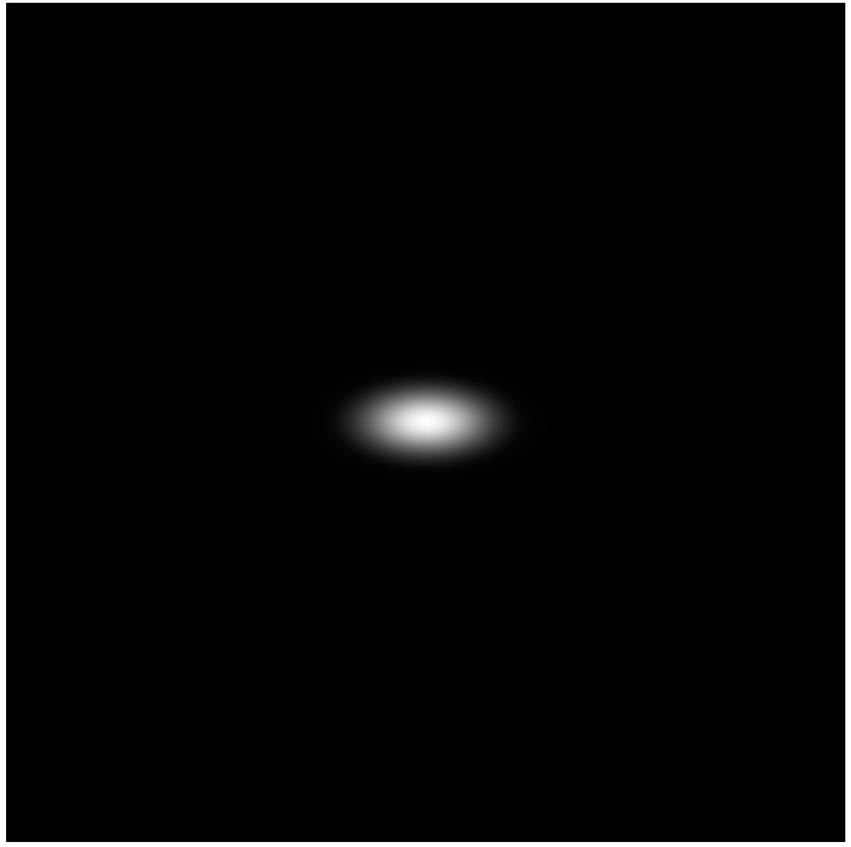


Spatial domain

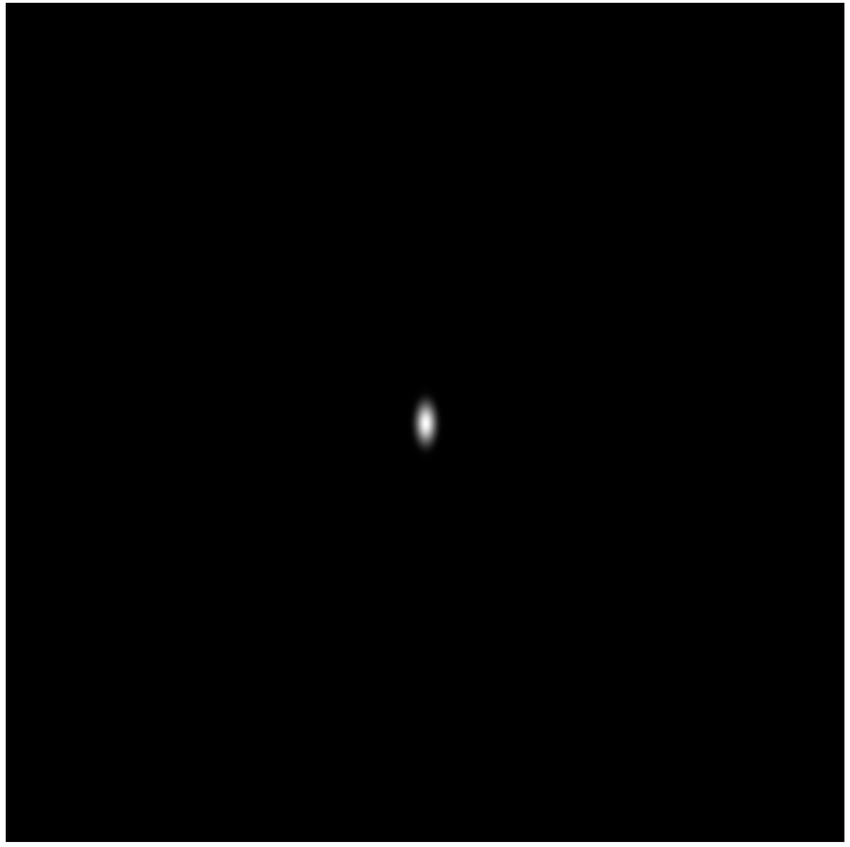


Frequency domain

$$\exp(-x^2/32^2) \times \exp(-y^2/16^2)$$



Spatial domain

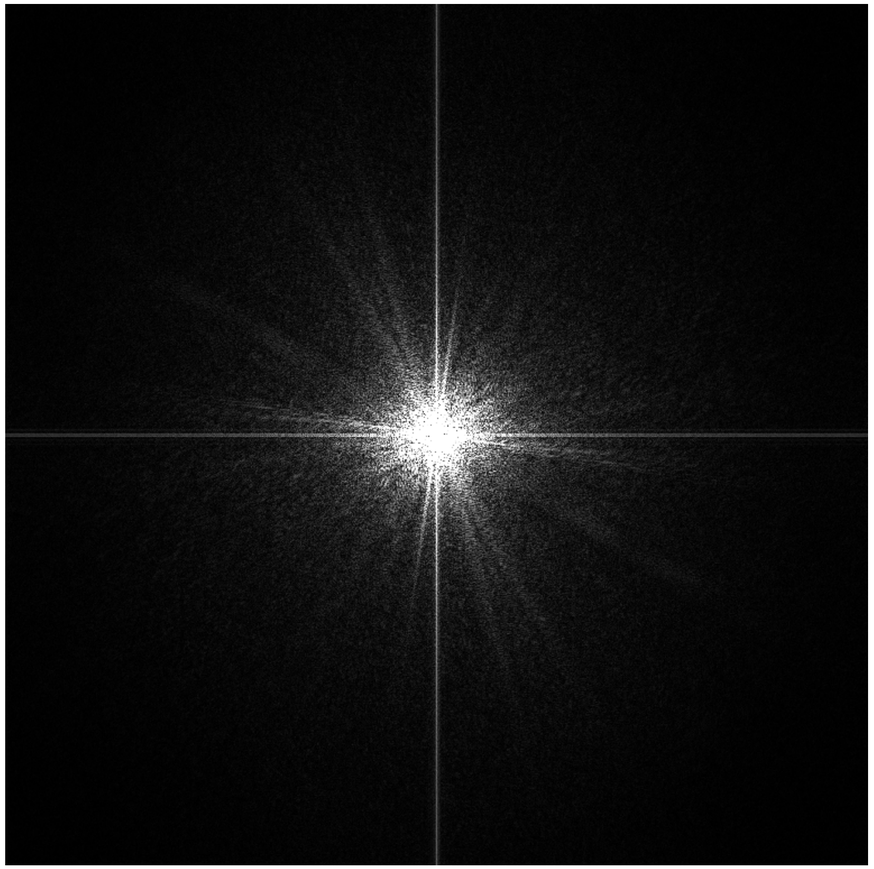


Frequency domain

Image filtering
(in the frequency domain)

Manipulating the frequency content of images

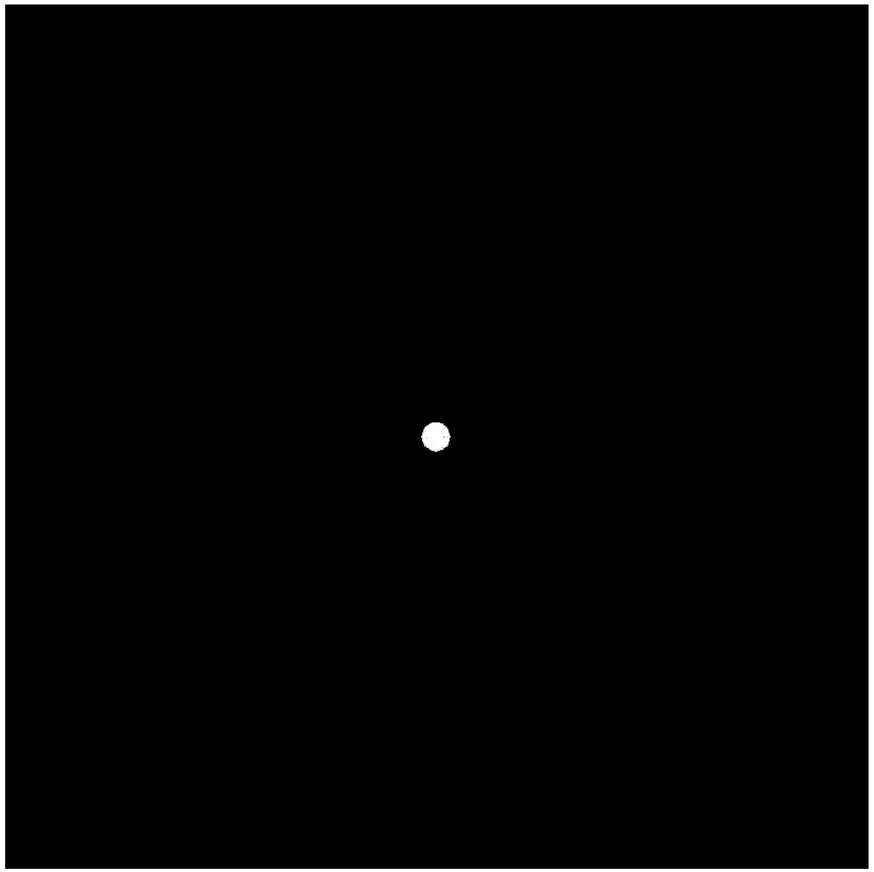
Spatial domain



Frequency domain

Low frequencies only (smooth gradients)

Spatial domain



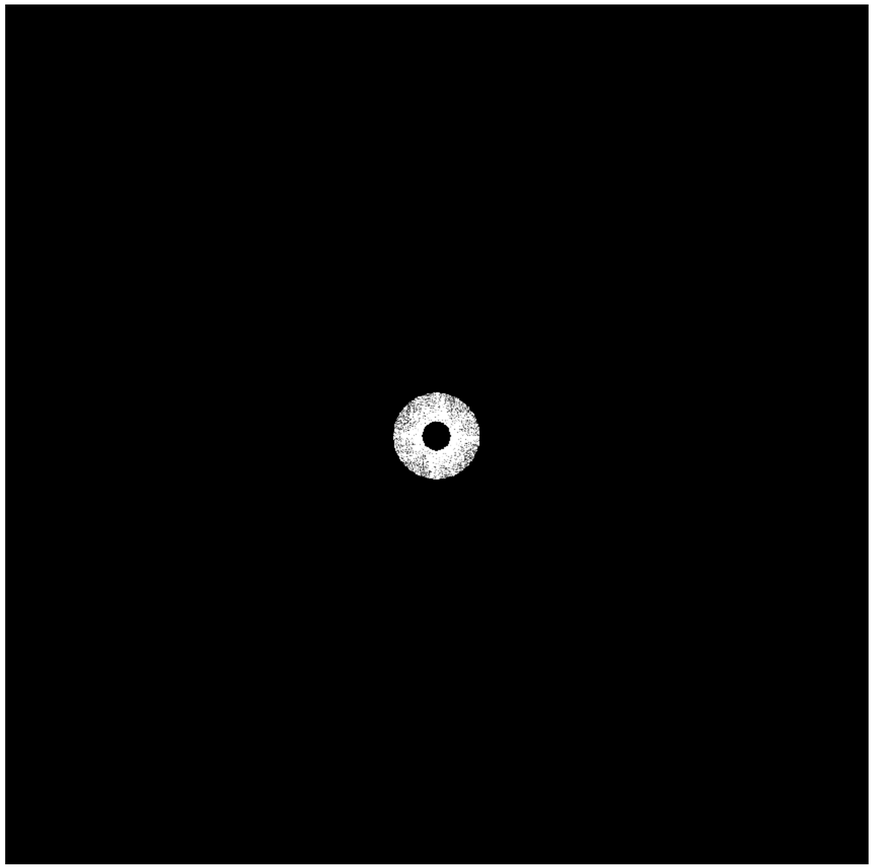
Frequency domain

(after low-pass filter)

All frequencies above cutoff have 0 magnitude

Mid-range frequencies

Spatial domain

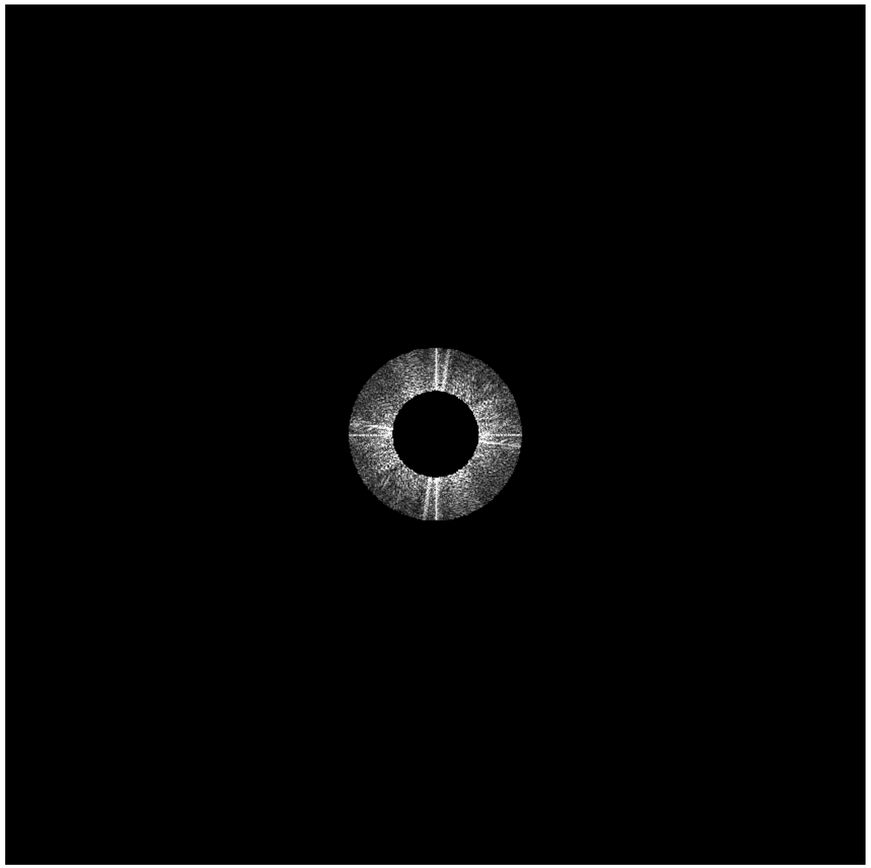


Frequency domain

(after band-pass filter)

Mid-range frequencies

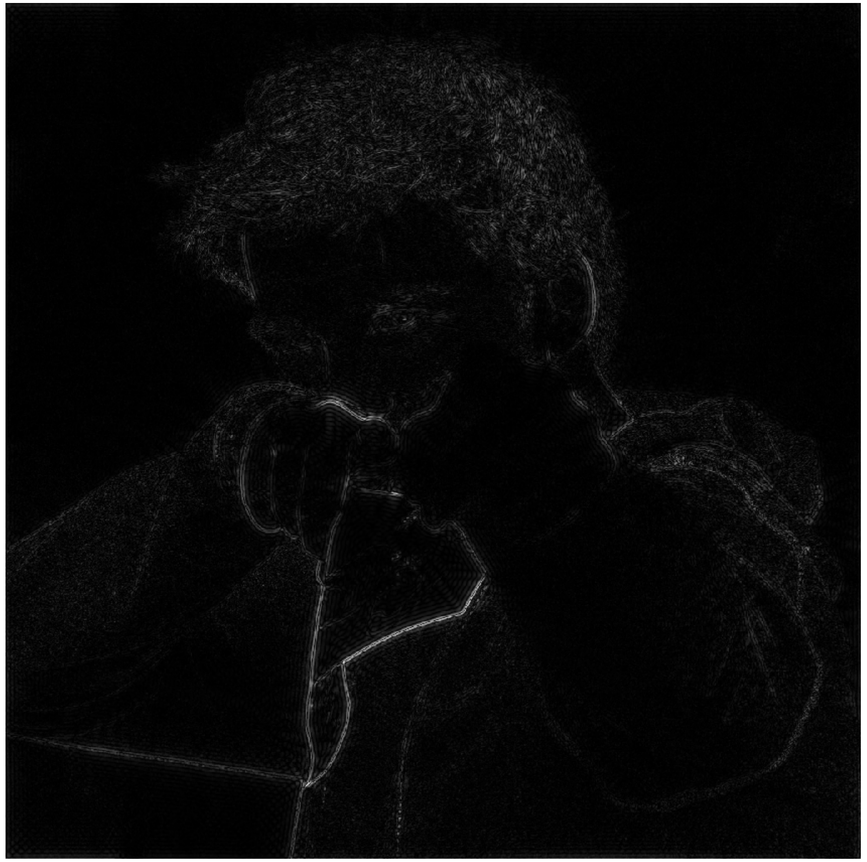
Spatial domain



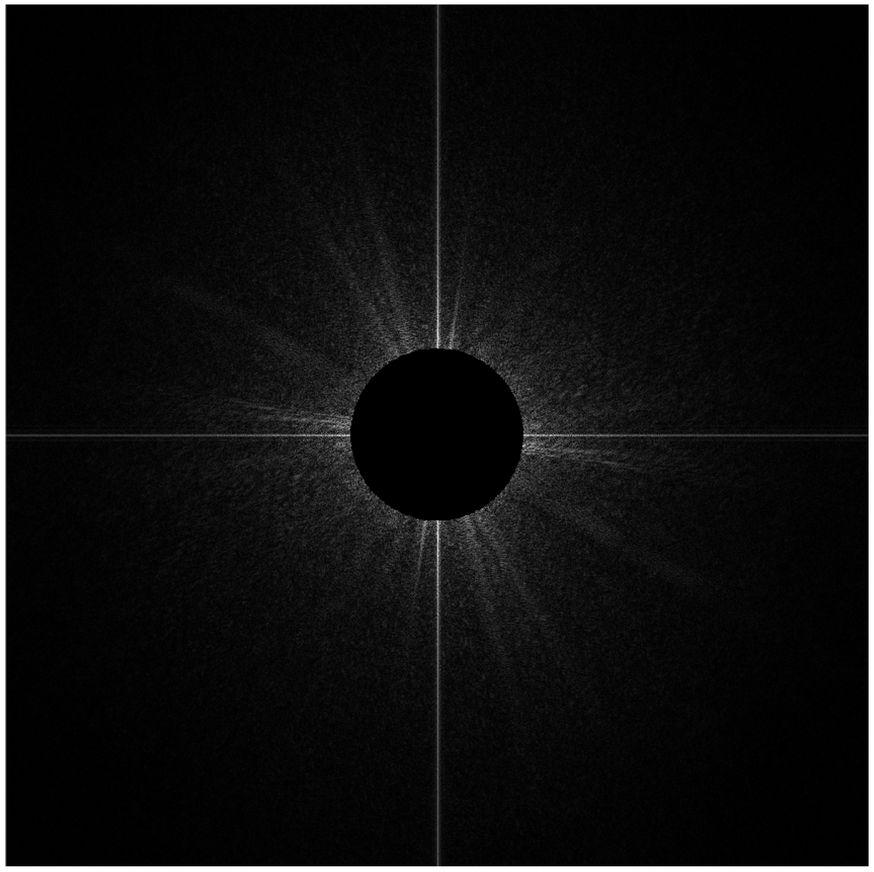
Frequency domain

(after band-pass filter)

High frequencies (edges)

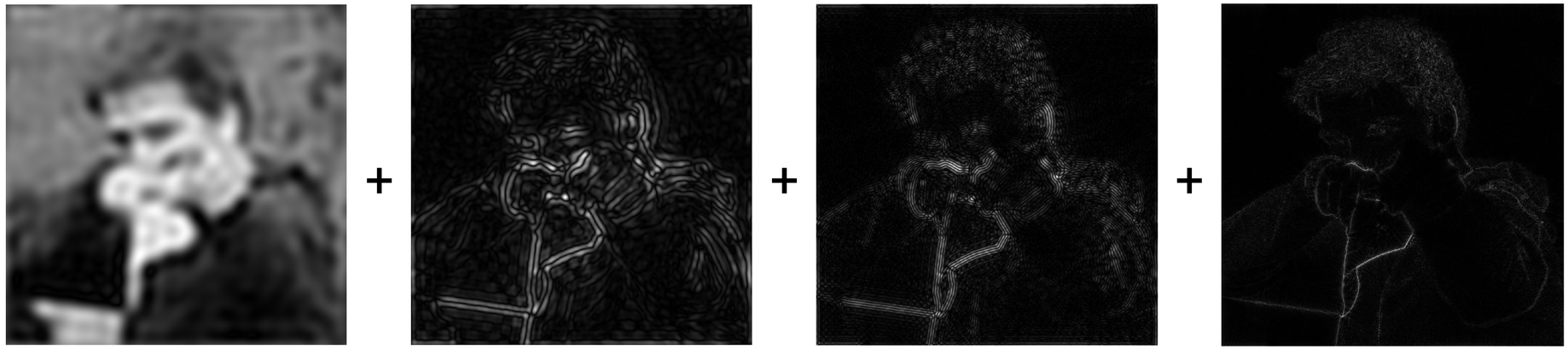


Spatial domain
(strongest edges)

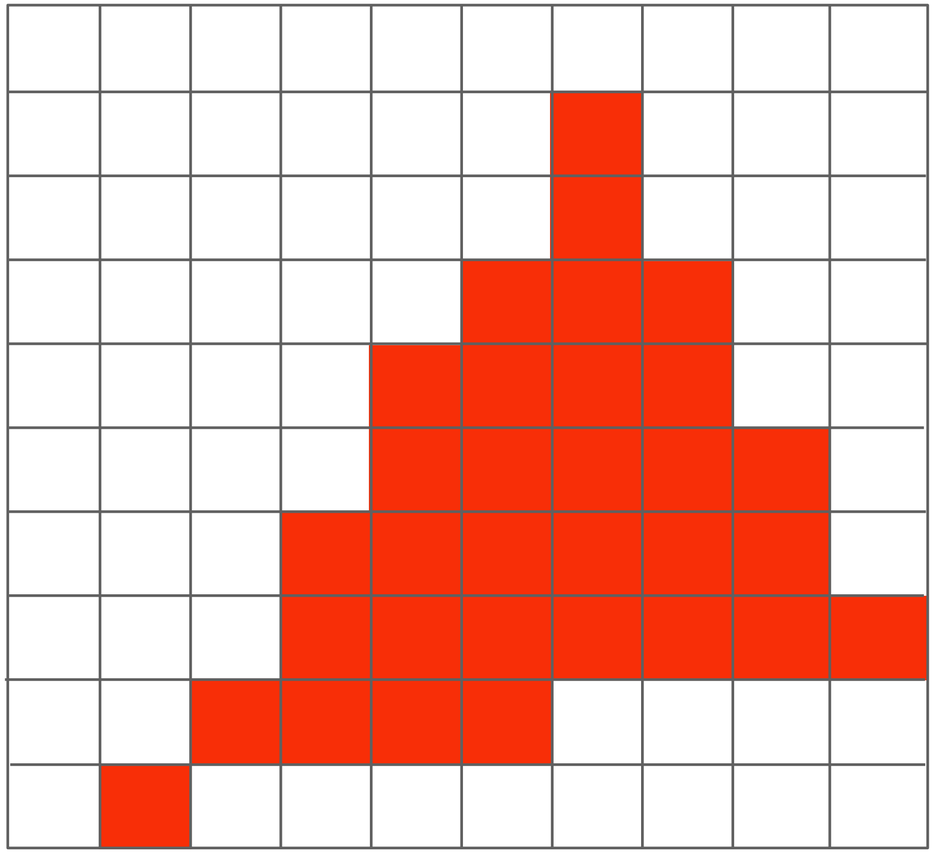


Frequency domain
(after high-pass filter)
All frequencies below threshold have 0 magnitude

An image as a sum of its frequency components

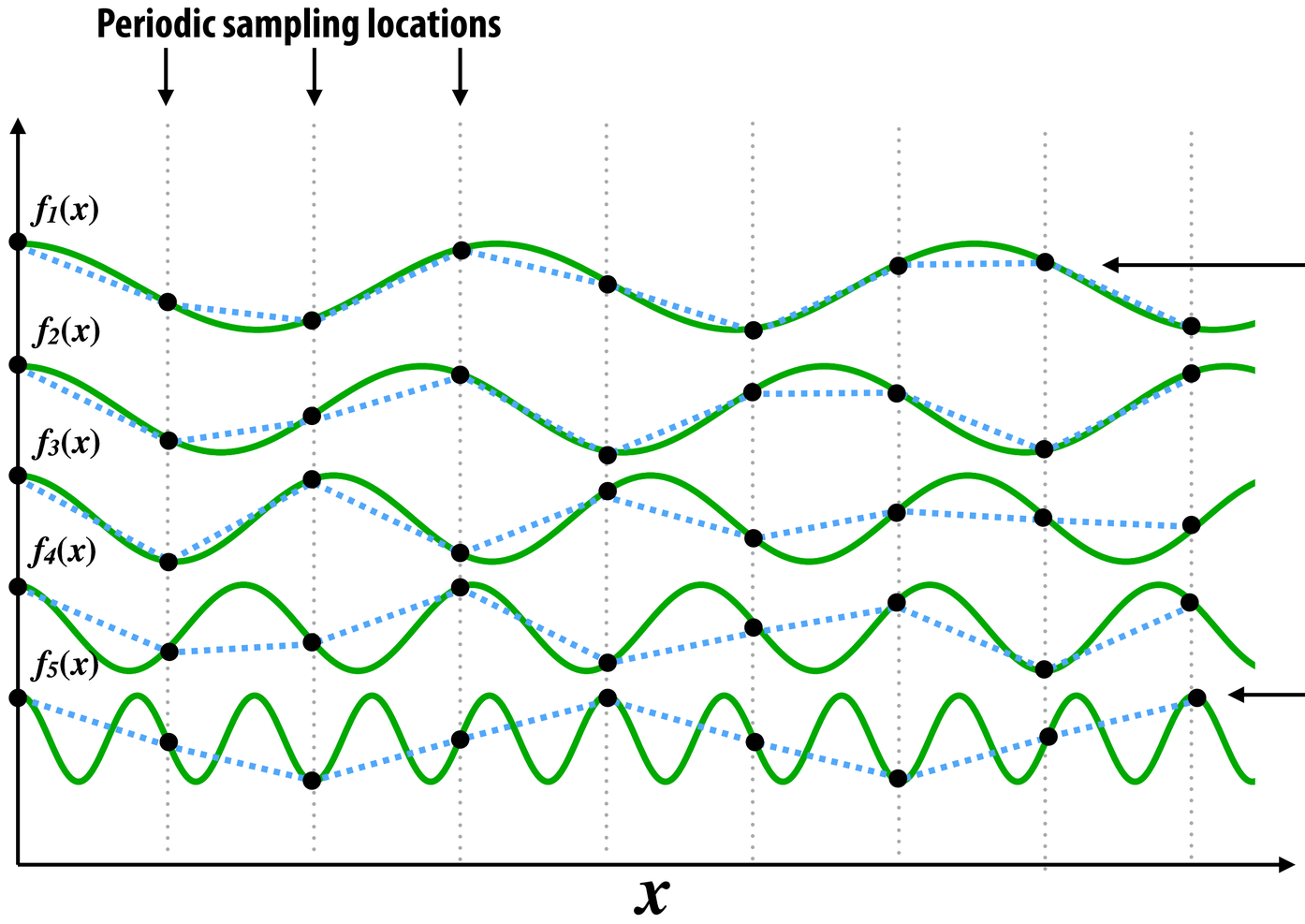


Back to our problem of artifacts in images



Jaggies!

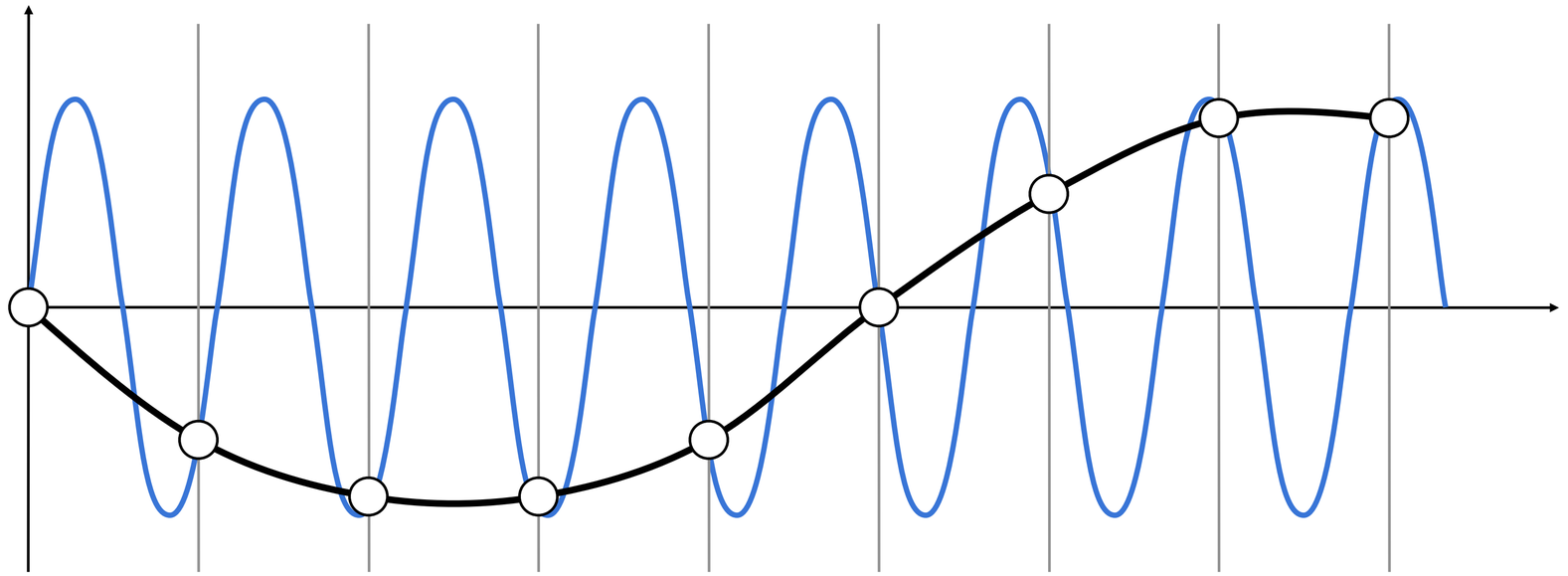
Higher frequencies need denser sampling



Low-frequency signal: sampled adequately for reasonable reconstruction

High-frequency signal is insufficiently sampled: reconstruction incorrectly appears to be from a low frequency signal

Undersampling creates frequency "aliases"

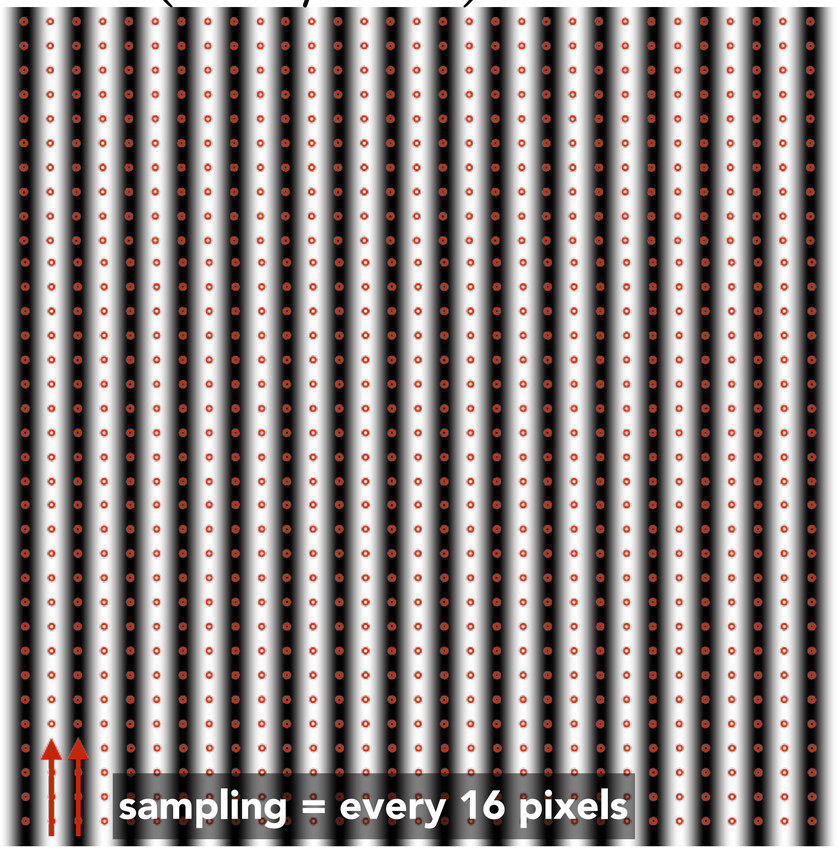


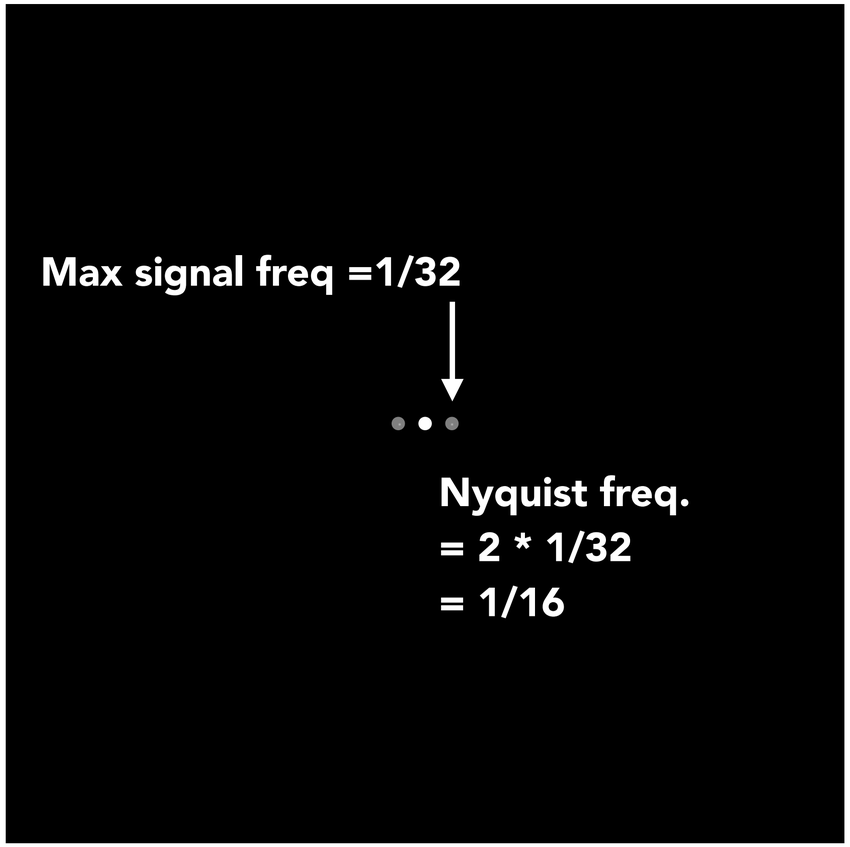
High-frequency signal is insufficiently sampled: samples erroneously appear to be from a low-frequency signal

Two frequencies that are indistinguishable at a given sampling rate are called "aliases"

Example: sampling rate vs signal frequency

$\sin(2\pi/32)x$ — frequency 1/32; 32 pixels per cycle





Spatial domain

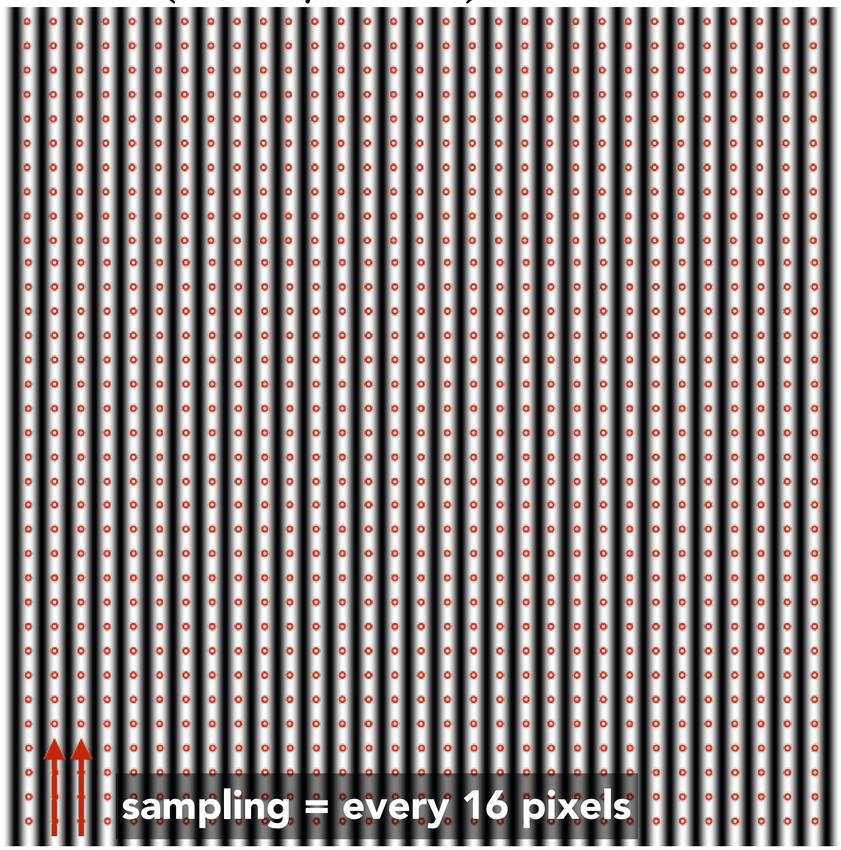
Frequency domain

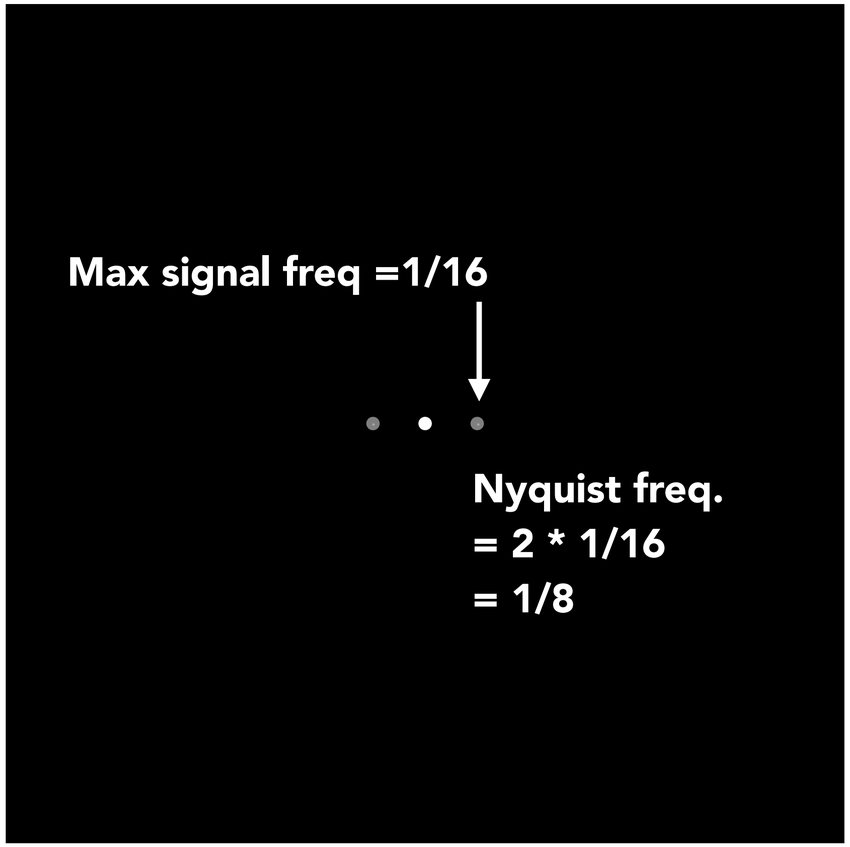
Sampling at twice the frequency of the signal: no aliasing! *

* technically no pre-aliasing. There is post-aliasing if reconstruction is not perfect

Example: sampling rate vs signal frequency

$\sin(2\pi/16)x$ — frequency 1/16; 16 pixels per cycle



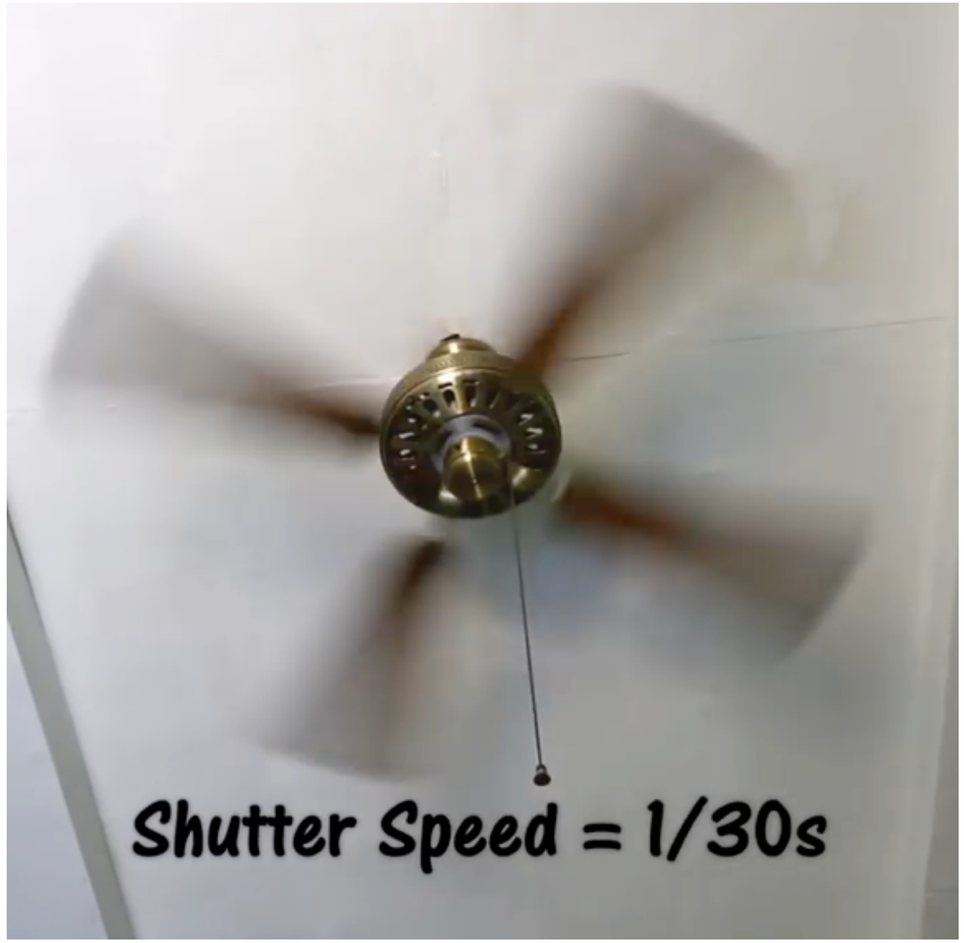


Sampling at same frequency as signal: dramatic aliasing! (due to undersampling)

Anti-aliasing idea: remove high frequency information from a signal before sampling it

Video: point vs antialiased sampling

Single point in time



Motion blurred

Video: point sampling in time

Credit: Aris & cams youtube, https://youtu.be/NoWwxTktoFs

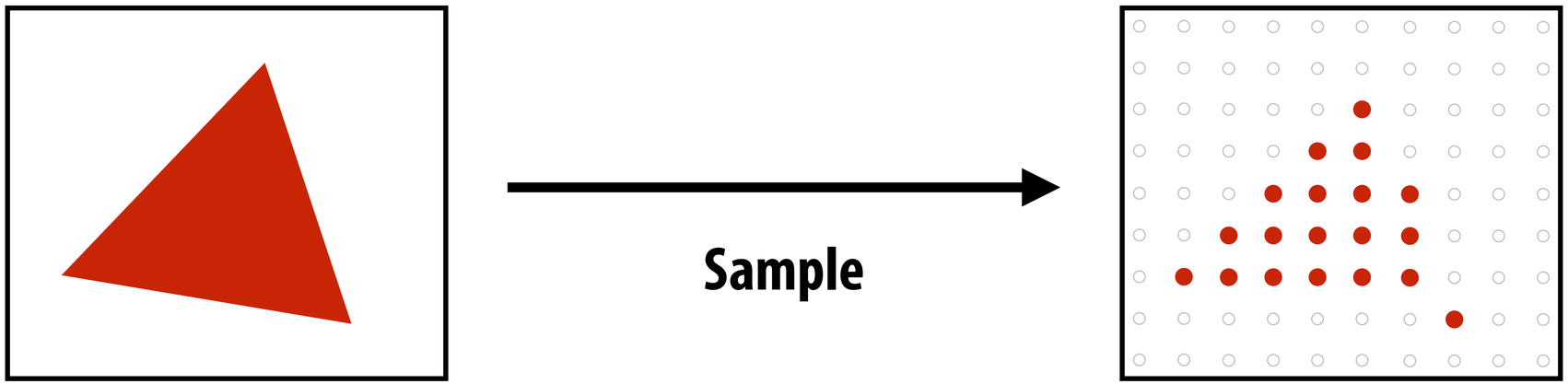
30 fps video. 1/800 second exposure is sharp in time, causes time aliasing.

Video: motion-blurred sampling

Credit: Aris & cams youtube, https://youtu.be/NoWwxTktoFs

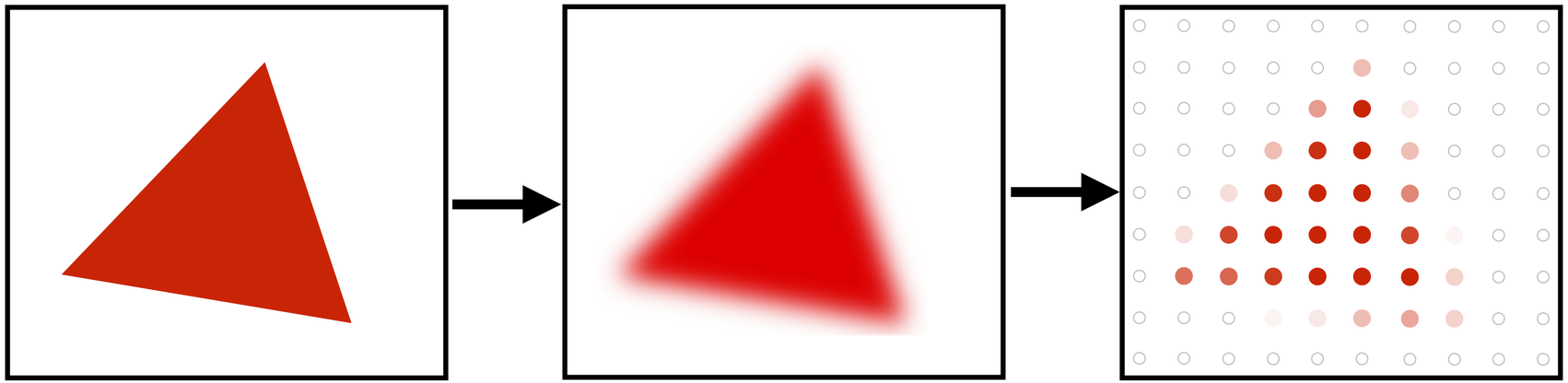
30 fps video. 1/30 second exposure is motion-blurred in time, reduces aliasing.

Rasterization is sampling in 2D space



Note jaggies in rasterized triangle
(pixel values are either red or white: sample is in or out of triangle)

Anti-aliasing by pre-filtering the signal

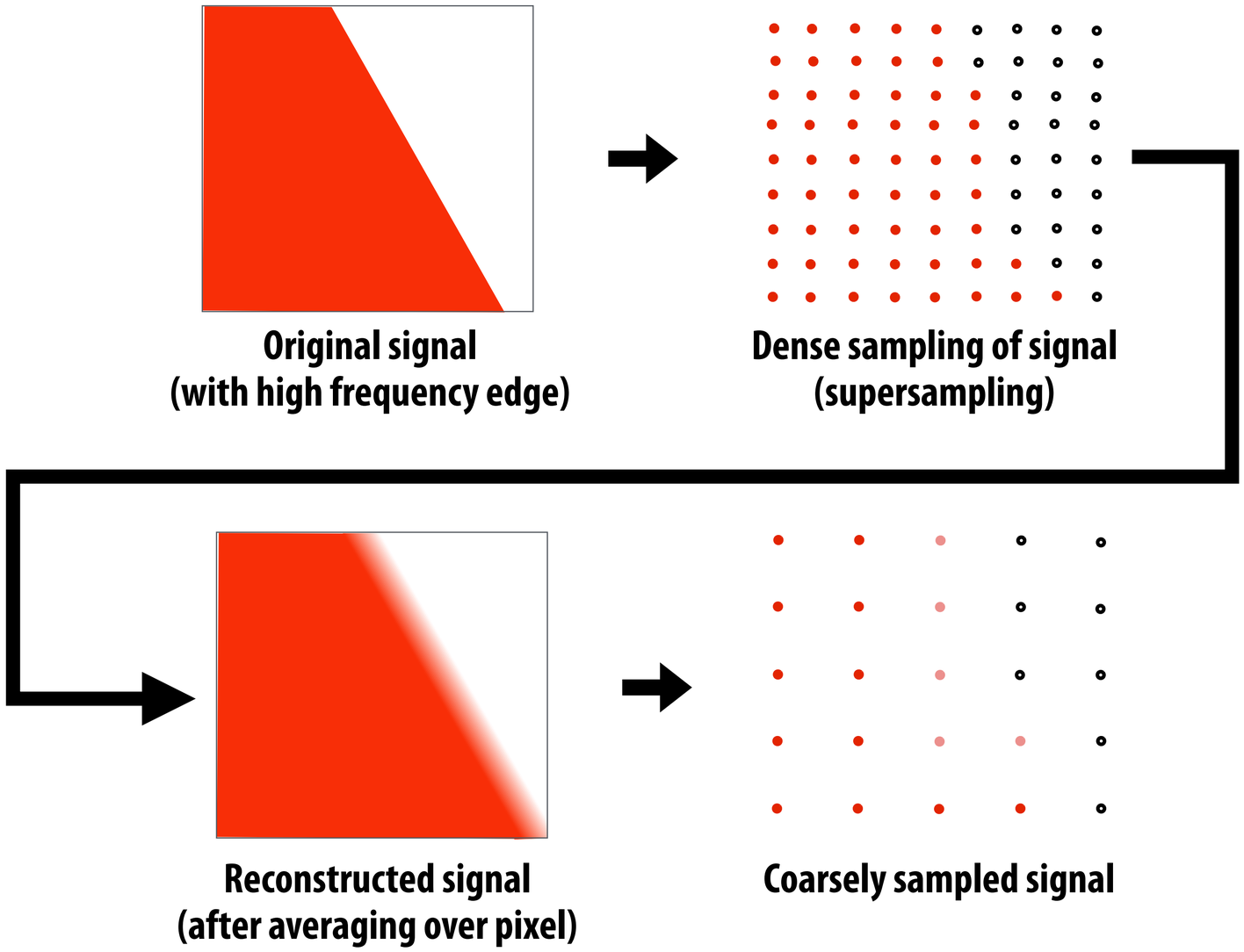


Pre-filter
(remove high frequency detail)

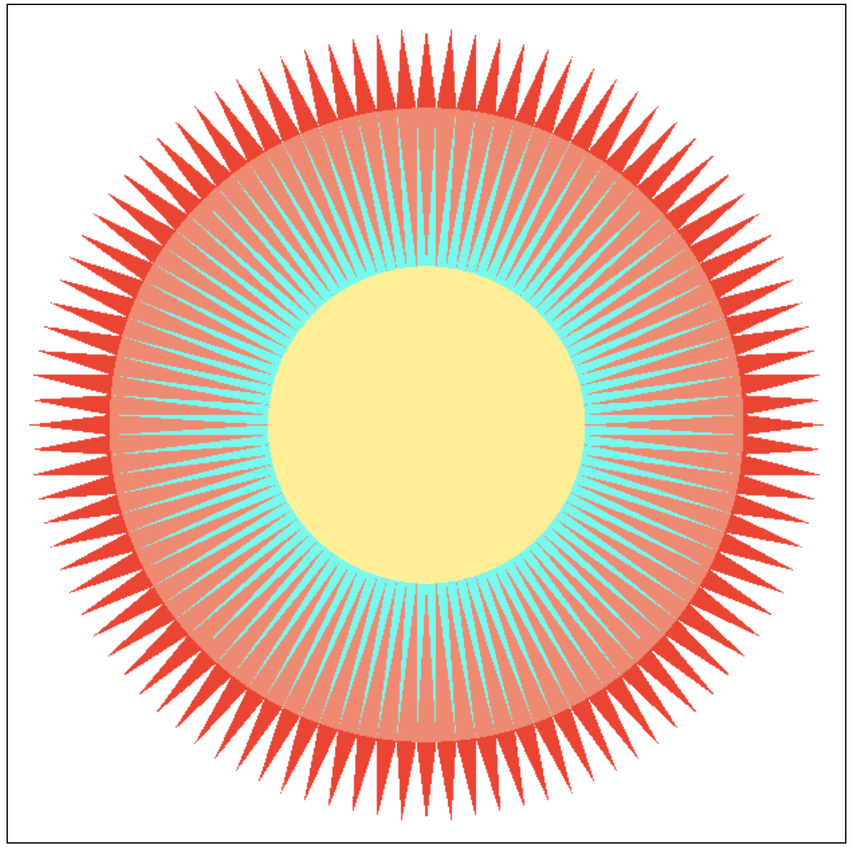
Sample

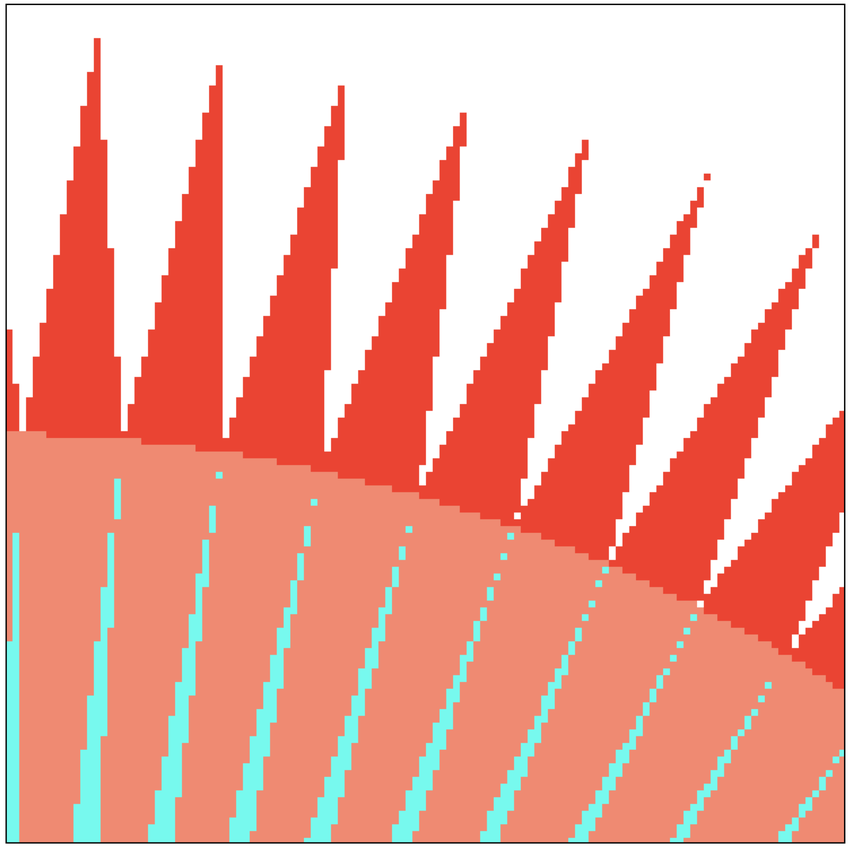
Note anti-aliased edges of rasterized triangle:
pixel values take intermediate values

Pre-filtering by "supersampling" then "blurring" (averaging)

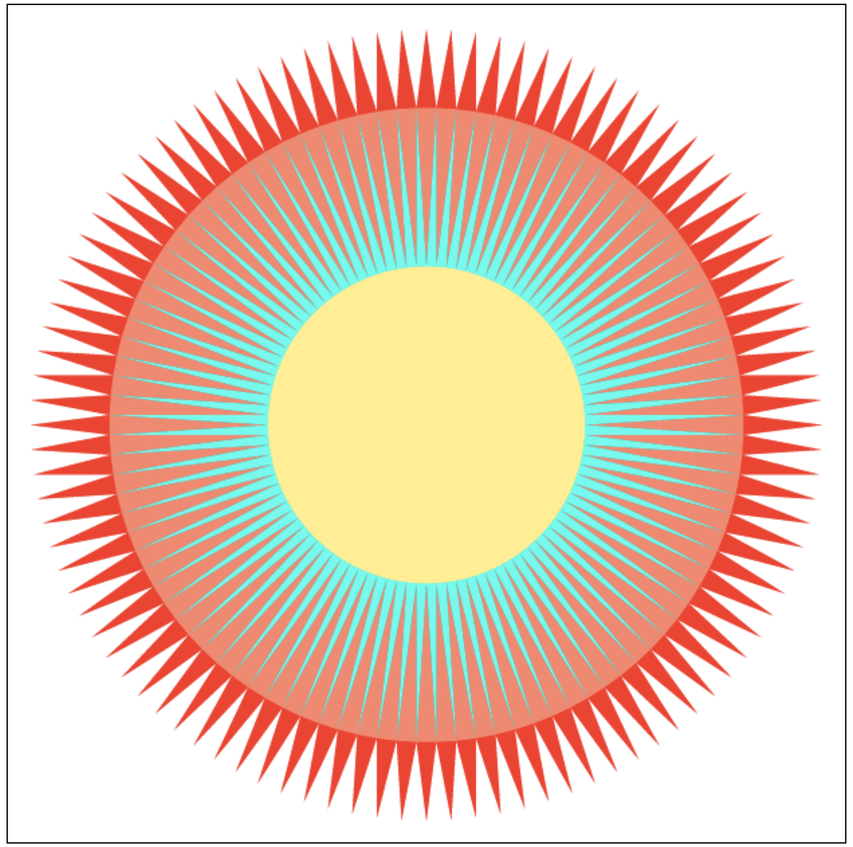


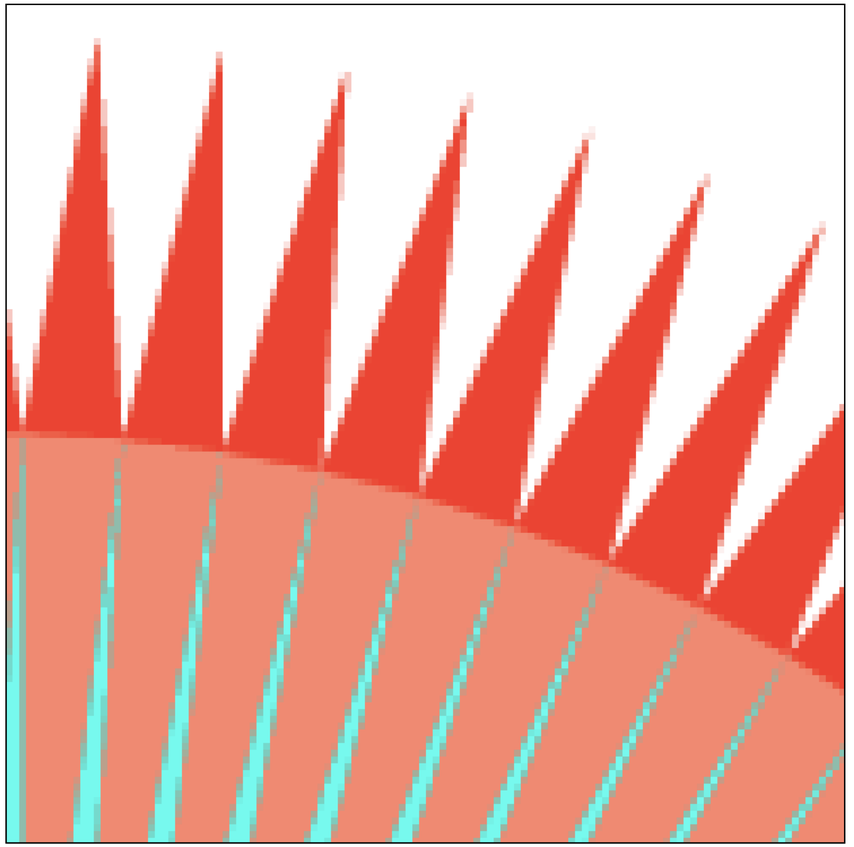
Images rendered using one sample per pixel



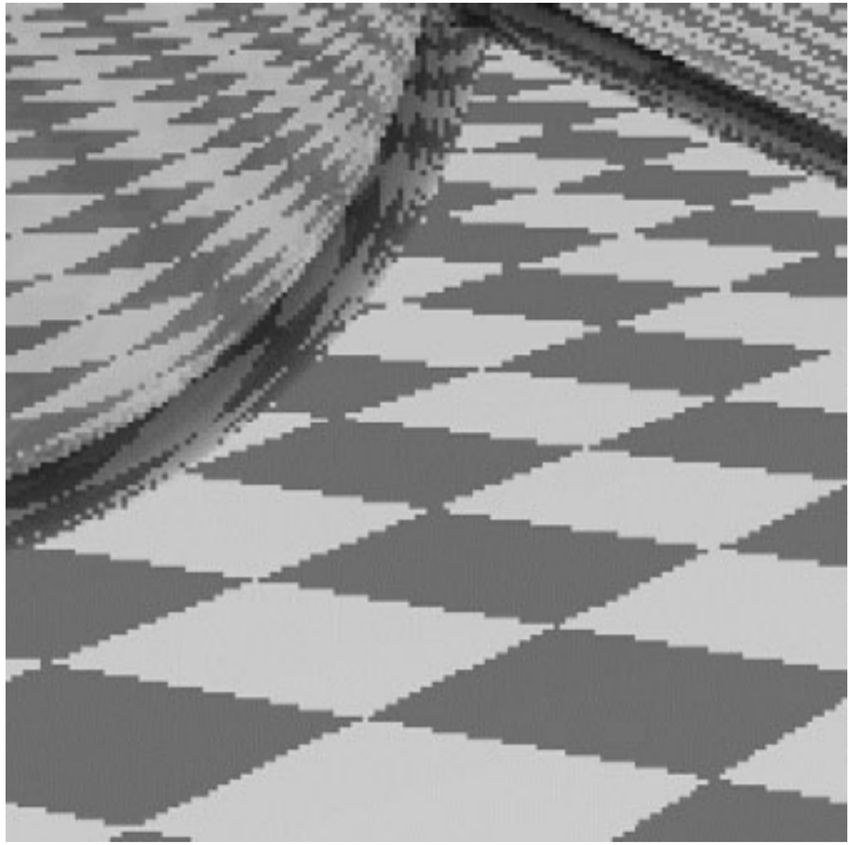


Anti-aliased results

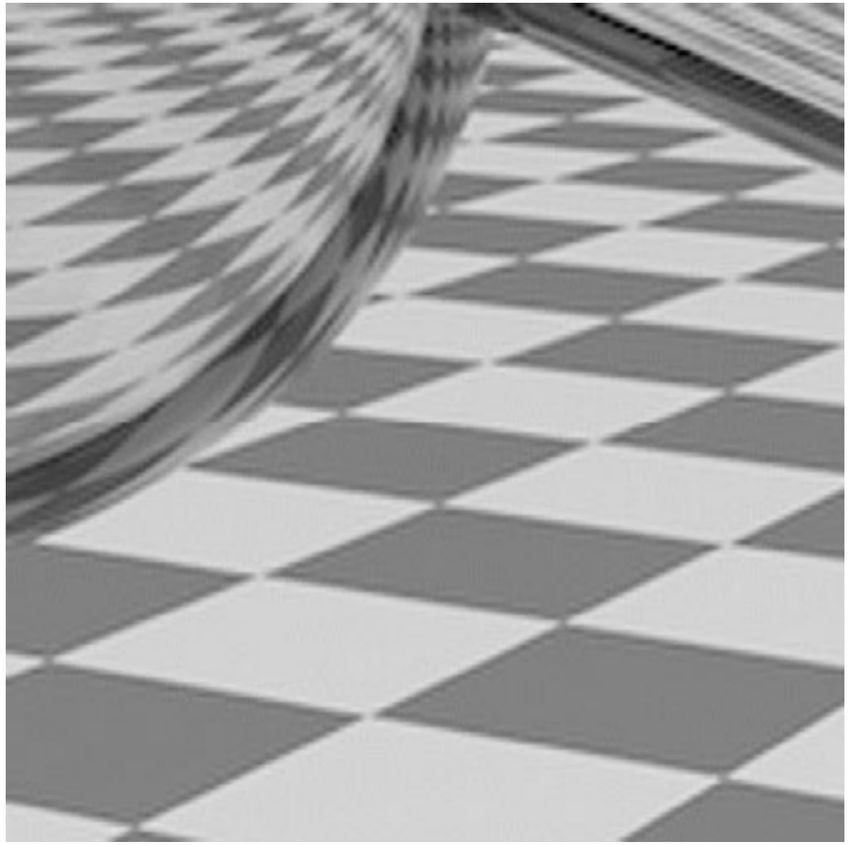




Benefits of anti-aliasing



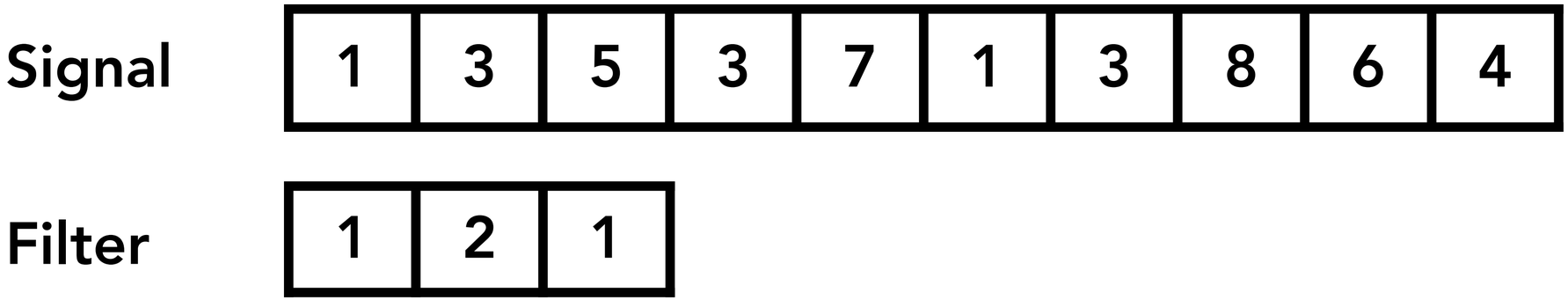
Jaggies



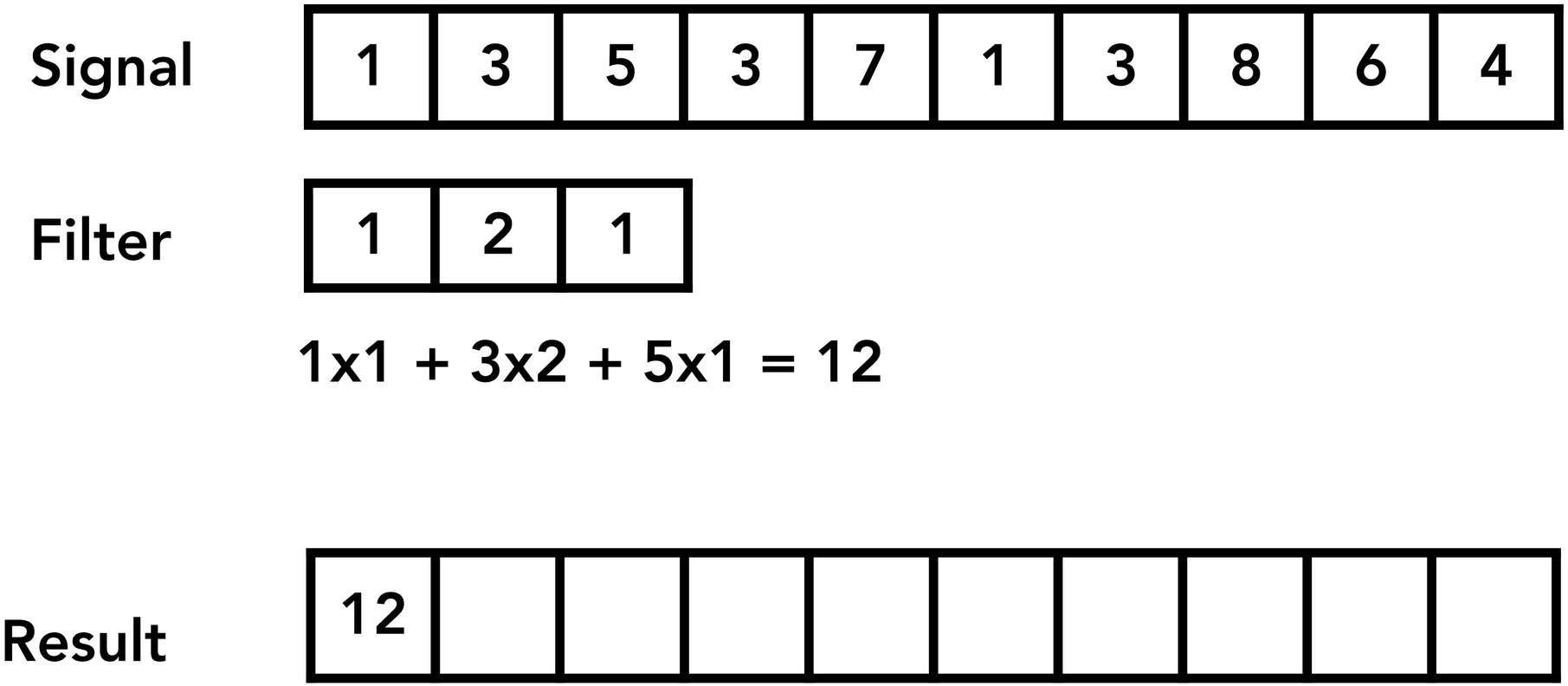
Pre-filtered

Filtering = convolution

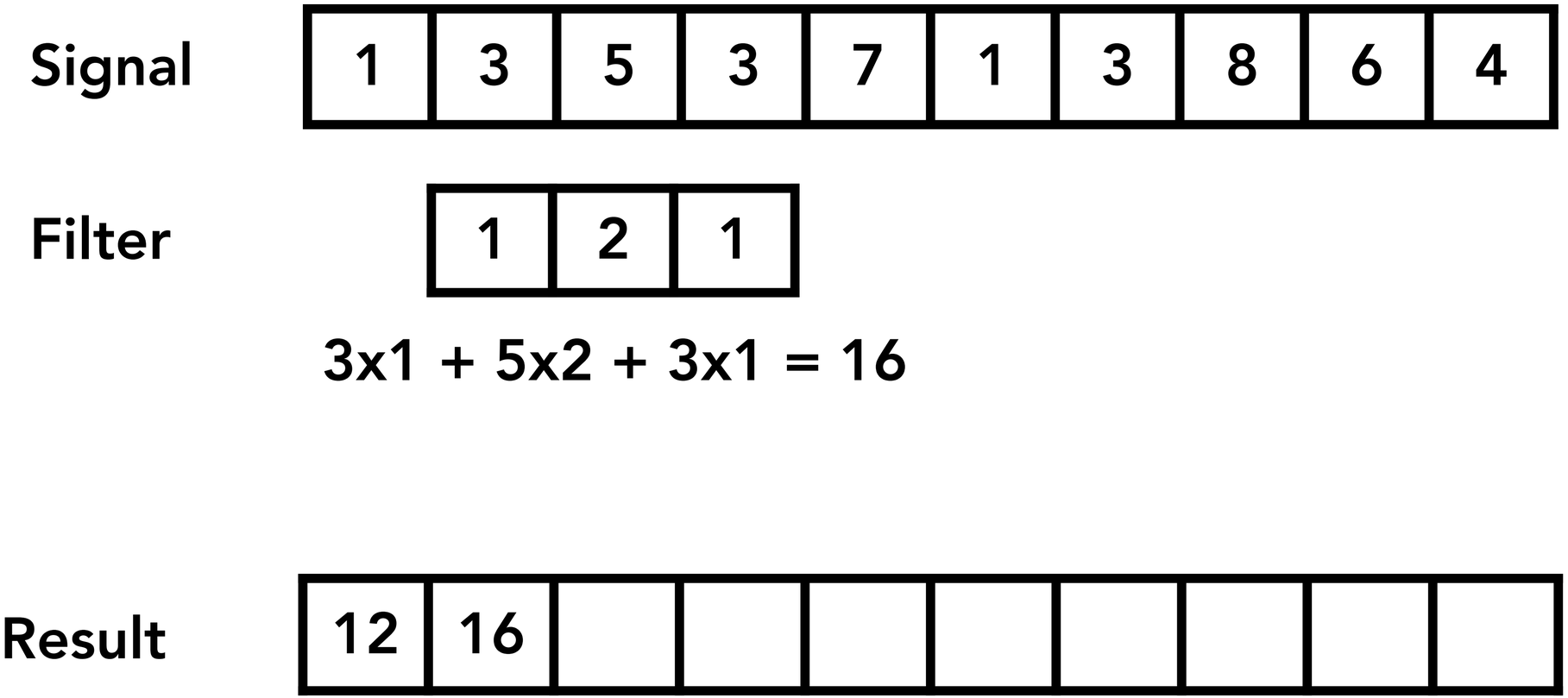
1D convolution



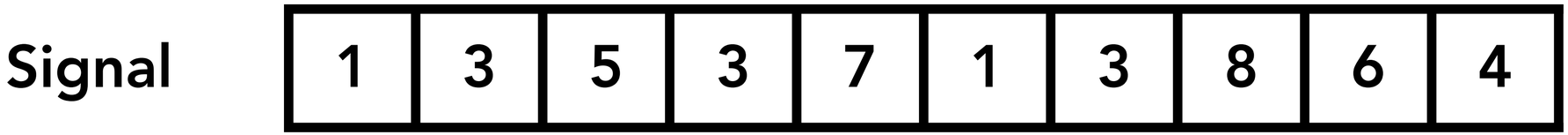
1D convolution



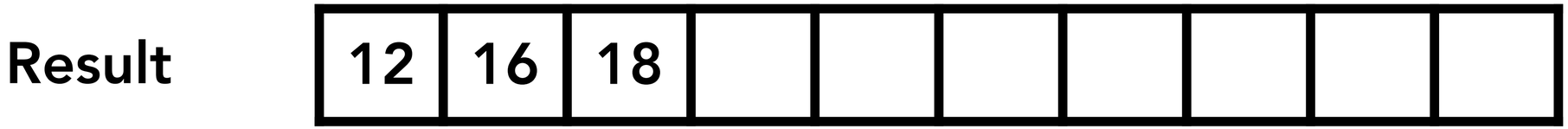
1D convolution



1D convolution



5x1 + 3x2 + 7x1 = 18



Box filter (used in a 2D convolution)

$$\frac{1}{9}$$

1	1	1
1	1	1
1	1	1

Example: 3x3 box filter

2D convolution with box filter blurs the image

Original image

Blurred
(convolve with box filter)

Hmm… this reminds me of a low-pass filter…

Discrete 2D convolution

$$(f * g)(x, y) = \sum_{i,j=-\infty}^{\infty} f(i, j) I(x - i, y - j)$$

output image

filter

input image

Consider $f(i, j)$ **that is nonzero only when:** $-1 \leq i, j \leq 1$

Then:

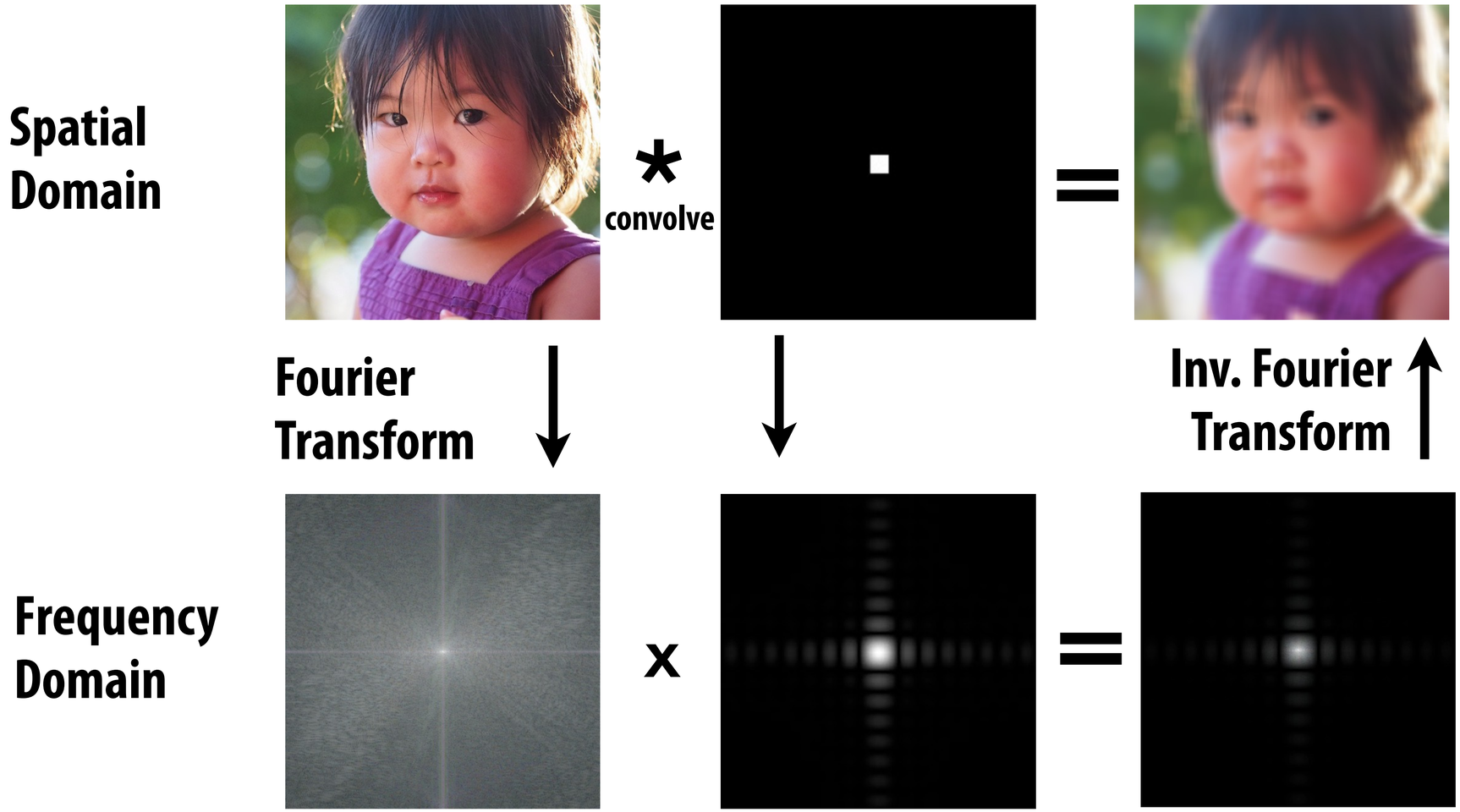
$$(f * g)(x, y) = \sum_{i,j=-1}^{1} f(i, j) I(x - i, y - j)$$

And we can represent f(i,j) as a 3x3 matrix of values where:

$$f(i, j) = \mathbf{F}_{i,j}$$ **(often called: "filter weights", "filter kernel")**

Convolution theorem

Convolution in the spatial domain is equal to multiplication in the frequency domain, and vice versa



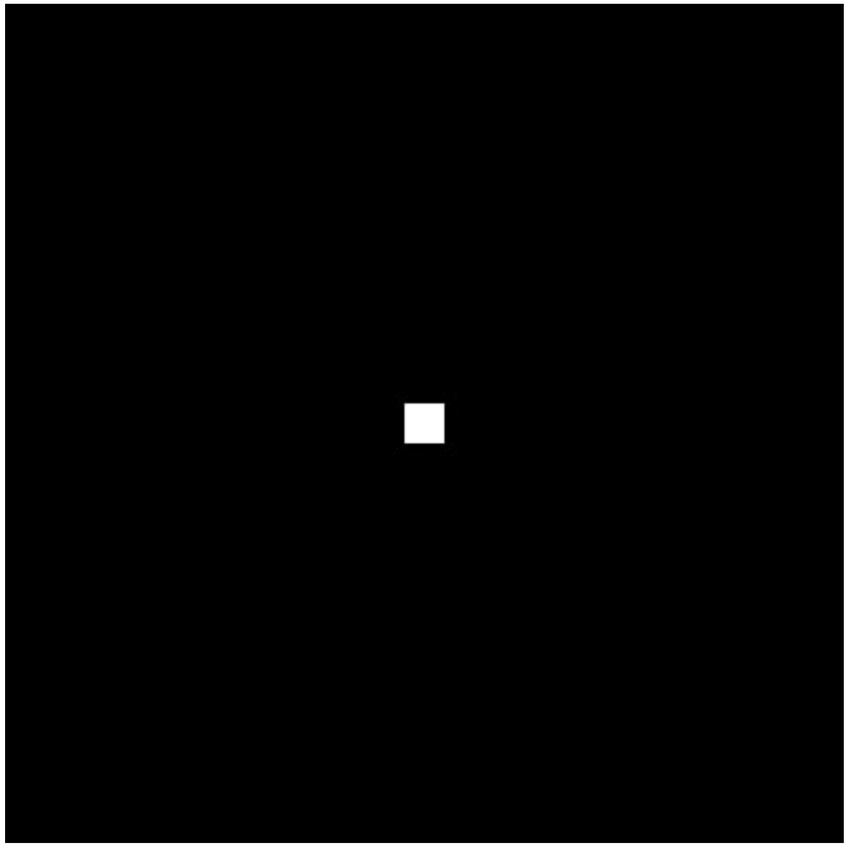
Convolution theorem

- Convolution in the spatial domain is equal to multiplication in the frequency domain, and vice versa

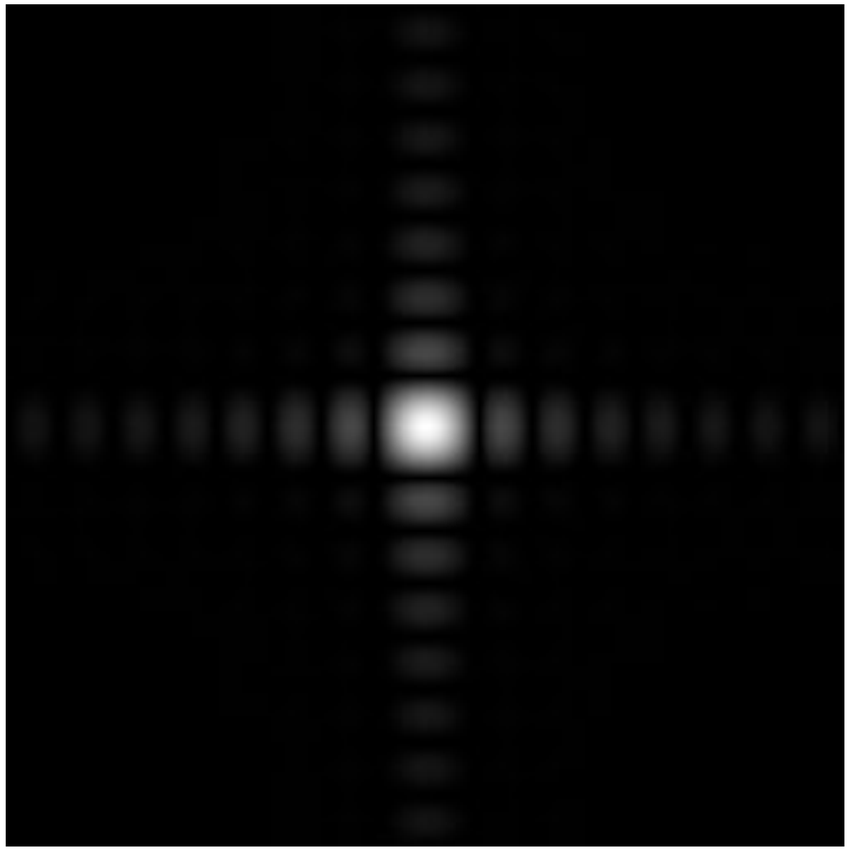
- Pre-filtering option 1:
  - Filter by convolution in the spatial domain

- Pre-filtering option 2:
  - Transform to frequency domain (Fourier transform)
  - Multiply by Fourier transform of convolution kernel
  - Transform back to spatial domain (inverse Fourier)

Box function = "low pass" filter

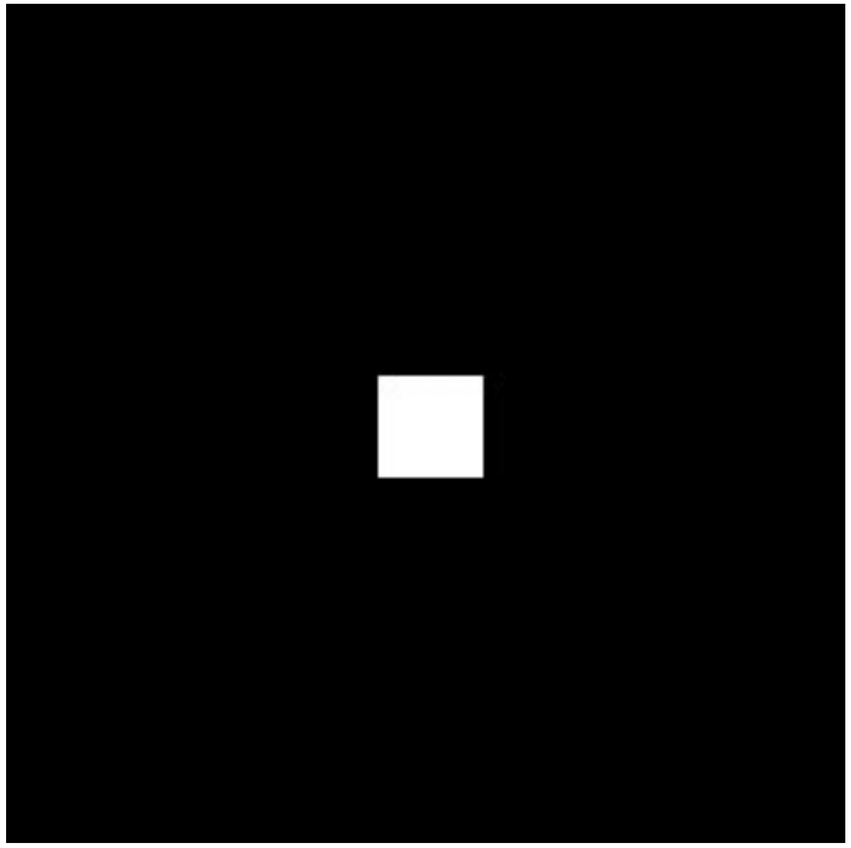


Spatial domain

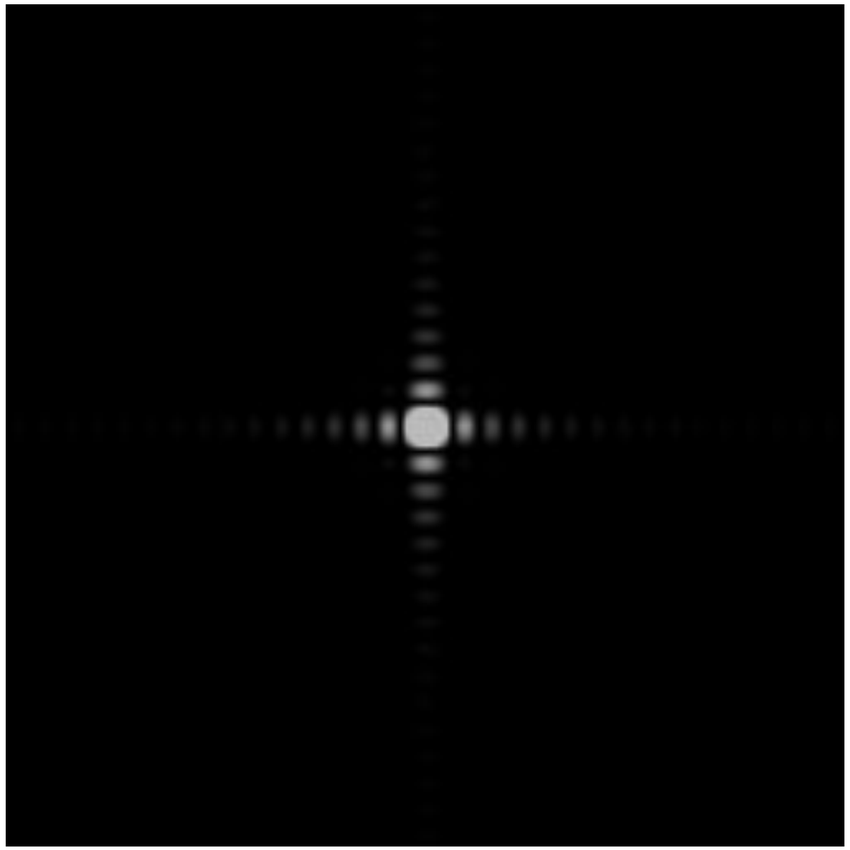


Frequency domain

Wider filter kernel = retain only lower frequencies



Spatial domain



Frequency domain

Wider filter kernel = lower frequencies

- As a filter is localized in the spatial domain, it spreads out in frequency domain

- Conversely, as a filter is localized in frequency domain, it spreads out in the spatial domain

How can we reduce aliasing error?

- Increase sampling rate
  - Higher resolution displays, sensors, framebuffers…
  - But: costly and may need very high resolution to sufficiently reduce aliasing

- Anti-aliasing
  - Simple idea: remove (or reduce) high frequencies before sampling
  - How to filter out high frequencies before sampling?

Anti-aliasing by averaging values in pixel area

- Convince yourself the following are the same:

- Option 1:
  - Convolve f(x,y) by a 1-pixel box-blur
  - Then sample at every pixel

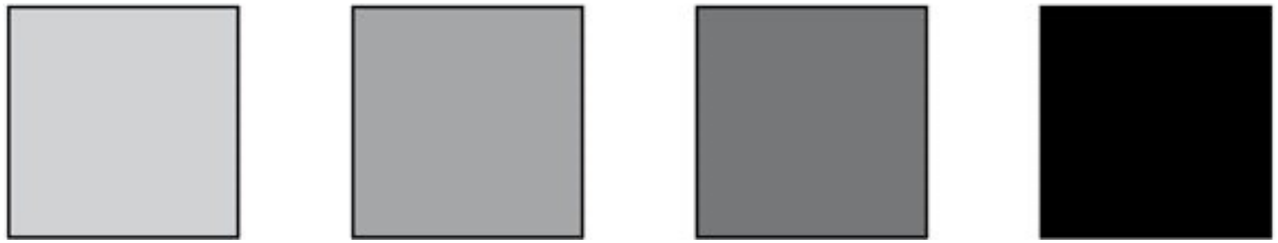
- Option 2:
  - Compute the average value of f(x,y) in the pixel

Anti-aliasing by computing average pixel value

In rasterizing one triangle, the value of f(x,y) = inside(tri,x,y) averaged over the area of a pixel is equal to the amount of the pixel covered by the triangle.

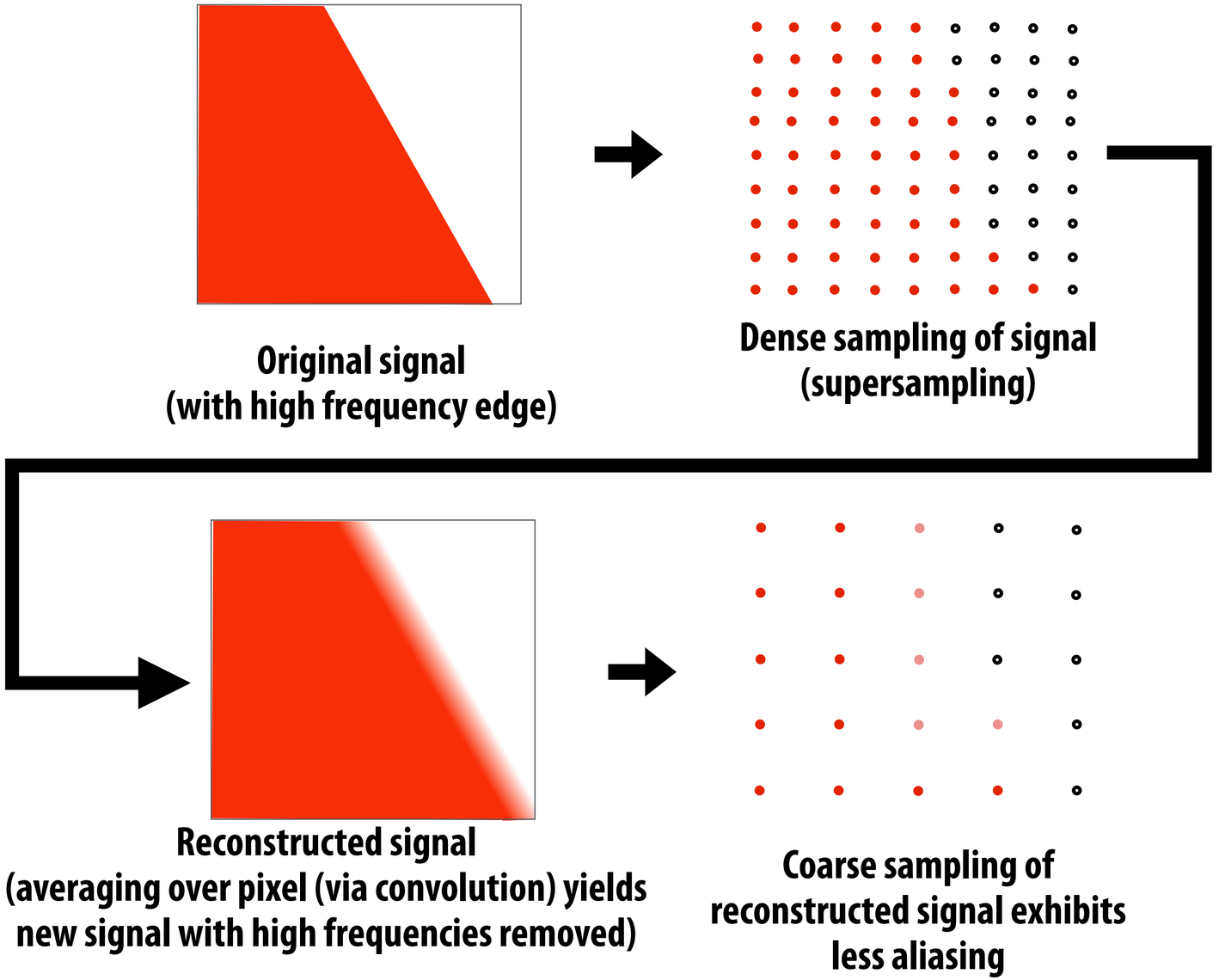
Original

Filtered



1 pixel width

Putting it all together: anti-aliasing via supersampling



Today's summary

- Drawing a triangle = sampling triangle/screen coverage
- Pitfall of sampling: aliasing
- Reduce aliasing by prefiltering signal
  - Supersample
  - Reconstruct via convolution (average coverage over pixel)
    - Higher frequencies removed
  - Sample reconstructed signal once per pixel

- There is much, much more to sampling theory and practice…

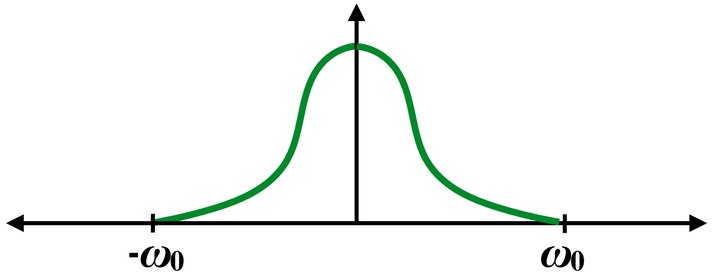
Acknowledgements

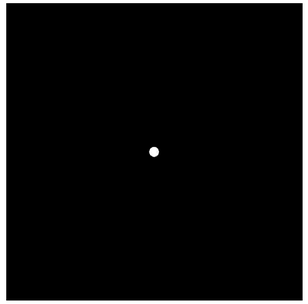
- Thanks to Ren Ng, Pat Hanrahan, Keenan Crane for slide materials

Bonus slides:
How much pre-filtering do we need to avoid aliasing?

Nyquist-Shannon theorem

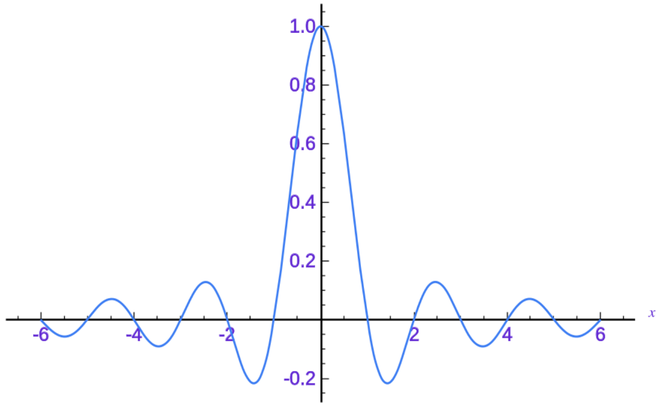
- Consider a band-limited signal: has no frequencies above ω_0
  - 1D: consider low-pass filtered audio signal
  - 2D: recall the blurred image example from a few slides ago





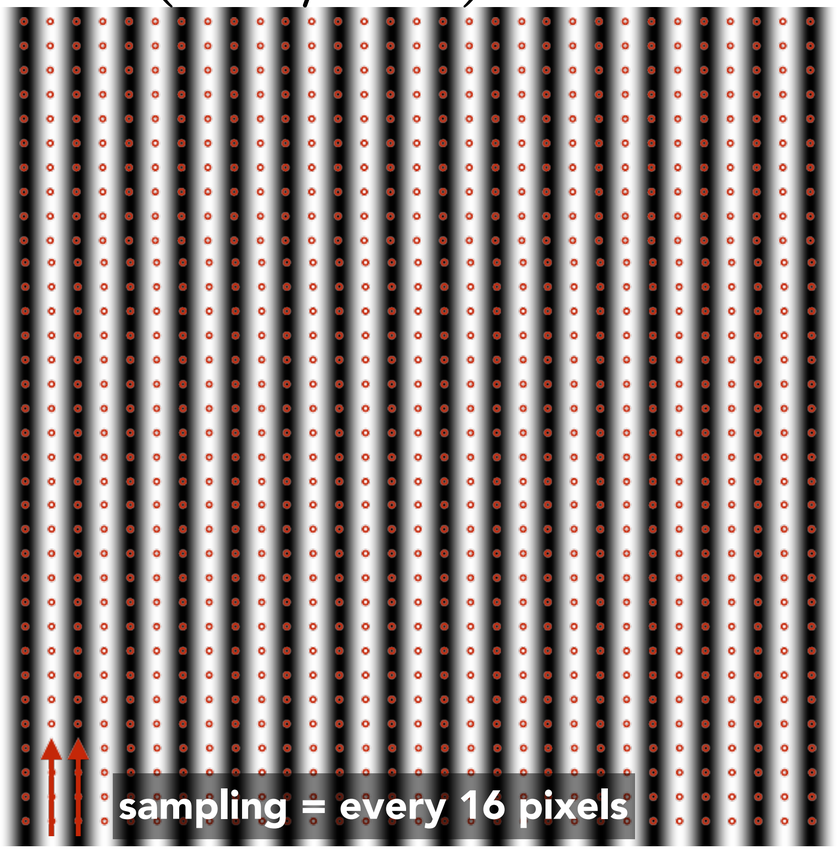
- The signal can be perfectly reconstructed if sampled with period $T = 1 / 2\omega_0$
- And reconstruction is performed using a "*sinc filter*"
  - Ideal filter with no frequencies above cutoff (*infinite extent!*)

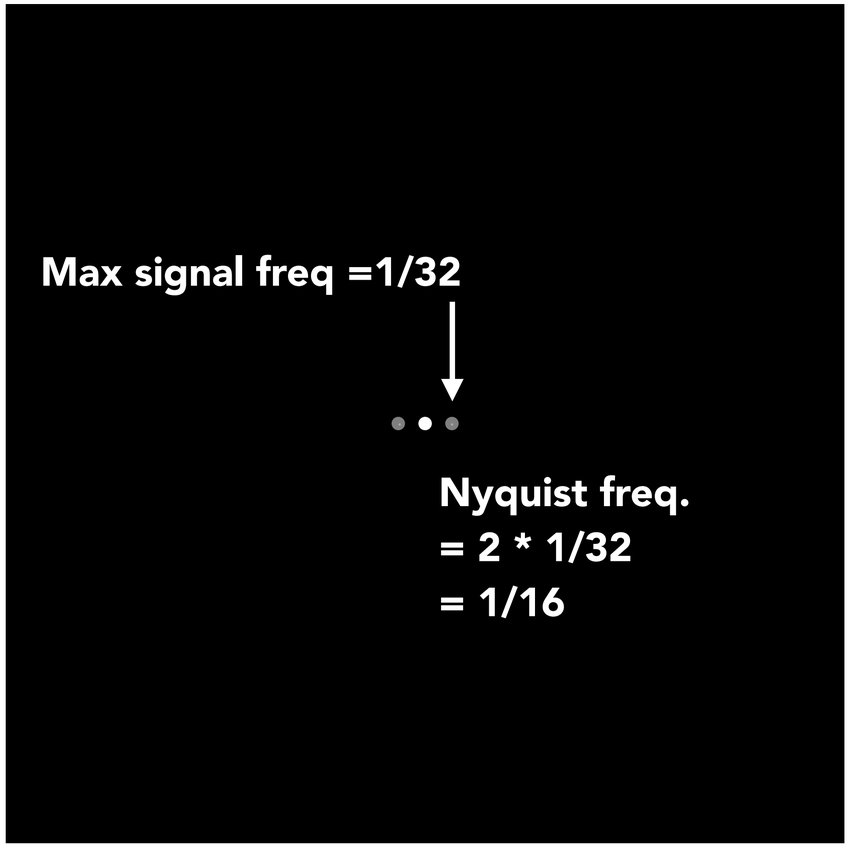
$$sinc(x) = \frac{sin(\pi x))}{\pi x}$$



Signal vs Nyquist frequency: example

$\sin(2\pi/32)x$ — frequency 1/32; 32 pixels per cycle





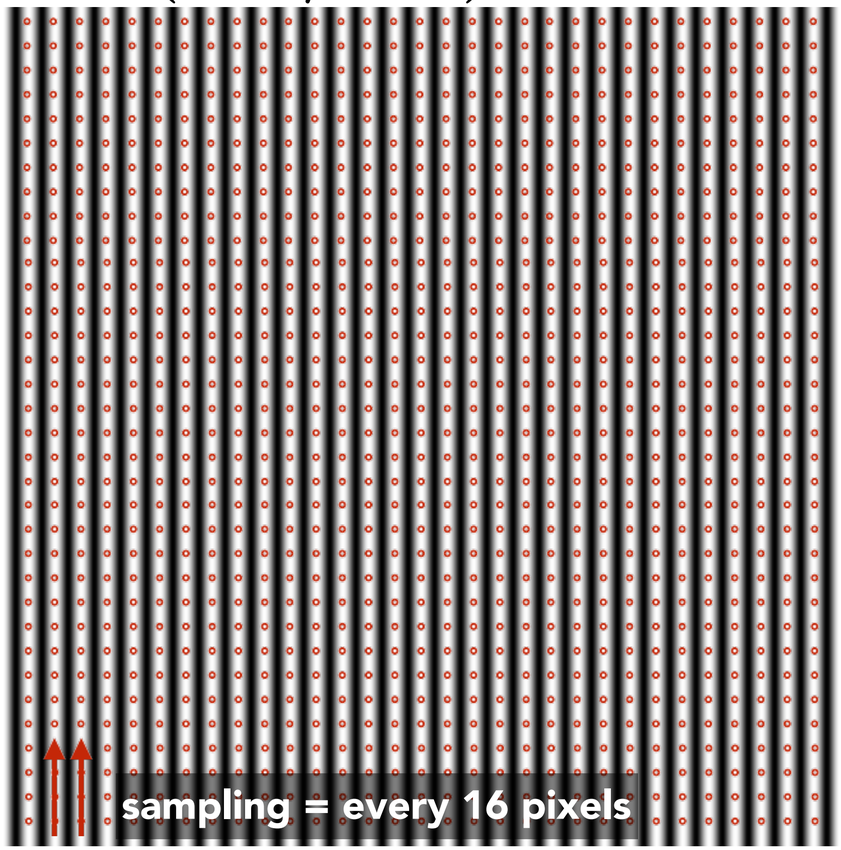
Spatial domain

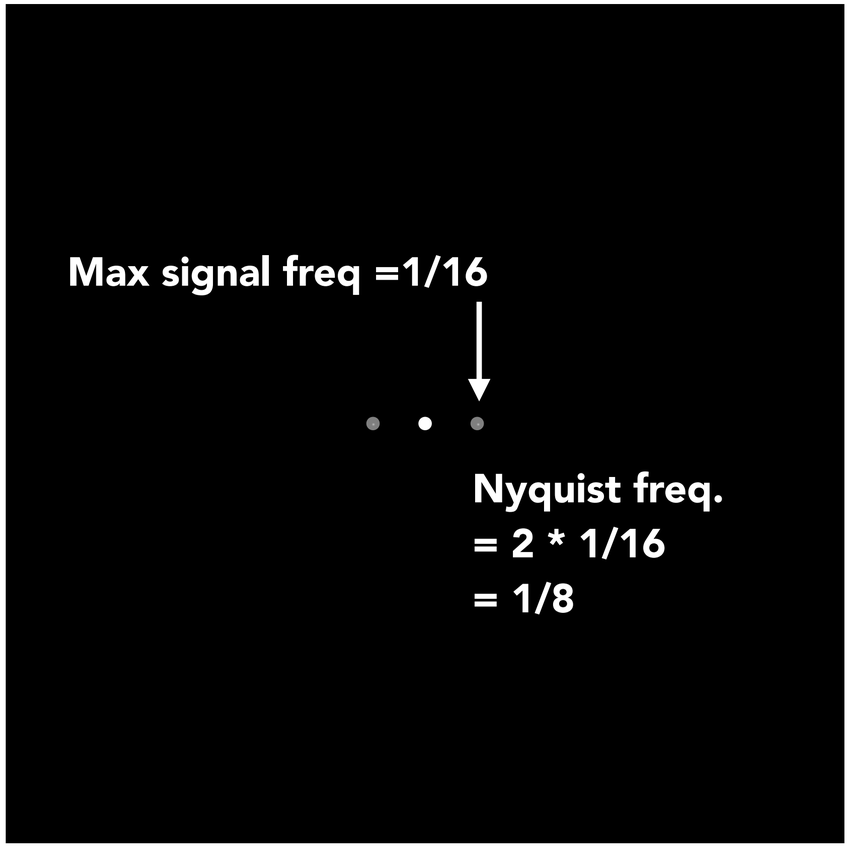
Frequency domain

No Aliasing!

Signal vs Nyquist frequency: example

$\sin(2\pi/16)x$ — frequency 1/16; 16 pixels per cycle





Aliasing! (due to undersampling)

Reminder: Nyquist theorem

- Recall: the Nyquist frequency is half the sampling frequency.

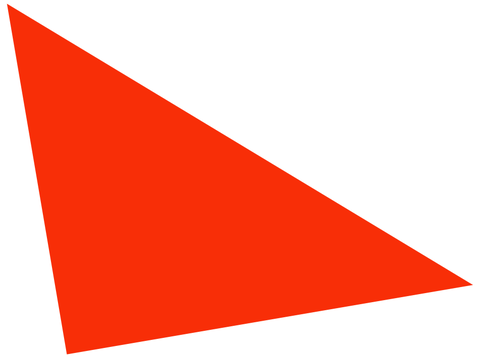
> **Theorem: We get no aliasing from frequencies in the signal that are less than the Nyquist frequency (which is defined as half the sampling frequency)**

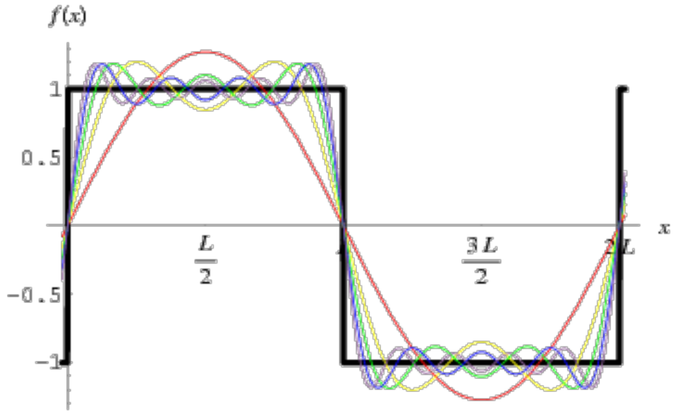
Consequence: sampling at twice the highest frequency in the signal will eliminate aliasing

Challenges of sampling-based approaches in graphics

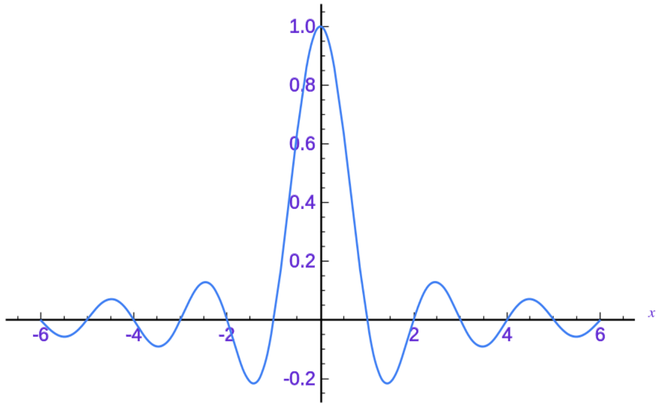
- Our signals are not always band-limited in computer graphics. Why?

Hint:

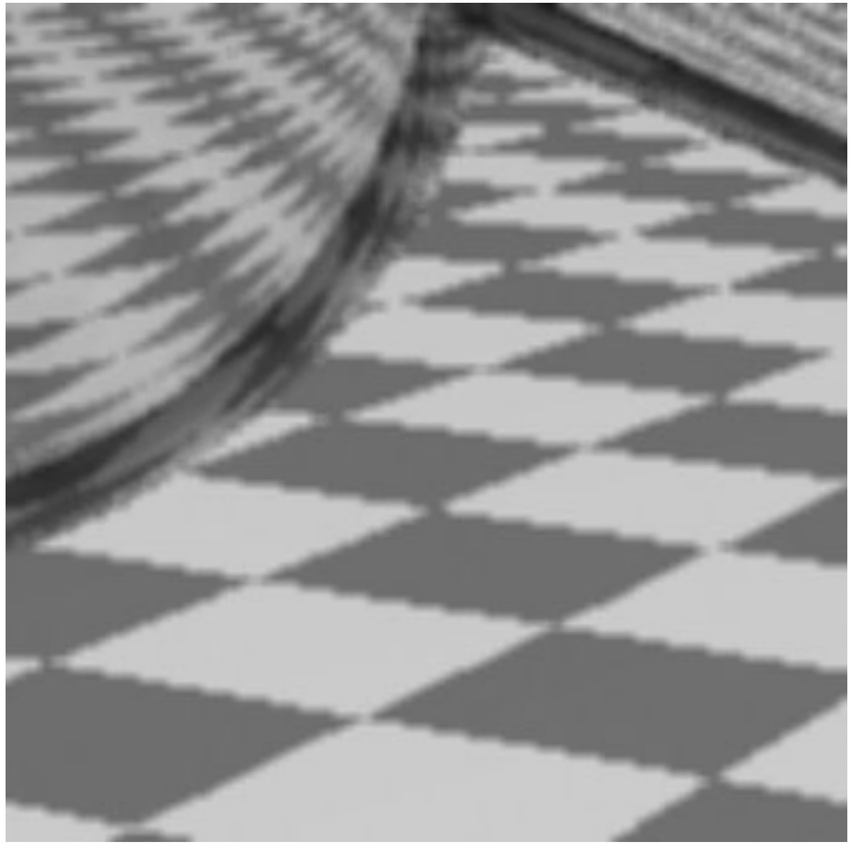




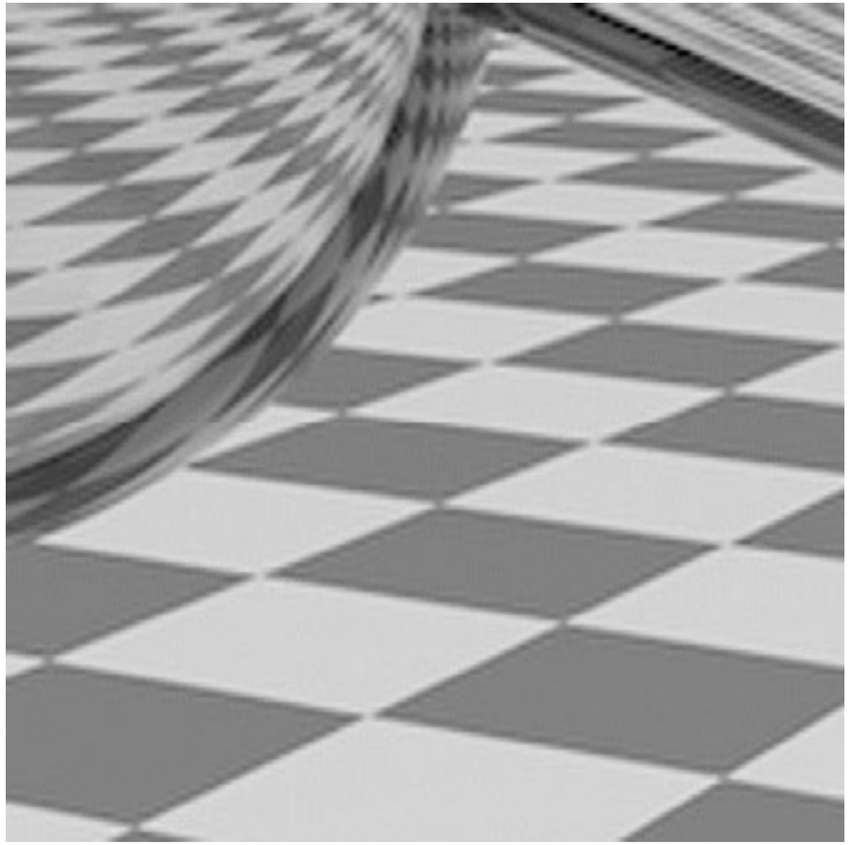
- Also, infinite extent of "ideal" reconstruction filter (sinc) is impractical for efficient implementations. Why?



Anti-aliasing vs blurring an aliased result



Blurred Jaggies
("Sample then blur jaggies")



Pre-filtered
("blur then sample")