Stanford CS248: Interactive Computer Graphics
Participation Exercise 7

Problem 1: Lagrangian Dynamics of a Bead on a Wire

In class we talked about Lagrange's equations of motion. In this problem you will derive the equations of motion for a bead of mass m on the spline curve at 3D position, $\boldsymbol{p} = \boldsymbol{C}(q) \in \mathbb{R}^3$, corresponding to independent spline parameter, $q \in \mathbb{R}$. The particle velocity is

$$\boldsymbol{v} = \dot{\boldsymbol{p}} = \frac{d}{dt}\boldsymbol{C}(q) = \frac{d\boldsymbol{C}}{dq}\frac{dq}{dt} = \boldsymbol{C}'(q)\,\dot{q} = \boldsymbol{J}\,\dot{q},$$

where the Jacobian, $\boldsymbol{J} = \boldsymbol{J}(q) = \boldsymbol{C}'(q) \in \mathbb{R}^3$, is the curve's tangent vector. In this problem, you will assume the spline location, q, is a function of time and compute the dynamics of $q(t)$ using Lagrange's equations of motion.

A. **Energy terms:** Given expressions for the position $\boldsymbol{p}(q)$ and velocity $\boldsymbol{v}(q, \dot{q})$ of the particle, write the particle's kinetic energy $K = K(q, \dot{q})$, and the gravitational potential energy $U(q)$ (up to an arbitrary constant) assuming a constant gravitational acceleration vector, $\boldsymbol{g} \in \mathbb{R}^3$. Express your energies in terms of q and $\dot{q}$ for subsequent differentiation.
Tip: You can express the squared two norm of a vector using a dot product, $\|\boldsymbol{v}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v}$.

B. **Plug and chug:** Substitute the Lagrangian $\mathcal{L} = K(q, \dot{q}) - U(q)$ into Lagrange's equations of motion. What is the resulting ODE satisfied by $q(t)$?