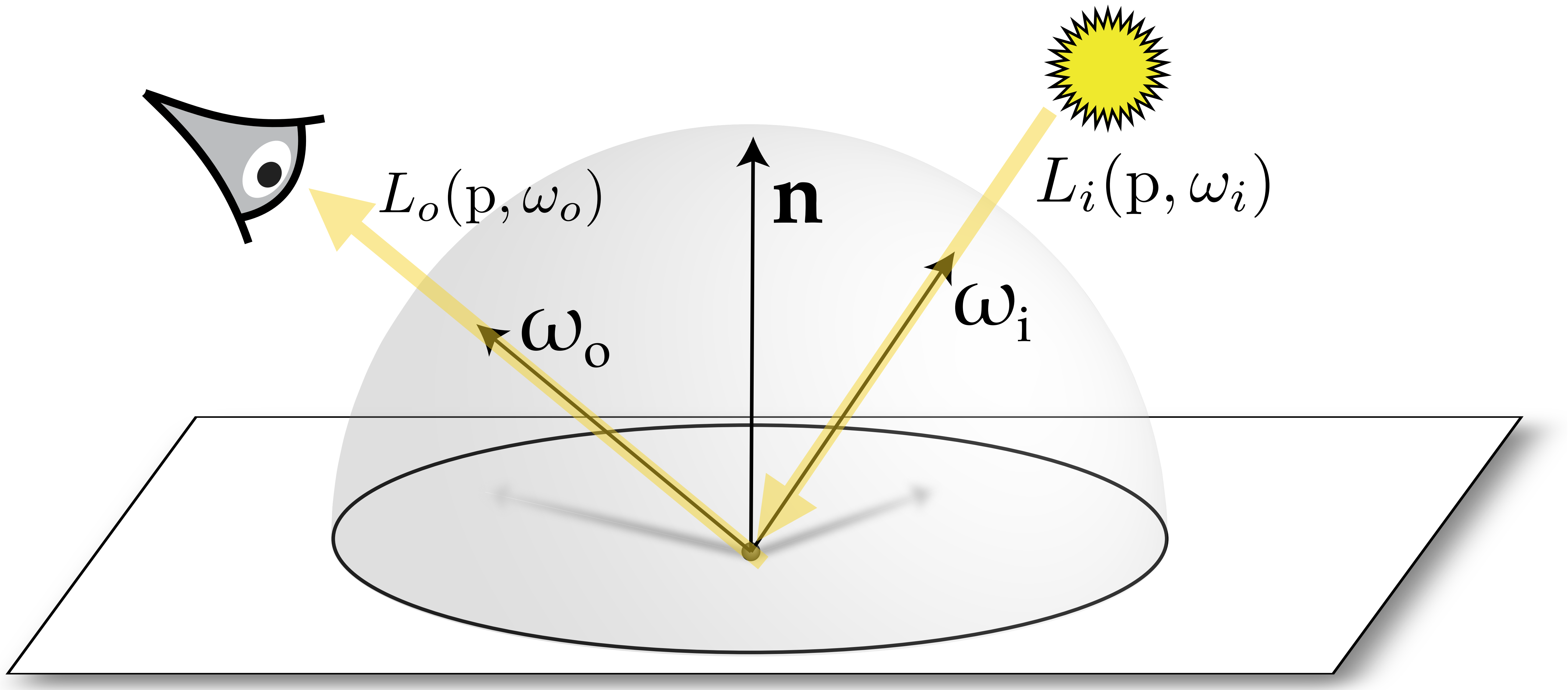
Lecture 11:

Materials (Part 2) + Numerical Integration Basics

Interactive Computer Graphics
Stanford CS248A, Winter 2024

Last time: the reflection equation

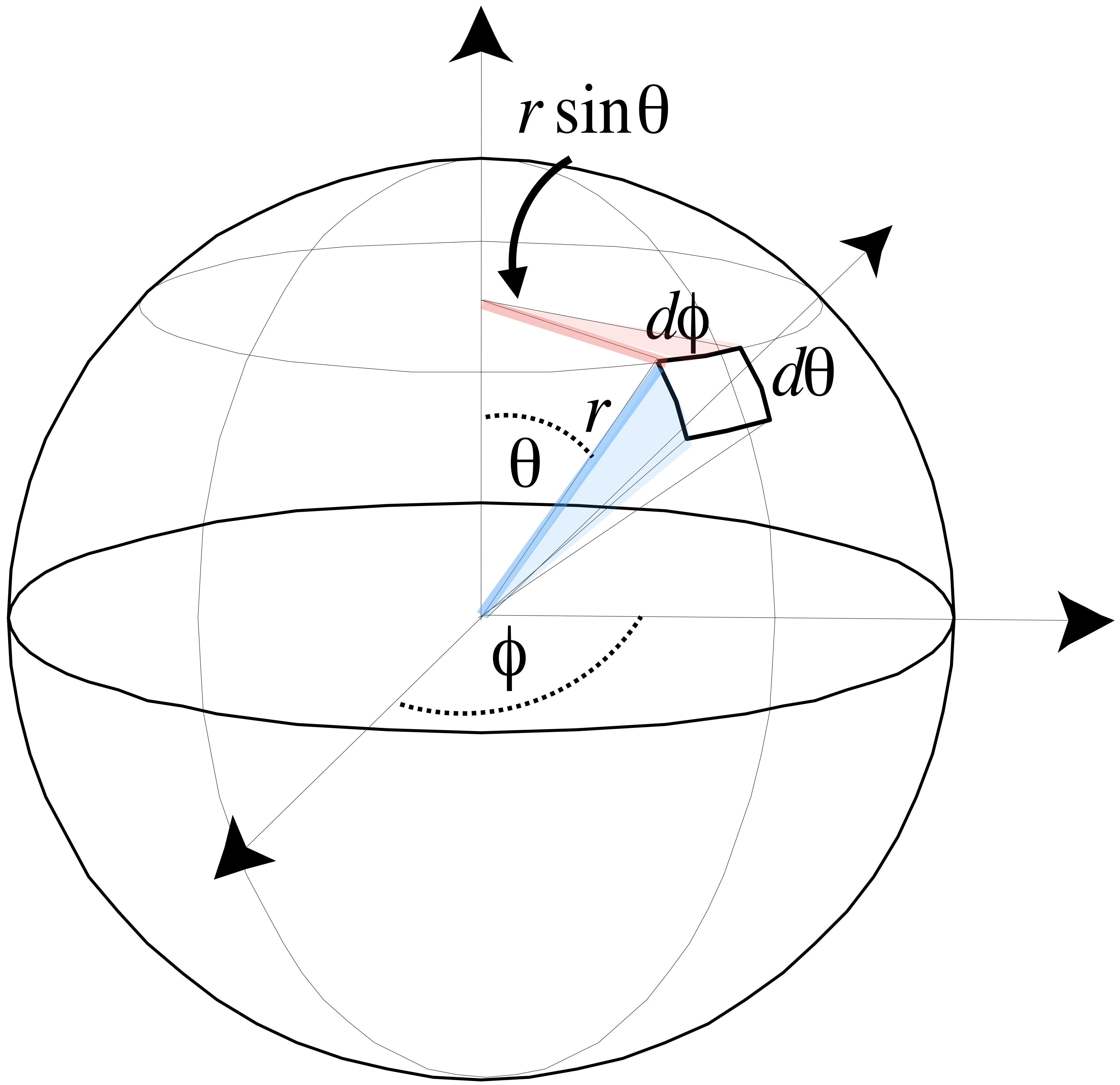


$$L_{\mathrm{o}}(\mathrm{p}, \omega_{\mathrm{o}}) = \int_{\Omega^2} \underbrace{f_{\mathrm{r}}(\mathrm{p}, \omega_{\mathrm{i}} \rightarrow \omega_{\mathrm{o}})}_{\textbf{BRDF}} \underbrace{L_{\mathrm{i}}(\mathrm{p}, \omega_{\mathrm{i}}) \cos\theta_{\mathrm{i}}}_{\textbf{Illumination}} \, \mathrm{d}\omega_{\mathrm{i}}$$

Review: radiometry and illumination

Review: differential solid angles

Sphere with radius r

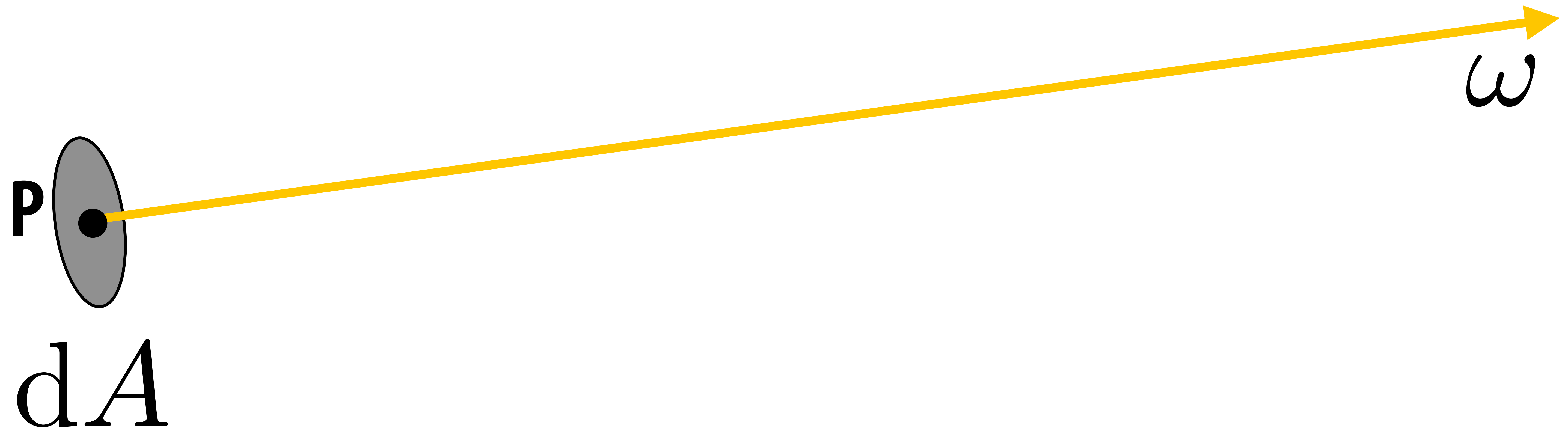


$$dA = (r\,d\theta)(r\sin\theta\,d\phi)$$
$$= r^2\sin\theta\,d\theta\,d\phi$$

$$d\omega = \frac{dA}{r^2} = \sin\theta\,d\theta\,d\phi$$

Review: radiance

Radiance (L) is energy along a ray defined by origin point *p* and direction ω

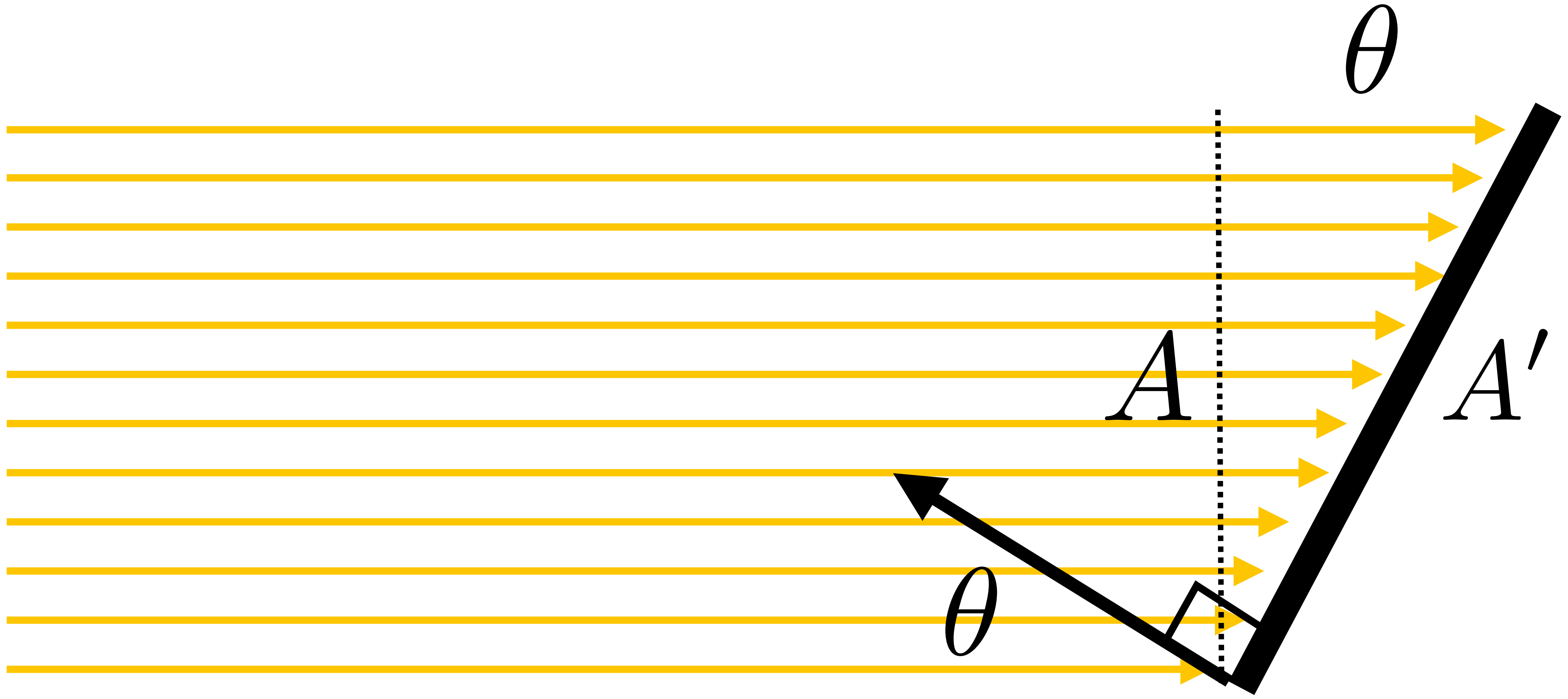


- **Radiance is the solid angle density of irradiance (irradiance per unit direction)**

  where ω denotes that the differential surface area is oriented to face in the direction

Review: irradiance = power per unit area

Irradiance at surface is proportional to cosine of angle between light direction and surface normal. (Lambert's Law)



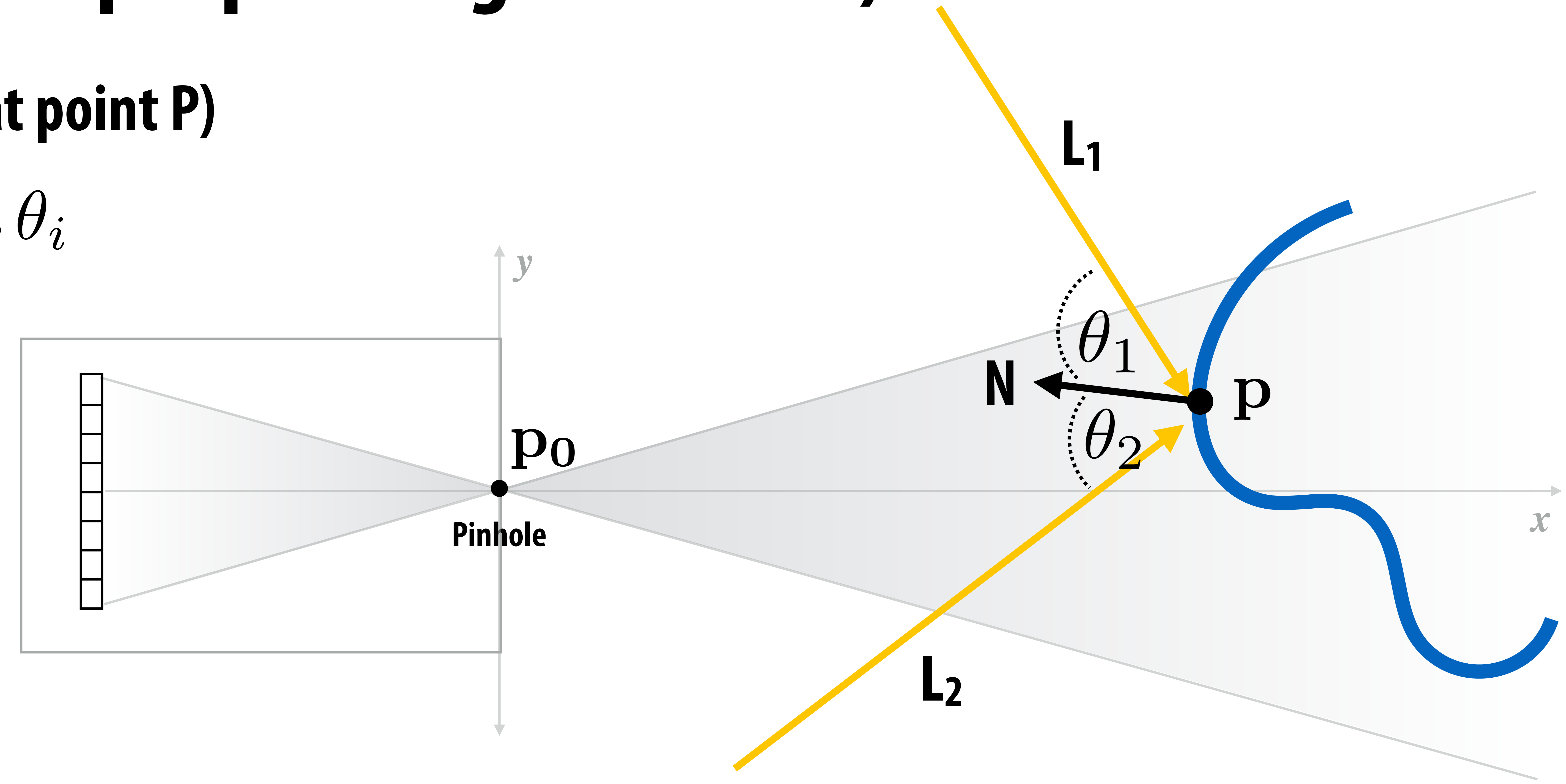
$$A = A' \cos\theta$$

$$E = \frac{\Phi}{A'} = \frac{\Phi \cos\theta}{A}$$

Review: how much light hits the surface at point p? (from multiple point light sources)

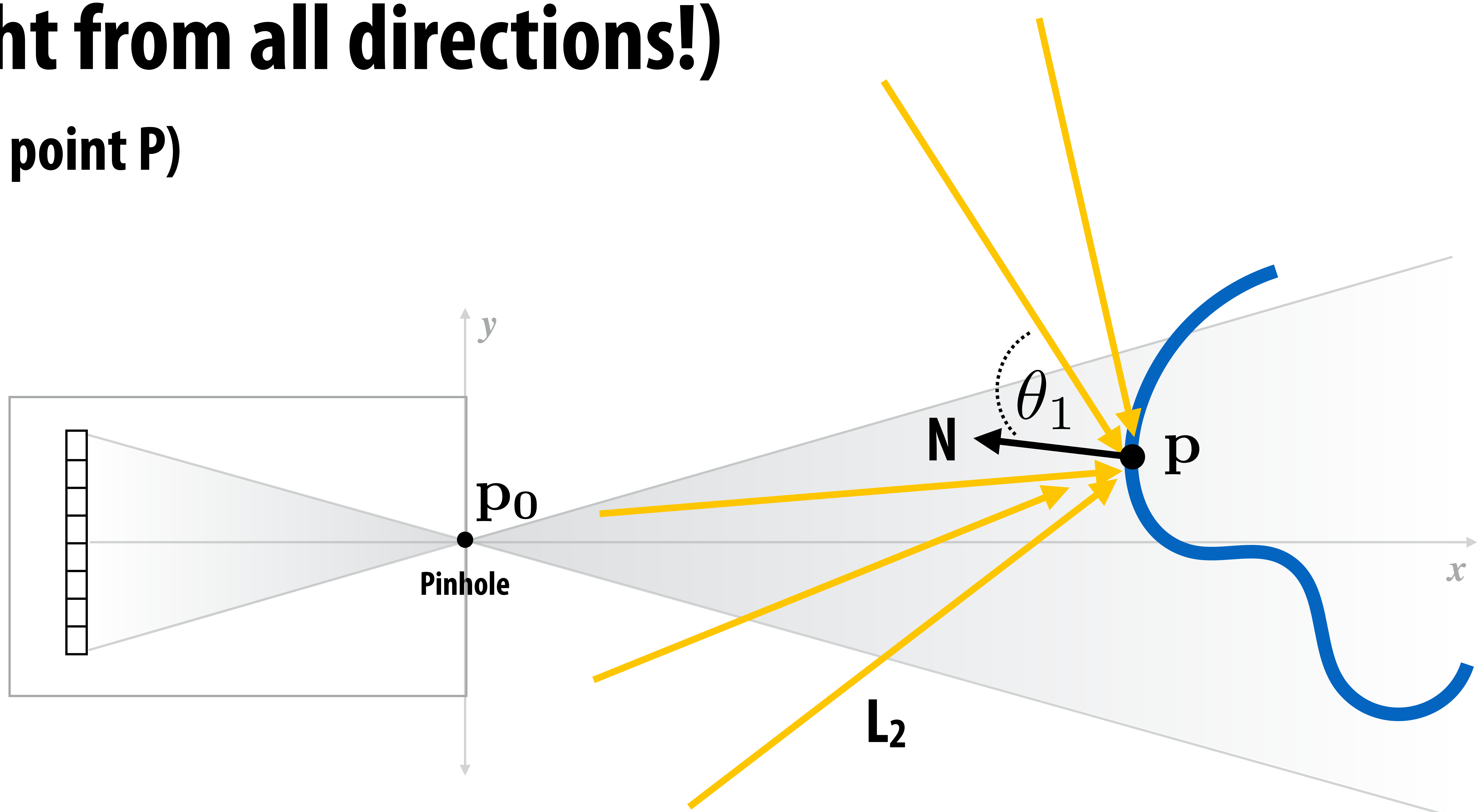
(irradiance at point P)

$$\sum_i L_i \cos\theta_i$$



How much light hits the surface at point p? (from light from all directions!)

(irradiance at point P)



$$\int_{S^2} L_i(\omega_i)\cos\theta_i d\omega = \int_0^{2\pi}\int_0^{\pi} L_i(\omega_i)\cos\theta_i \sin\theta_i d\theta d\phi$$

Irradiance at point X from a uniform area source

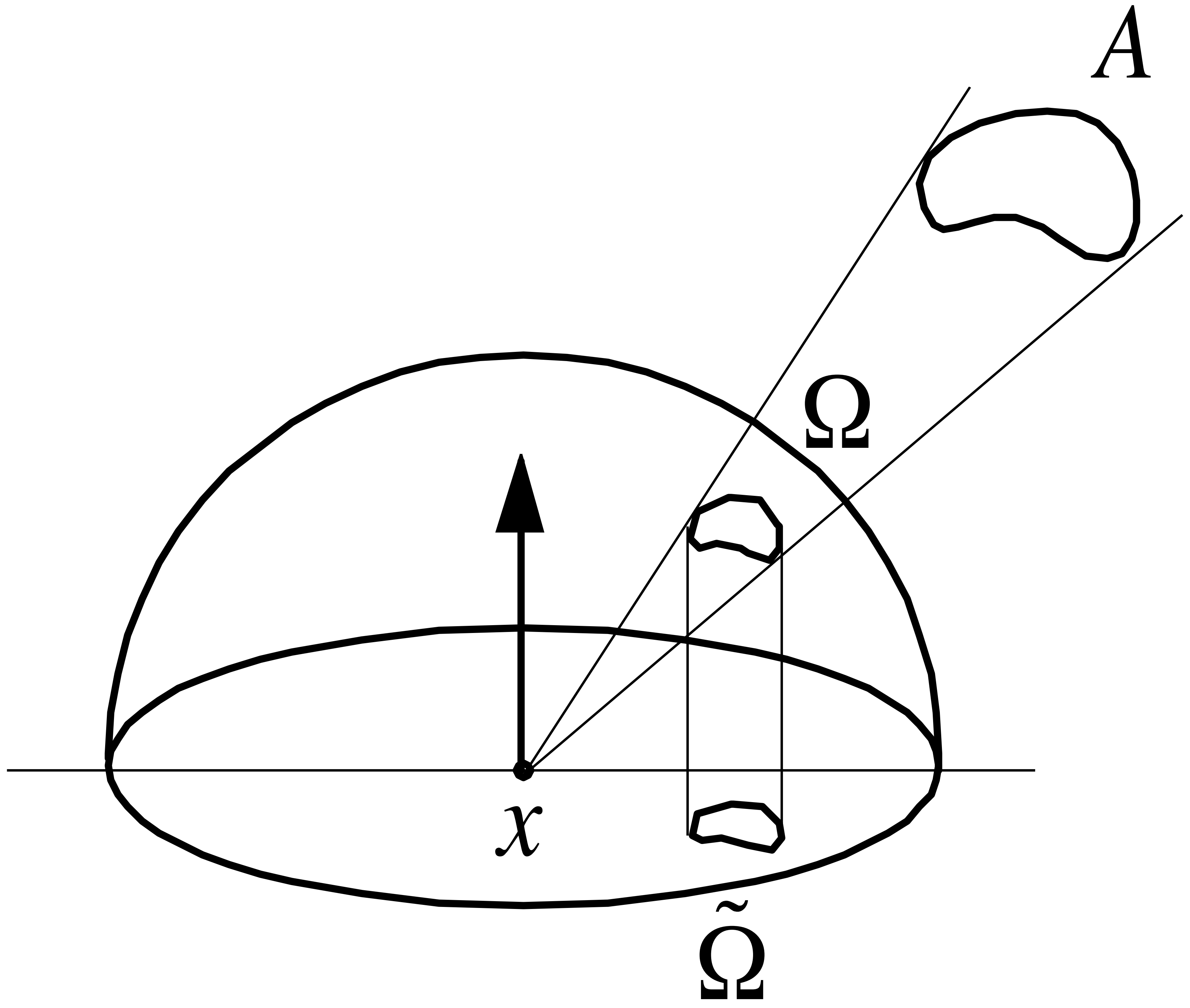
$$E(\mathrm{x}) = \int_{H^2} L(\omega)\ \cos\theta \, \mathrm{d}\omega$$

$$= L \int_{\Omega} \cos\theta \, \mathrm{d}\omega$$

$$= L\tilde{\Omega}$$

Constant
(it's a uniform source)

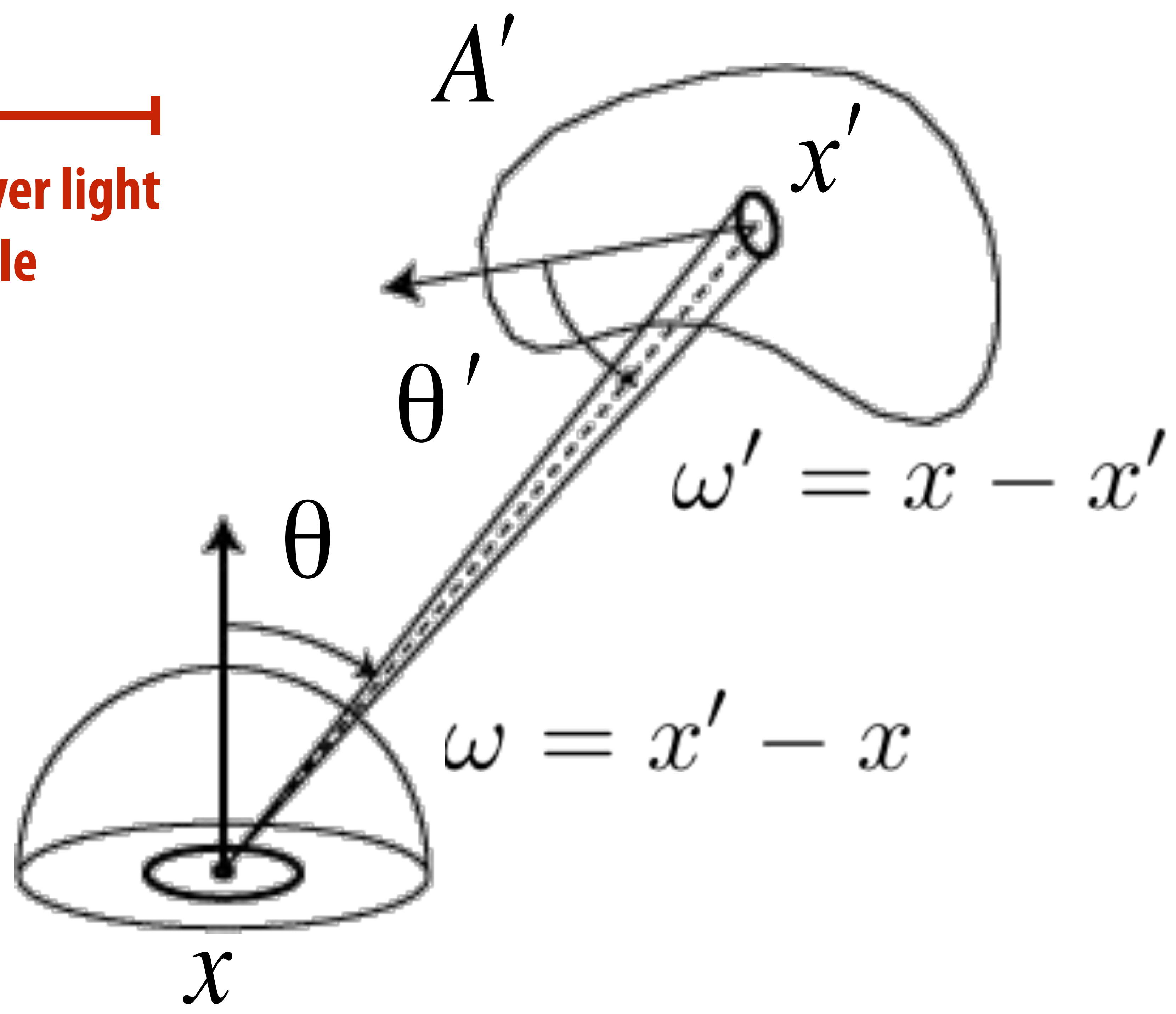
Total projected solid angle



Irradiance at point X from uniform area source

$$E(x) = \int_{H^2} L_i(x,\omega)\ \cos\theta\, d\omega = \int_{A'} L \frac{\cos\theta \cos\theta'}{|x - x'|^2} dA'$$

Reparameterization: now integrate over light source area, instead of solid angle



Integral reparameterization:

$$d\omega = \frac{\cos\theta'}{|x - x'|^2} dA'$$

Radiance leaving light from x′ in direction ω′ = radiance arriving at surface at x from ω.
(assuming that ω is pointing at the light)

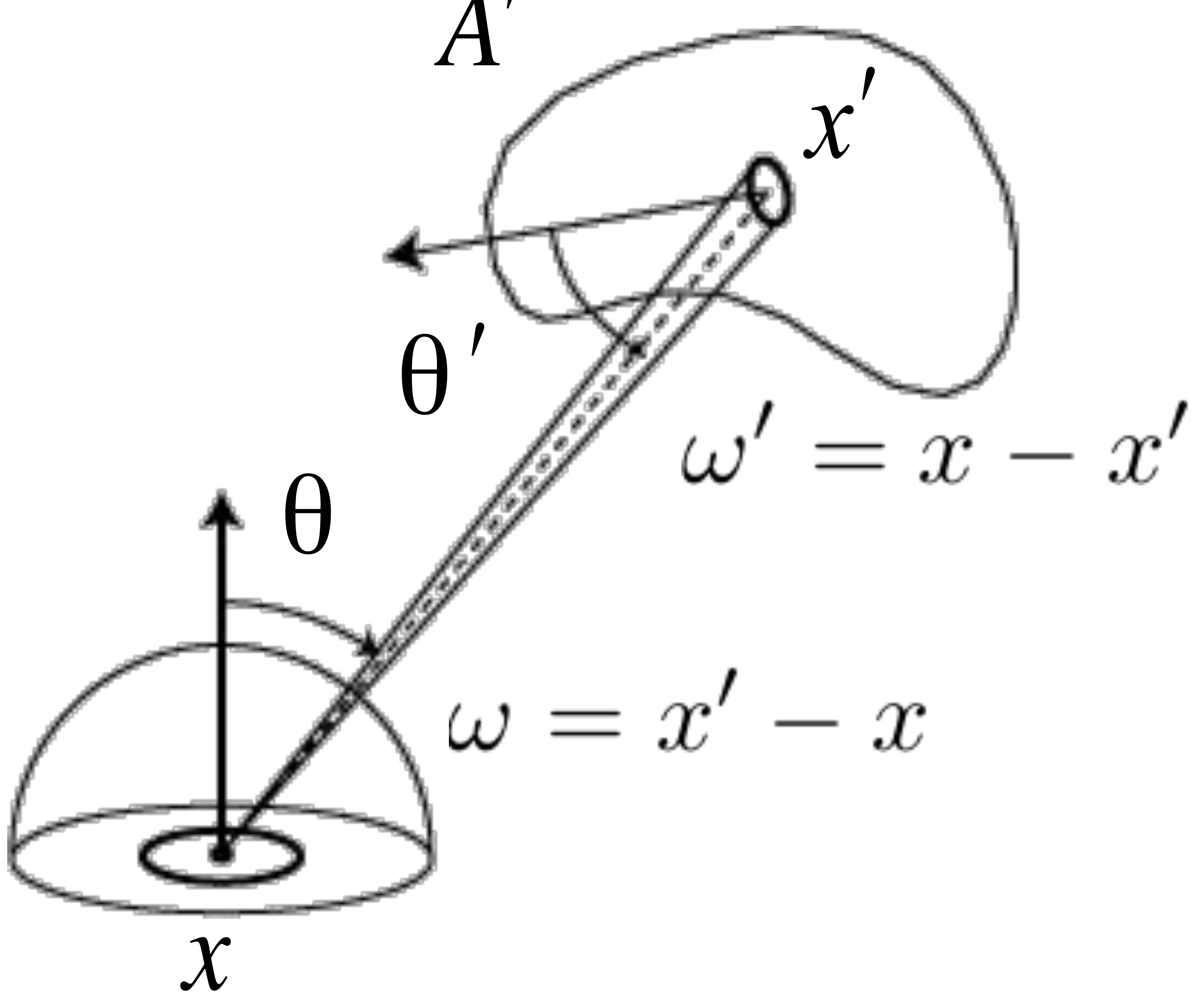
$$L_i(x,\omega) = L_o(x',\omega') = L$$

Materials
(Back to slides from last lecture)

Numerical Integration

Many examples of needing to compute integrals already in this lecture

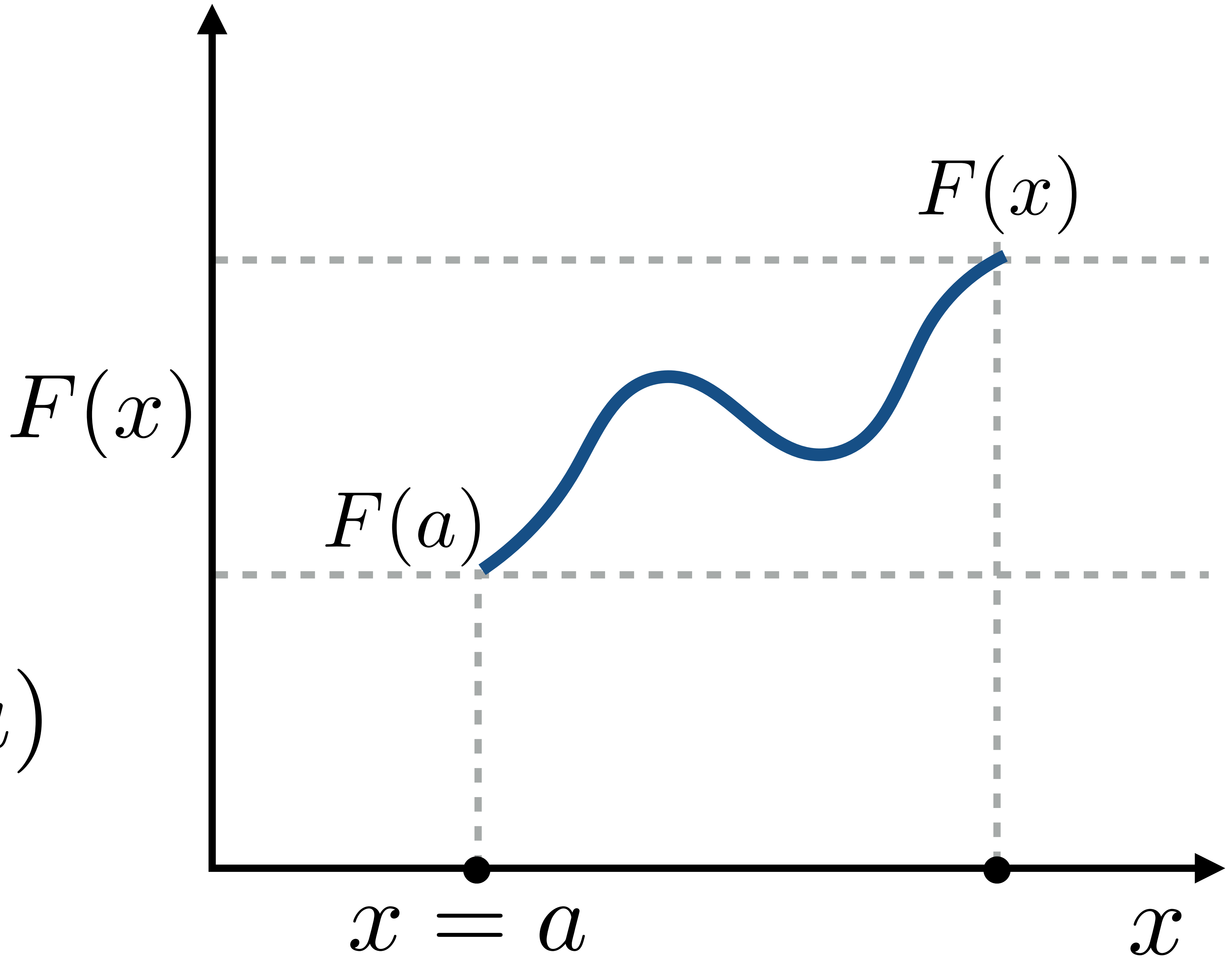
$$E(x) = \int_{H^2} L \, \cos\theta \, d\omega = \int_{A'} L \frac{\cos\theta \cos\theta'}{|x - x'|^2} dA'$$



Review: fundamental theorem of calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

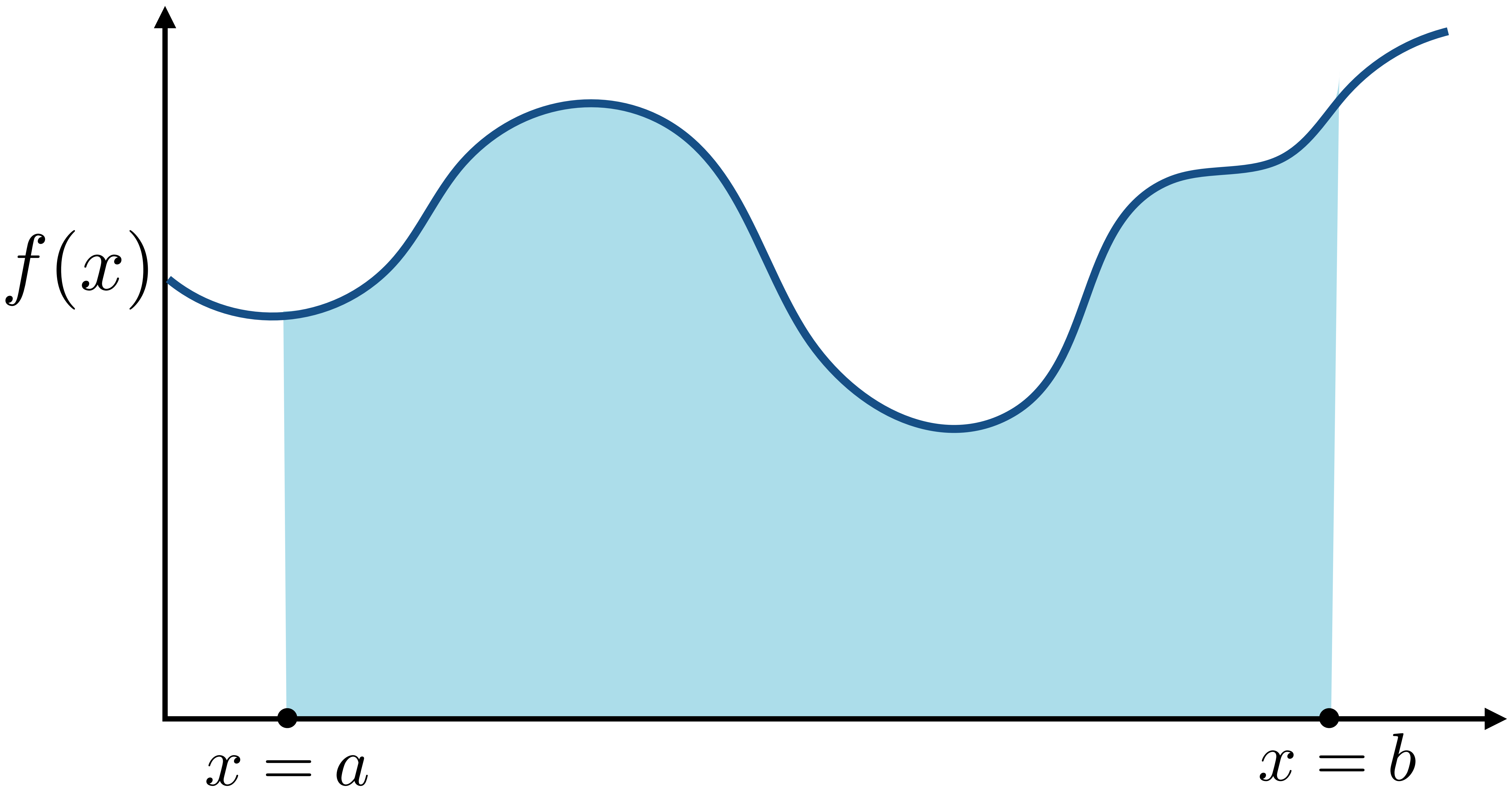
$$f(x) = \frac{d}{dx}F(x)$$



$$\int_a^x f(t)dt = F(x) - F(a)$$

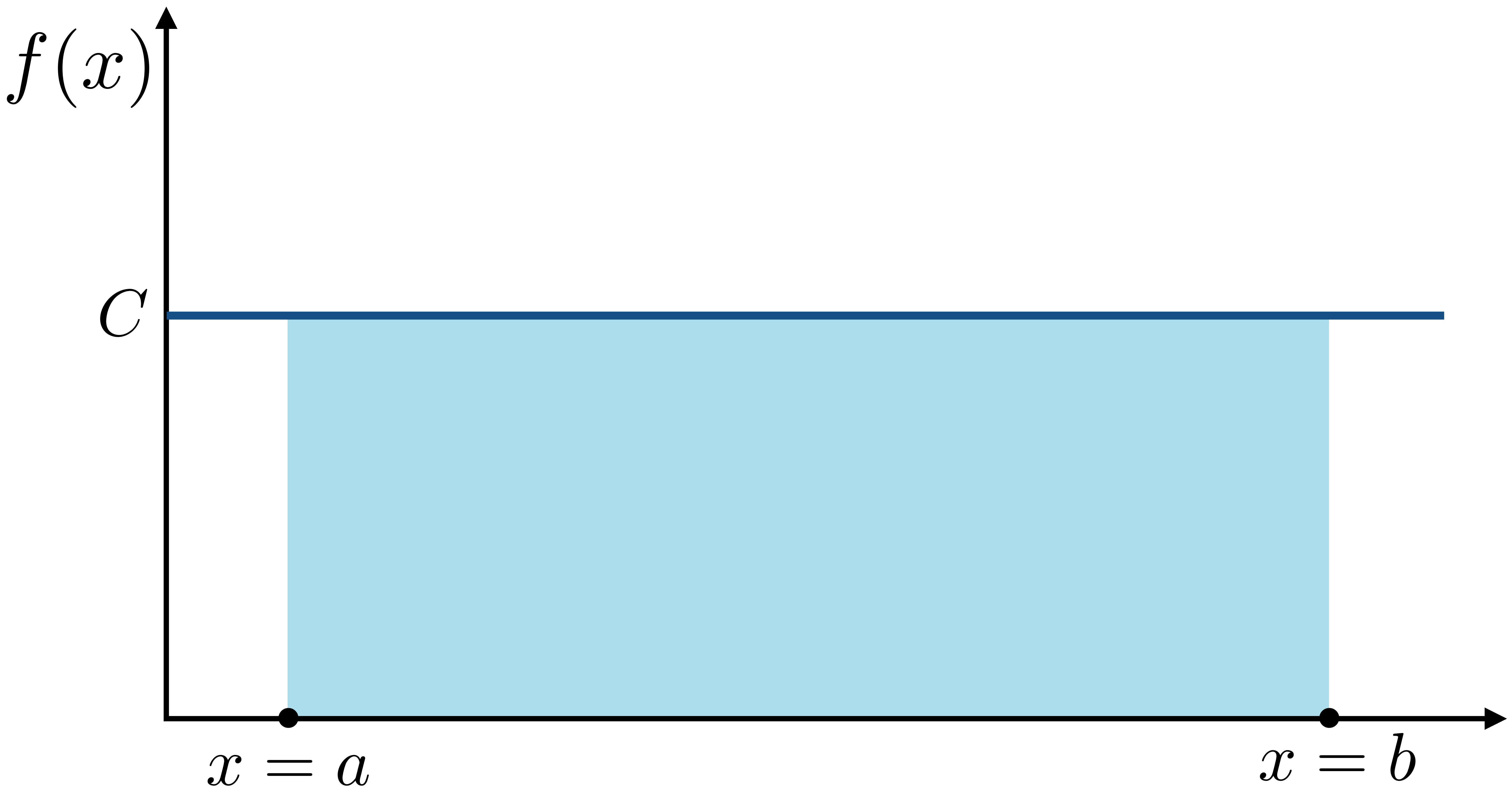
Definite integral as "area under curve"

$$\int_a^b f(x)dx$$



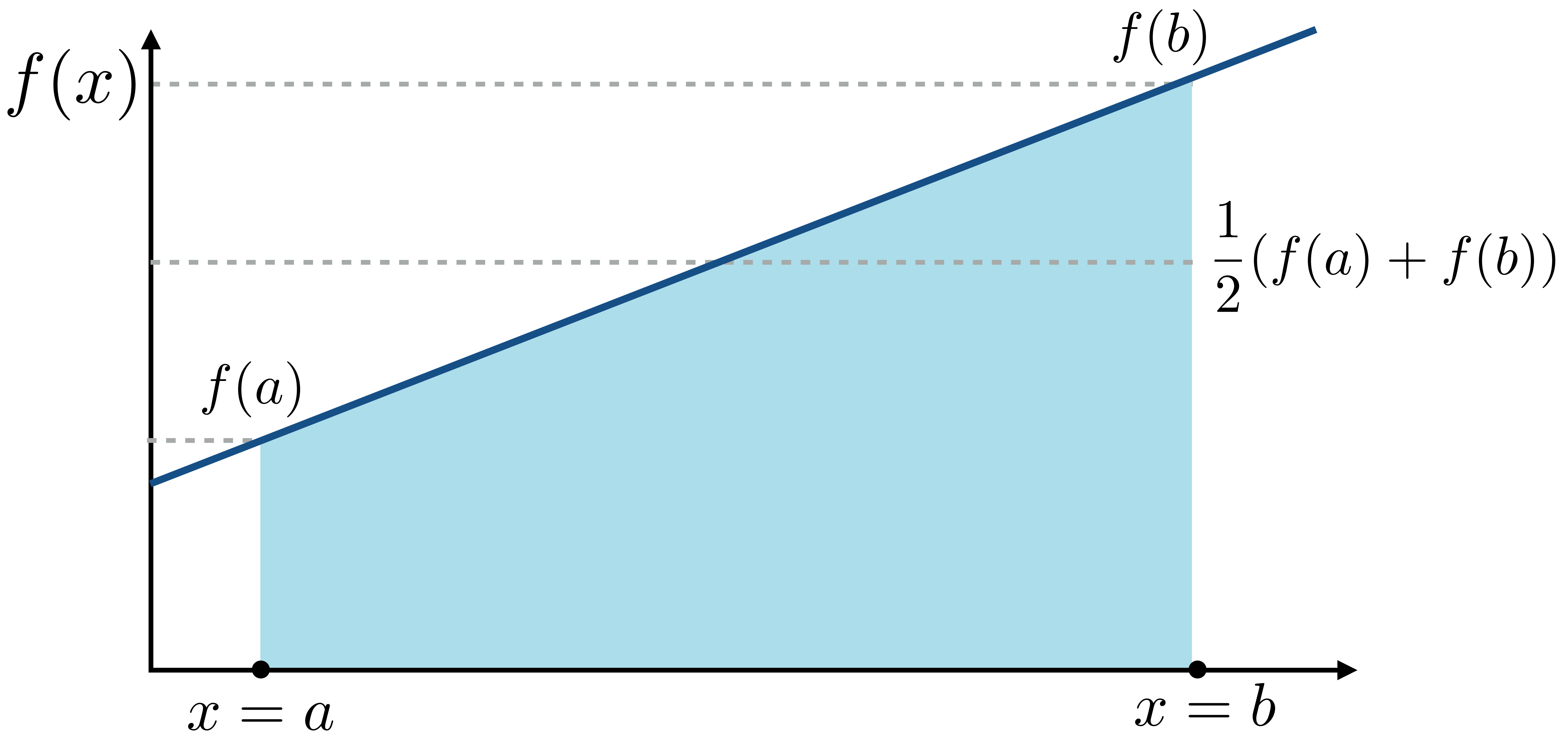
Simple case: constant function

$$\int_a^b C\,dx = (b-a)C$$



Affine function: $f(x) = cx + d$

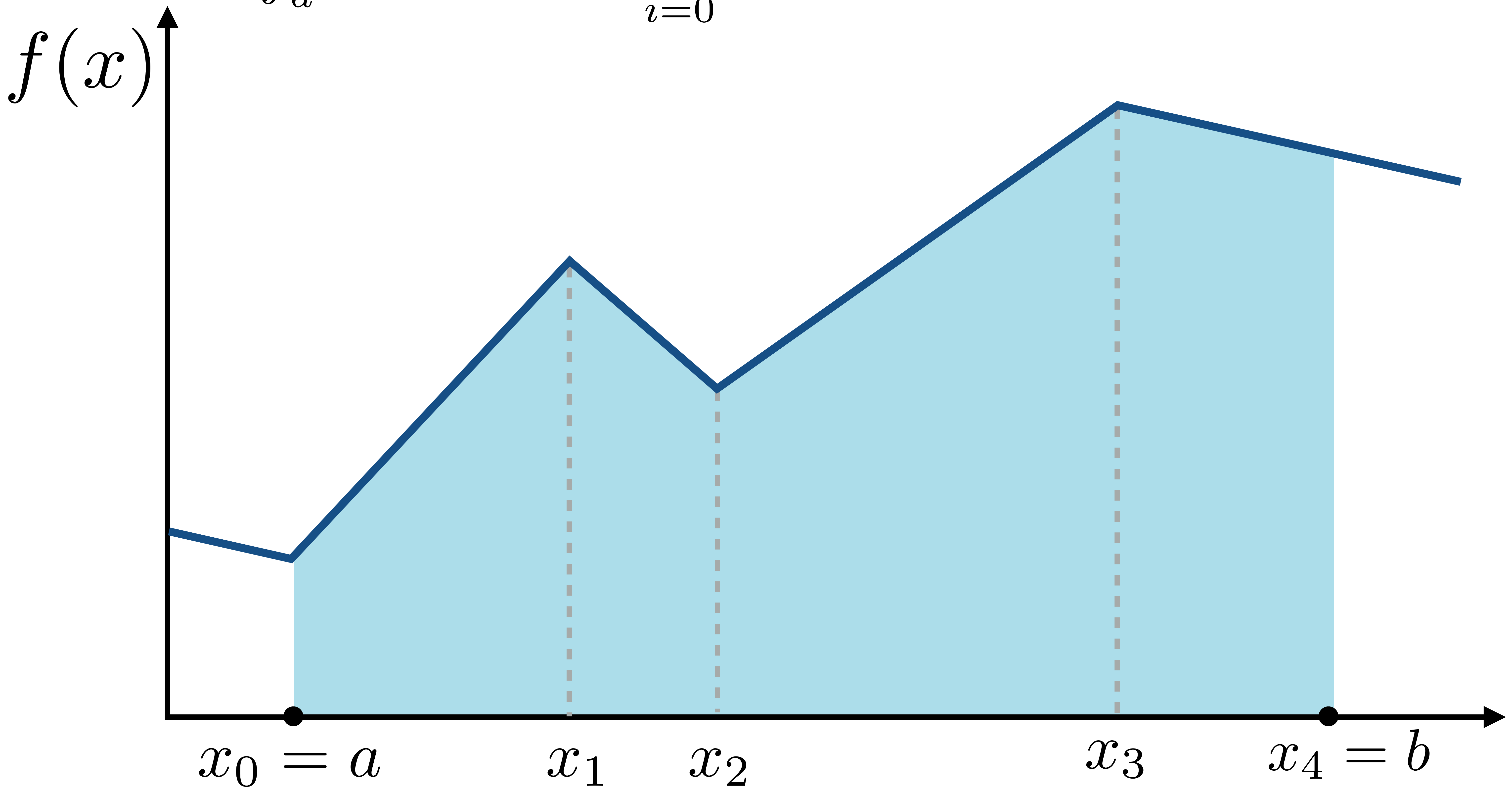
$$\int_a^b f(x)dx = \frac{1}{2}(f(a) + f(b))(b - a)$$



Piecewise affine function

Sum of integrals of individual affine components

$$\int_a^b f(x)dx = \frac{1}{2}\sum_{i=0}^{n-1}(x_{i+1} - x_i)(f(x_i) + f(x_{i+1}))$$



Piecewise affine function

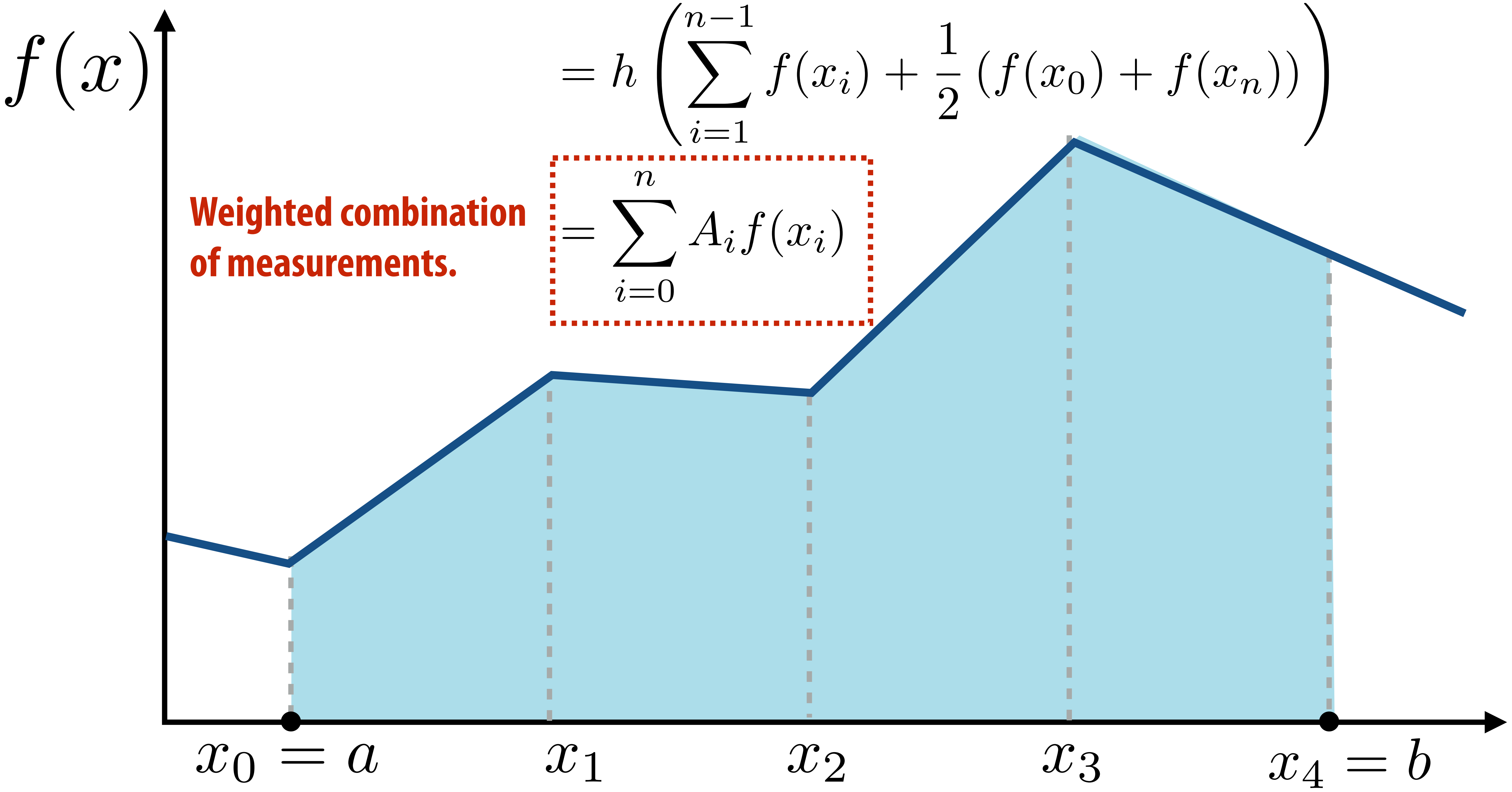
If N-1 segments are of equal length: $h = \frac{b-a}{n-1}$

$$\int_a^b f(x)dx = \frac{h}{2}\sum_{i=0}^{n-1}(f(x_i) + f(x_{i+1}))$$

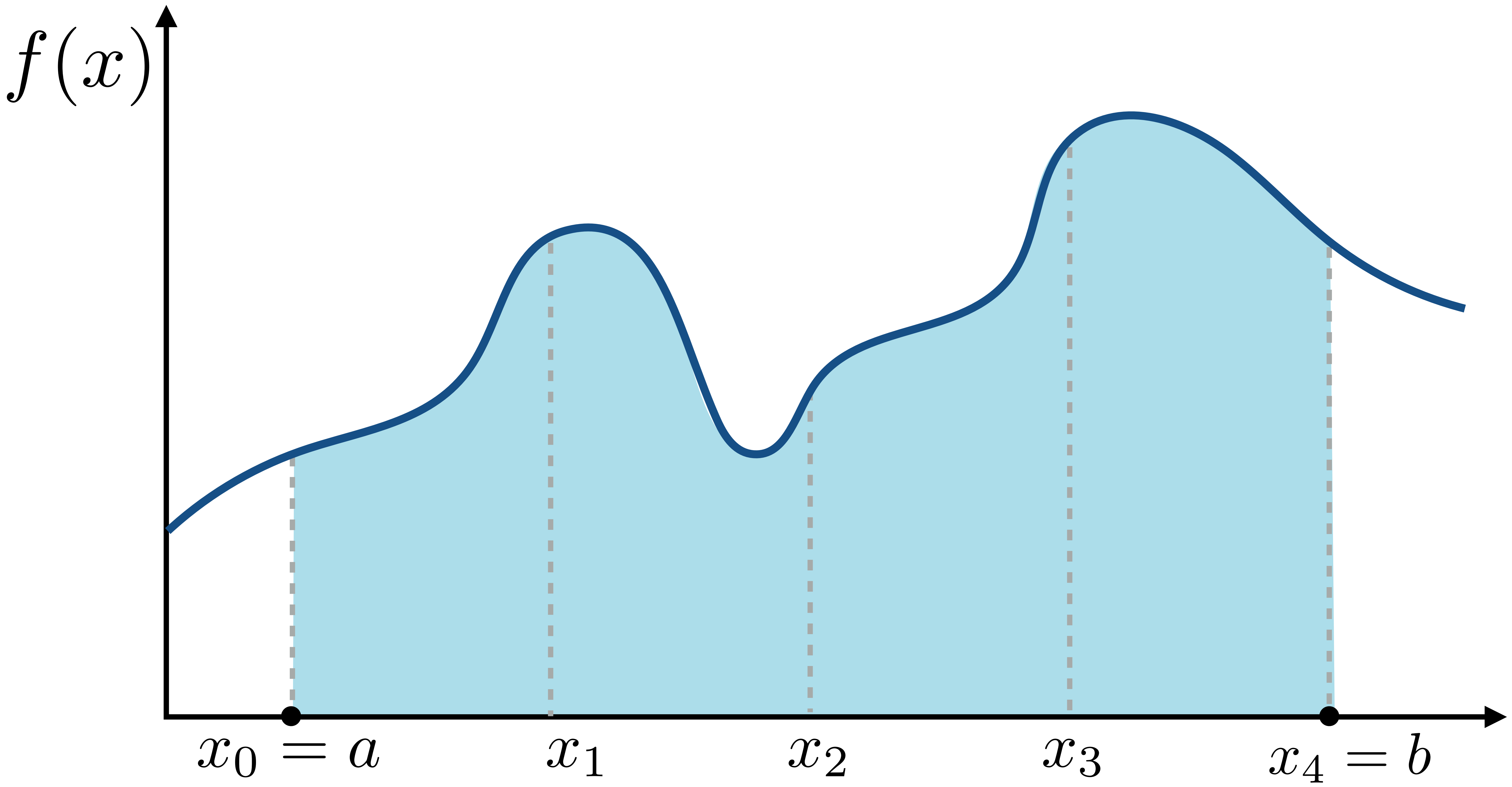
$$= h\left(\sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}(f(x_0) + f(x_n))\right)$$

$$= \sum_{i=0}^{n} A_i f(x_i)$$

Weighted combination of measurements.



Arbitrary function f(x)?

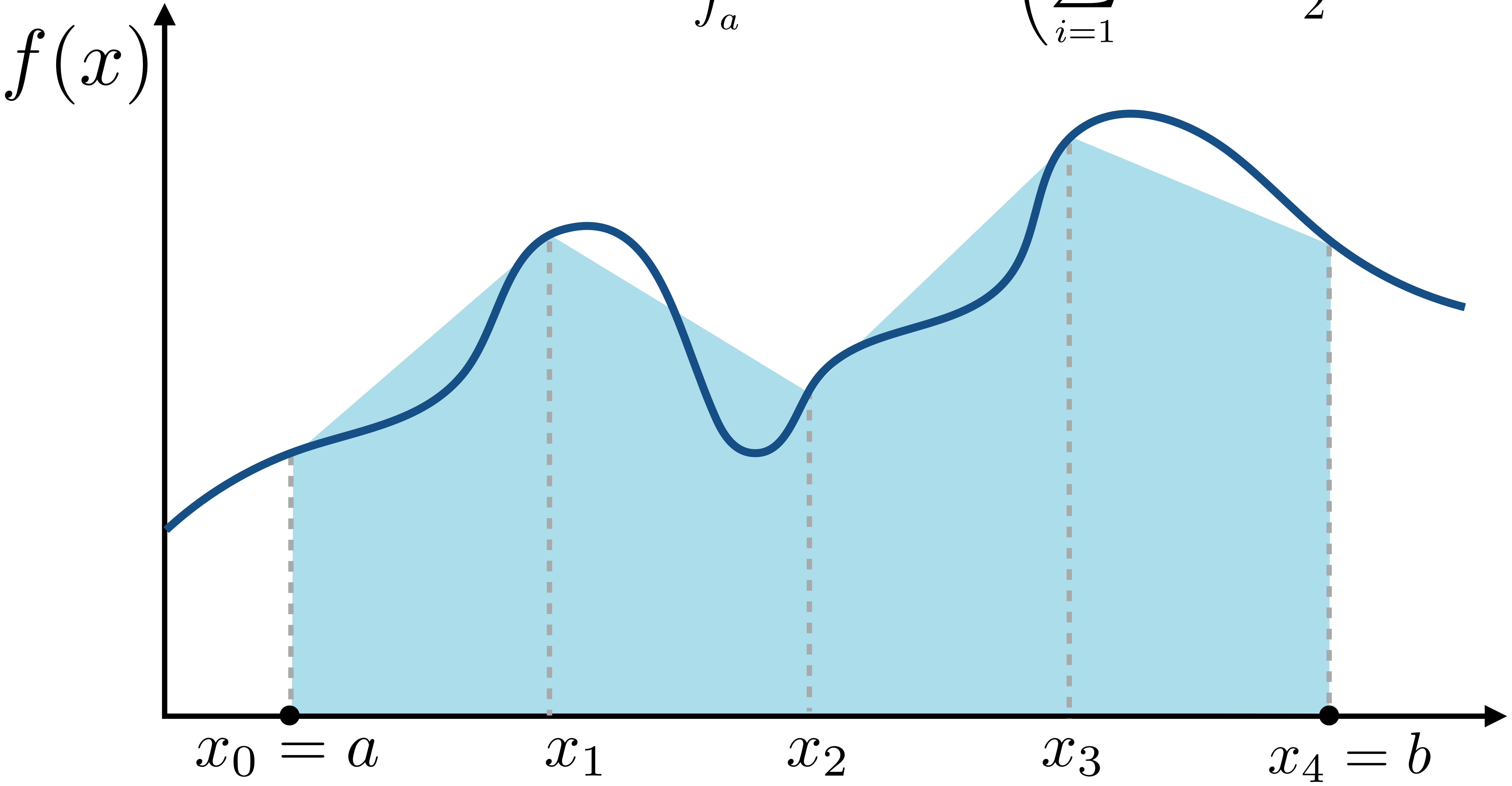


Trapezoidal rule

<u>Approximate</u> integral of f(x) by assuming function is piecewise linear

For equal length segments: $h = \dfrac{b-a}{n-1}$

$$\int_a^b f(x)dx = h\left(\sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}\left(f(x_0) + f(x_n)\right)\right)$$

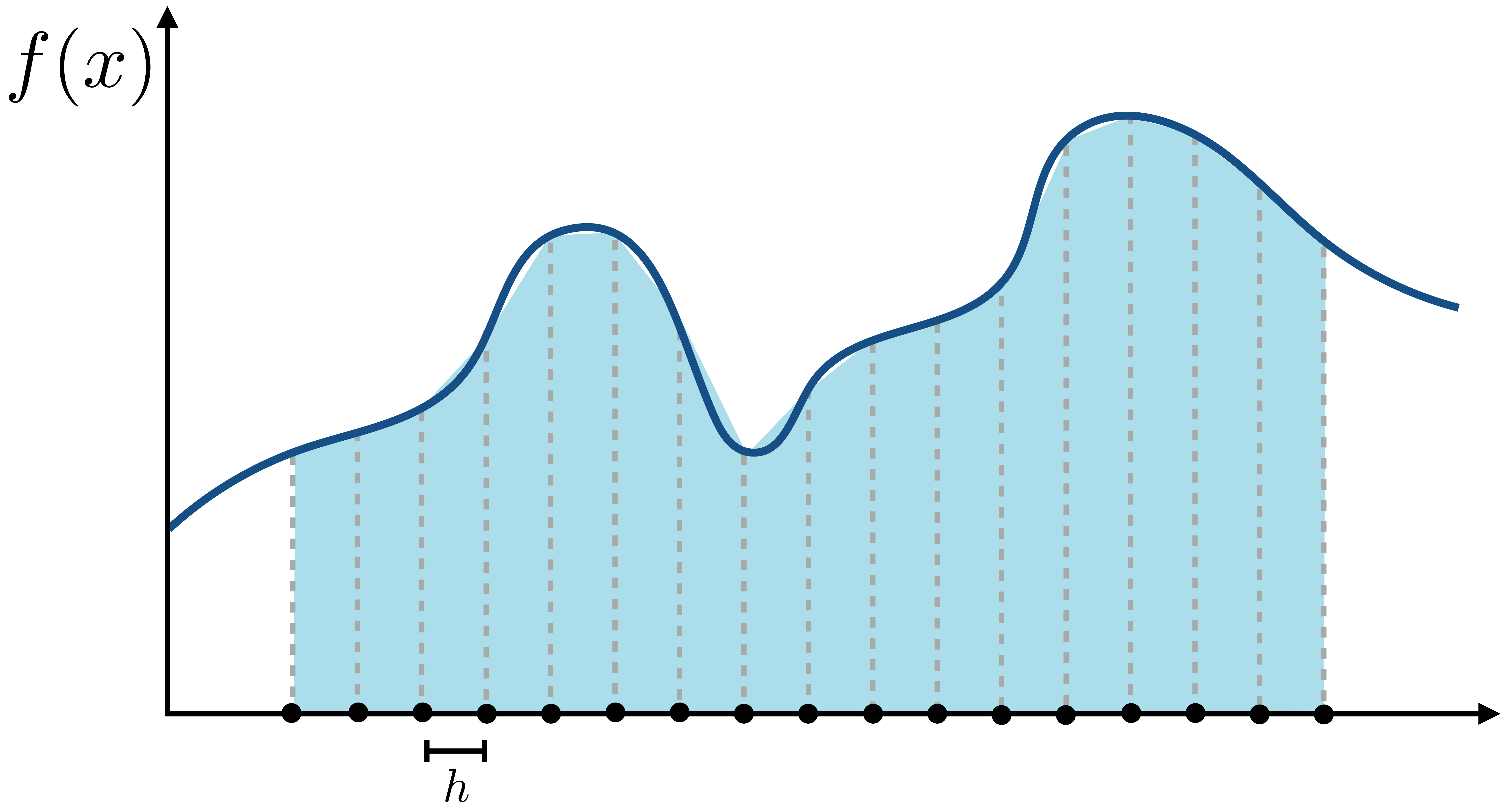


Trapezoidal rule

Consider cost and accuracy of estimate as $n \to \infty$ **(or** $h \to 0$**)**

Work: $O(n)$

Error can be shown to be: $O(h^2) = O(\frac{1}{n^2})$ **(for f(x) with continuous second derivative)**



Integration in 2D

Consider integrating $f(x,y)$ using the trapezoidal rule
(apply rule twice: when integrating in *x* and in *y*)

$$\int_{a_y}^{b_y}\int_{a_x}^{b_x} f(x,y)dxdy = \int_{a_y}^{b_y}\left(O(h^2)+\sum_{i=0}^{n}A_i f(x_i,y)\right)dy$$

First application of rule

$$= O(h^2)+\sum_{i=0}^{n}A_i\int_{a_y}^{b_y} f(x_i,y)dy$$

$$= O(h^2)+\sum_{i=0}^{n}A_i\left(O(h^2)+\sum_{j=0}^{n}A_j f(x_i,y_j)\right)$$

Second application

$$= O(h^2)+\sum_{i=0}^{n}\sum_{j=0}^{n}A_iA_j f(x_i,y_j)$$

Errors add, so error still: $O(h^2)$
But work is now: $O(n^2)$
(*n* x *n* set of measurements)

Must perform much more work in 2D to get same error bound on integral!

In K-D, let $N = n^k$

Error goes as: $O\left(\frac{1}{N^{2/k}}\right)$

Monte Carlo integration

Monte Carlo numerical integration

- **Estimate value of integral using random sampling of function**
  - **Value of estimate depends on random samples used**
  - **But algorithm gives the correct value of integral "on average"**

- **Only requires function to be evaluated at random points on its domain**
  - **Applicable to functions with discontinuities, functions that are impossible to integrate directly**

- **Error of estimate is independent of the dimensionality of the integrand**
  - **Depends on the number of random samples used:** $O(n^{1/2})$

Monte Carlo algorithms

- Advantages
  - Easy to implement
  - Easy to think about (but be careful of subtleties)
  - Robust when used with complex integrands (lights, BRDFs) and domains (shapes)
  - Efficient for high-dimensional integrals
  - Efficient when only need solution at a few points

- Disadvantages
  - Noisy
  - Slow (many samples needed for convergence)

Review: random variables

X random variable. Represents a distribution of potential values

$X \sim p(x)$ probability density function (PDF). Describes relative probability of a random process choosing value x

Uniform PDF: all values over a domain are equally likely

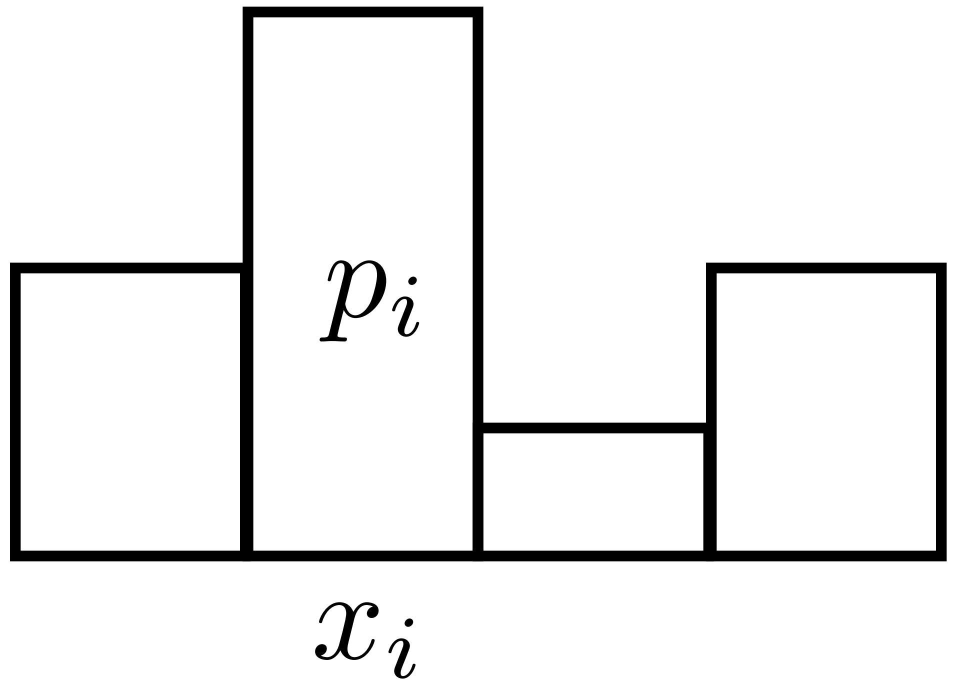
e.g., for an unbiased die

X **takes on values 1,2,3,4,5,6**

$$p(1) = p(2) = p(3) = p(4) = p(5) = p(6)$$

Discrete probability distributions

n **discrete values** x_i



With probability p_i

Requirements of a PDF:

$$p_i \geq 0$$

$$\sum_{i=1}^{n} p_i = 1$$

Six-sided die example: $p_i = \frac{1}{6}$

Think: p_i **is the probability that a random measurement of** X **will yield the value** x_i

X **takes on the value** x_i **with probability** p_i

Cumulative distribution function (CDF)

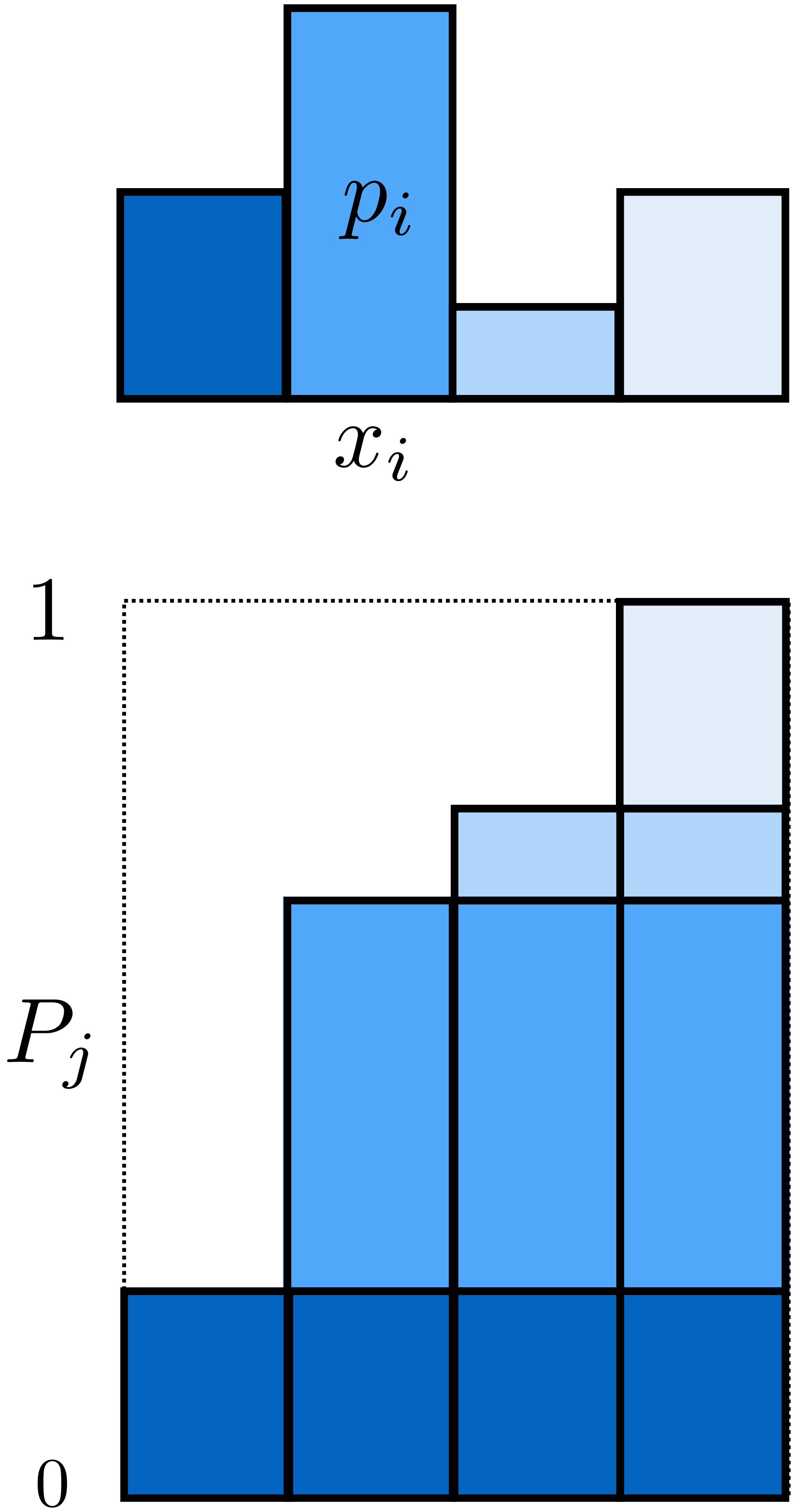
(For a discrete probability distribution)

Cumulative PDF: $P_j = \sum_{i=1}^{j} p_i$

where:

$$0 \leq P_i \leq 1$$

$$P_n = 1$$



Sampling from discrete probability distributions

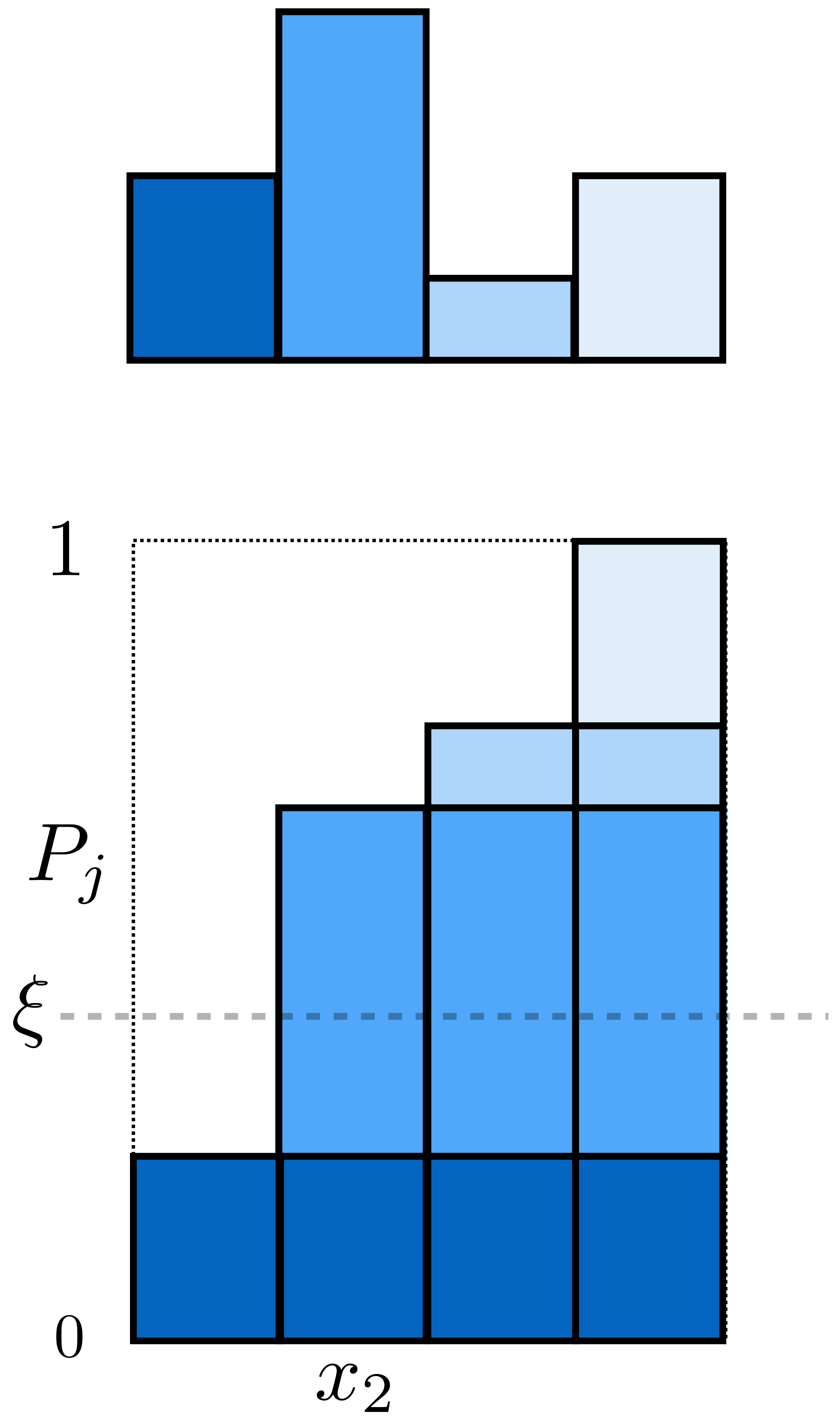
How do we generate samples of a discrete random variable (with a known PDF?)

To randomly select an event, select x_i if

$$P_{i-1} < \xi \le P_i$$

↑

Uniform random variable $\in [0, 1)$



Continuous probability distributions

PDF $p(x)$

$$p(x) \geq 0$$

CDF $P(x)$

$$P(x) = \int_0^x p(x)\,\mathrm{d}x$$

$$P(x) = \Pr(X < x)$$

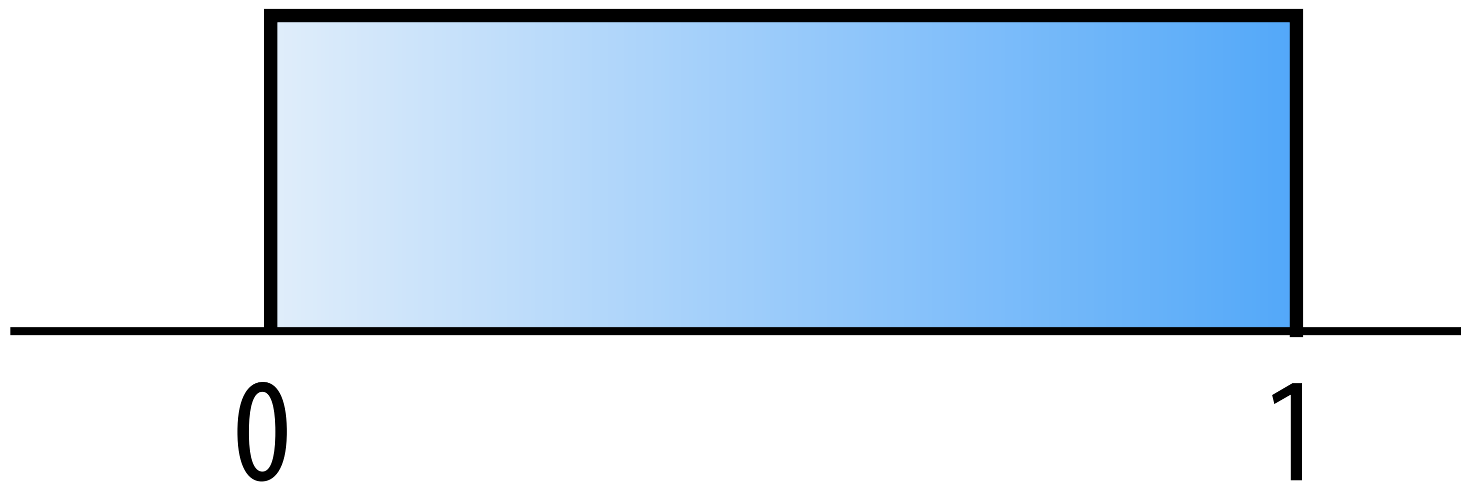
$$P(1) = 1$$

$$\Pr(a \leq X \leq b) = \int_a^b p(x)\,\mathrm{d}x$$

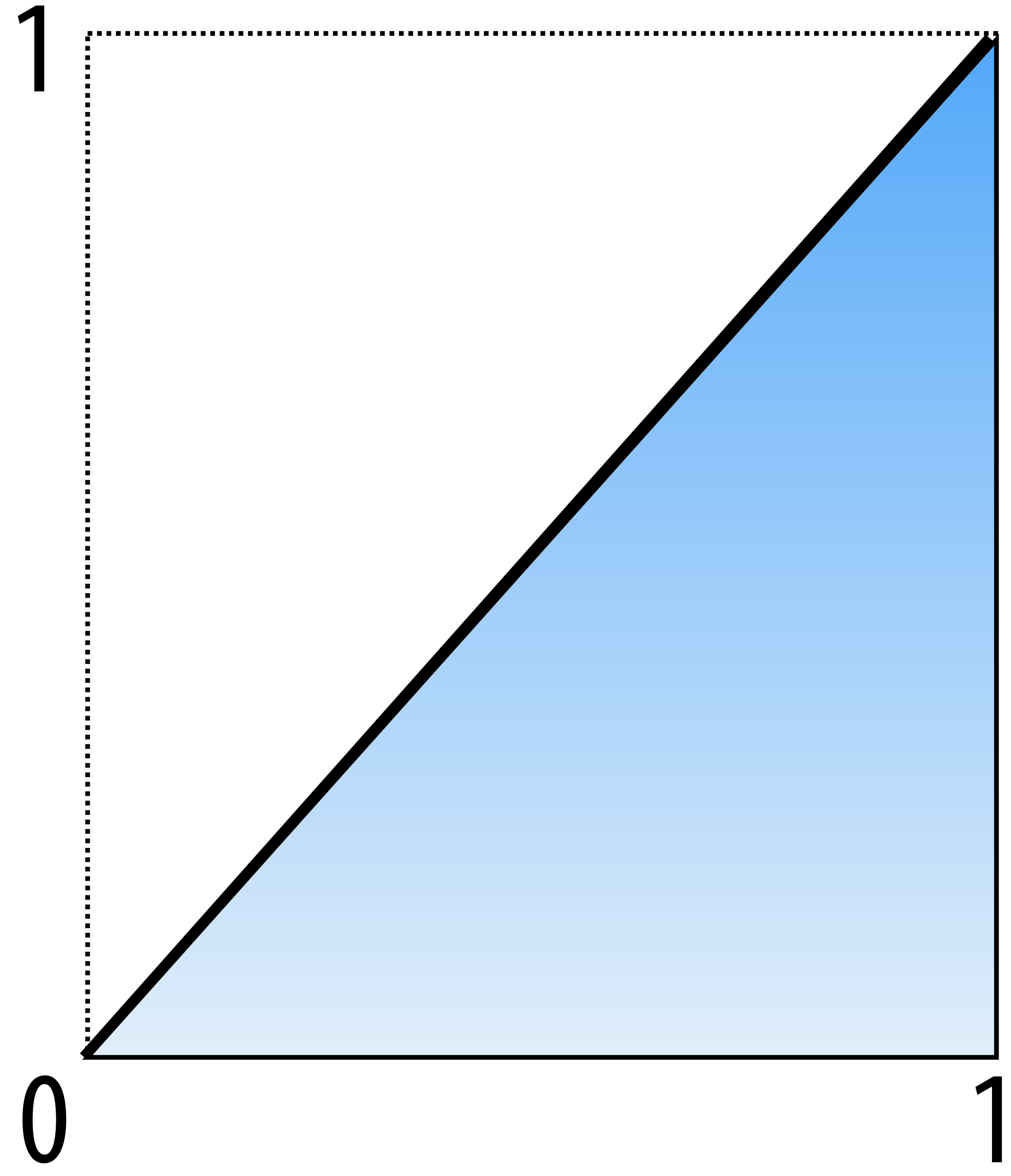
$$= P(b) - P(a)$$

Uniform distribution: p(x) = c

(for random variable X defined on [0,1] domain)



CDF P(x)



Sampling continuous random variables using the inversion method

Cumulative probability distribution function

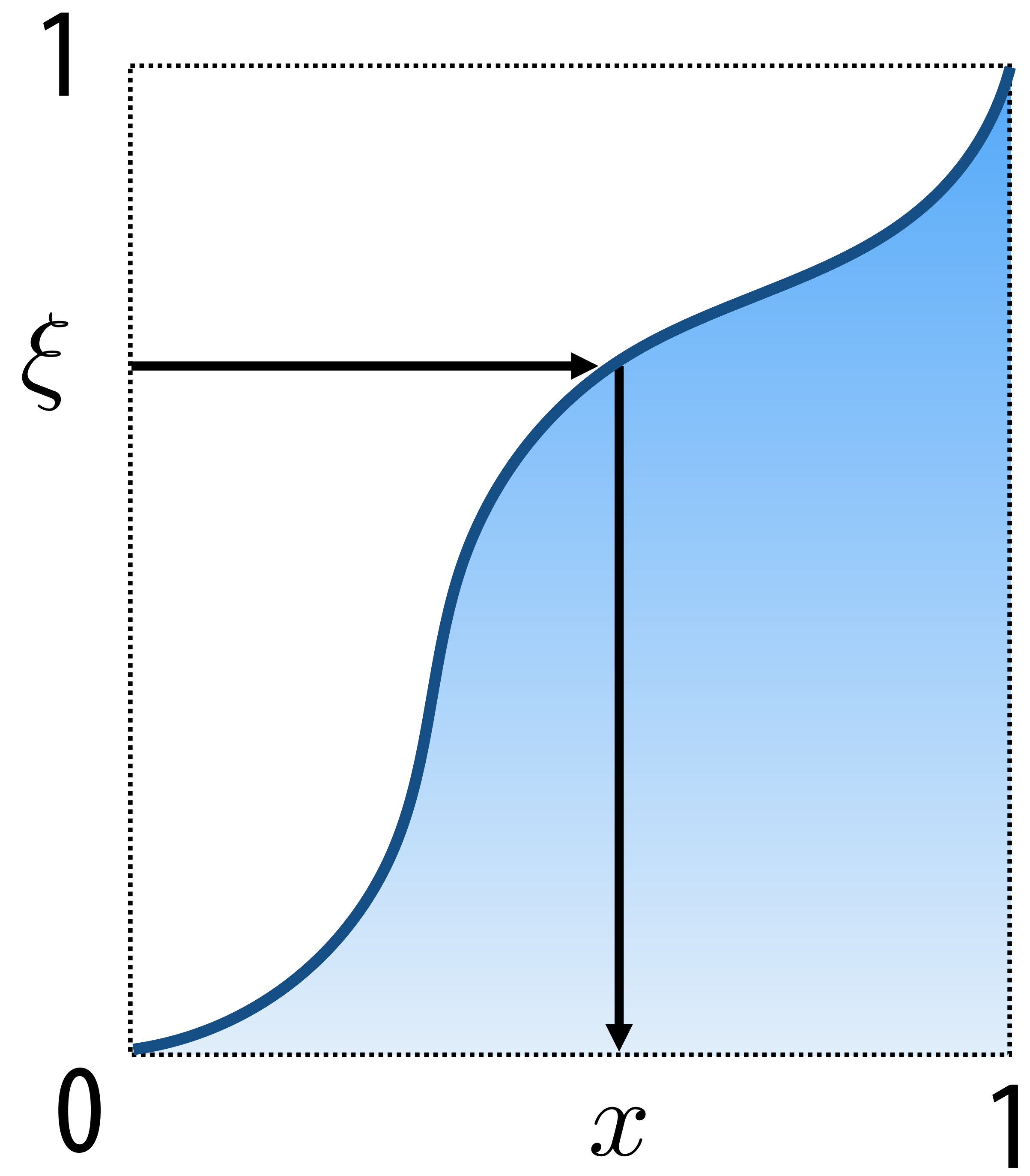
$$P(x) = \Pr(X < x)$$

Construction of samples:

Solve for $x = P^{-1}(\xi)$

Must know the formula for:

1. **The integral of** $p(x)$
2. **The inverse function** $P^{-1}(x)$

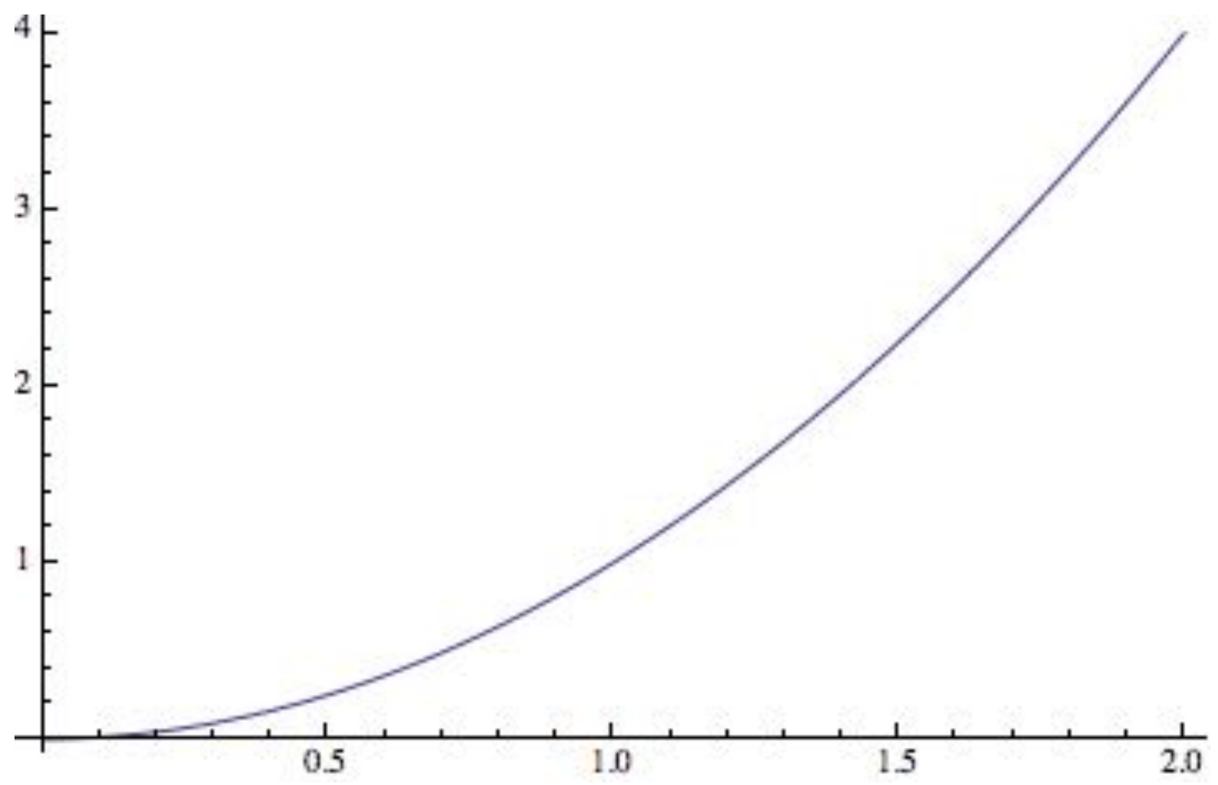


Example: applying the inversion method

Given:

Relative density of probability of random variable taking on value *x* over [0,2] domain

$$f(x) = x^2 \qquad x \in [0, 2]$$



Compute PDF from f(x):

$$\begin{aligned} 1 &= \int_0^2 c\, f(x)\, \mathrm{d}x \\ &= c(F(2) - F(0)) \qquad\qquad F(x) = \frac{1}{3}x^3 \\ &= c\frac{1}{3}2^3 \\ &= \frac{8c}{3} \longrightarrow \quad c = \frac{3}{8}, \qquad p(x) = \frac{3}{8}x^2 \end{aligned}$$

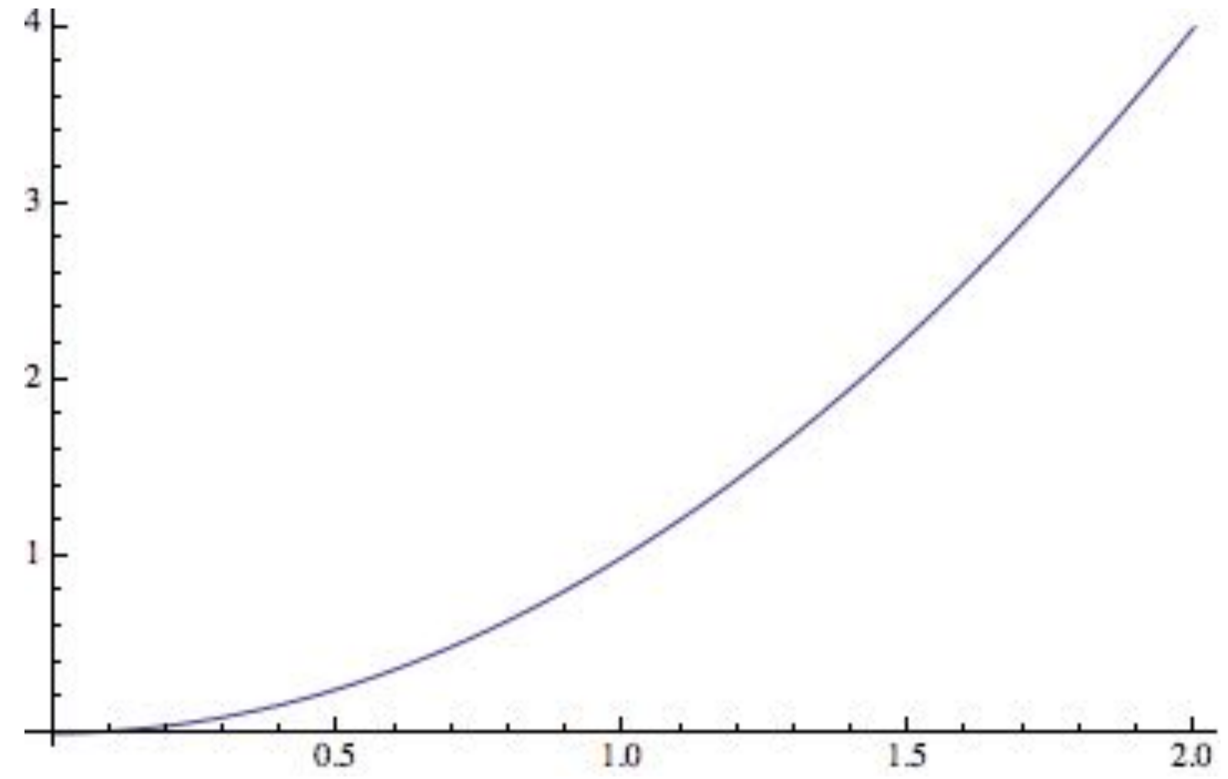
Probability density function (integrates to 1)

Example: applying the inversion method

Given:

$$f(x) = x^2 \quad x \in [0, 2]$$

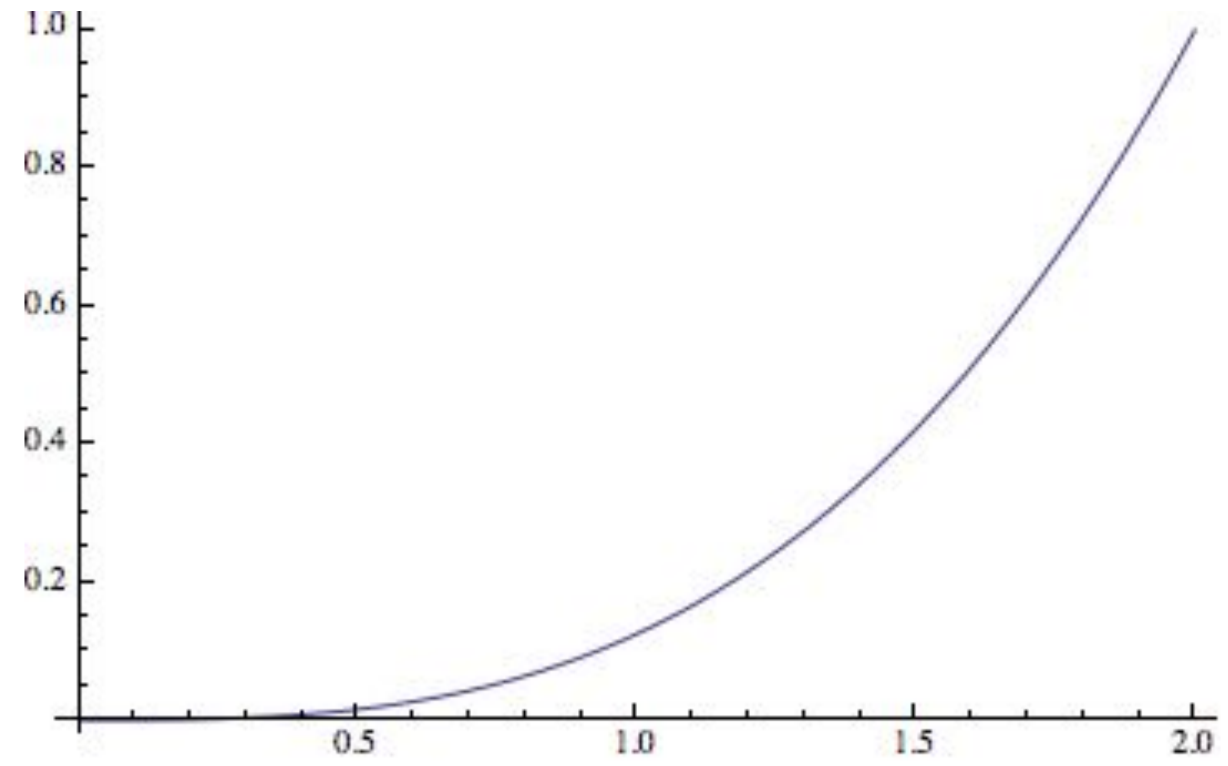
$$p(x) = \frac{3}{8}x^2$$



Compute CDF:

$$P(x) = \int_0^x p(x)\,\mathrm{d}x$$

$$= \frac{x^3}{8}$$



Example: applying the inversion method

Given:

$$f(x) = x^2 \qquad x \in [0, 2]$$

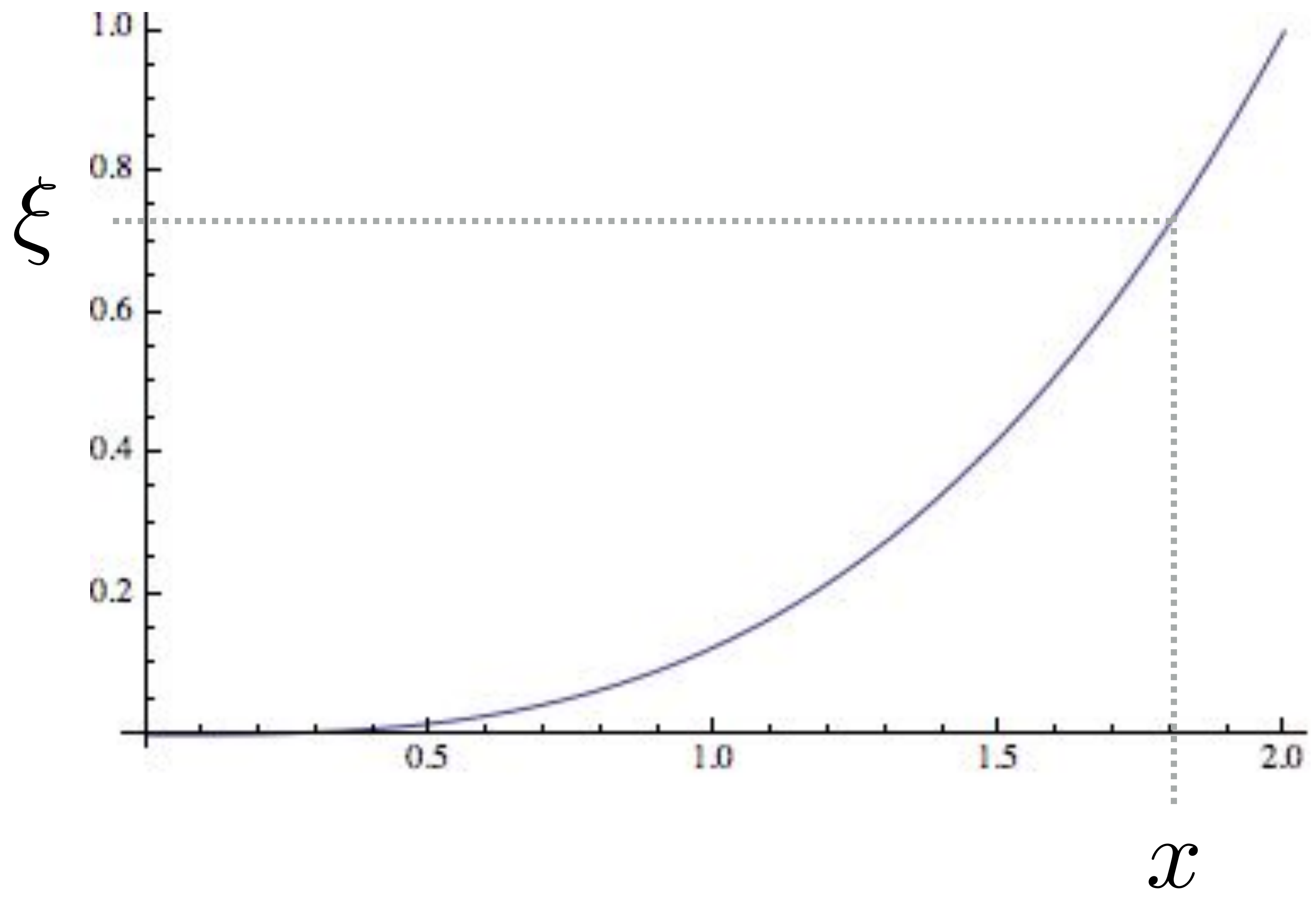
$$p(x) = \frac{3}{8}x^2$$

$$P(x) = \frac{x^3}{8}$$

Sample from $p(x)$

$$\xi = P(x) = \frac{x^3}{8}$$

$$x = \sqrt[3]{8\xi}$$



How do we uniformly sample the unit circle?

(Choose any point P=(px, py) in circle with equal probability)

Uniformly sampling unit circle: first try

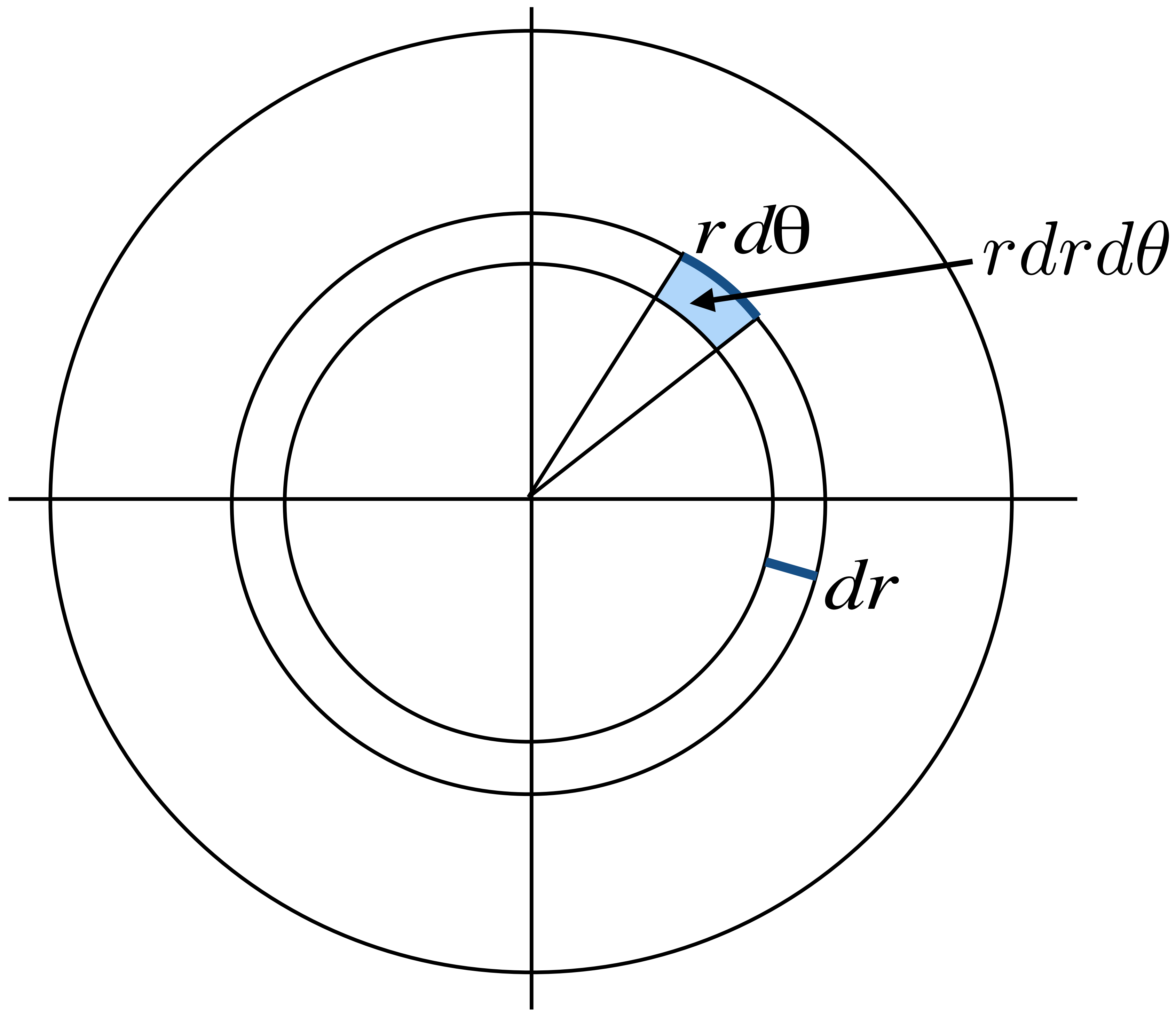
- θ = uniform random angle between 0 and 2π
- r = uniform random radius between 0 and 1
- **Return point:** $(r\cos\,\theta, r\sin\,\theta)$

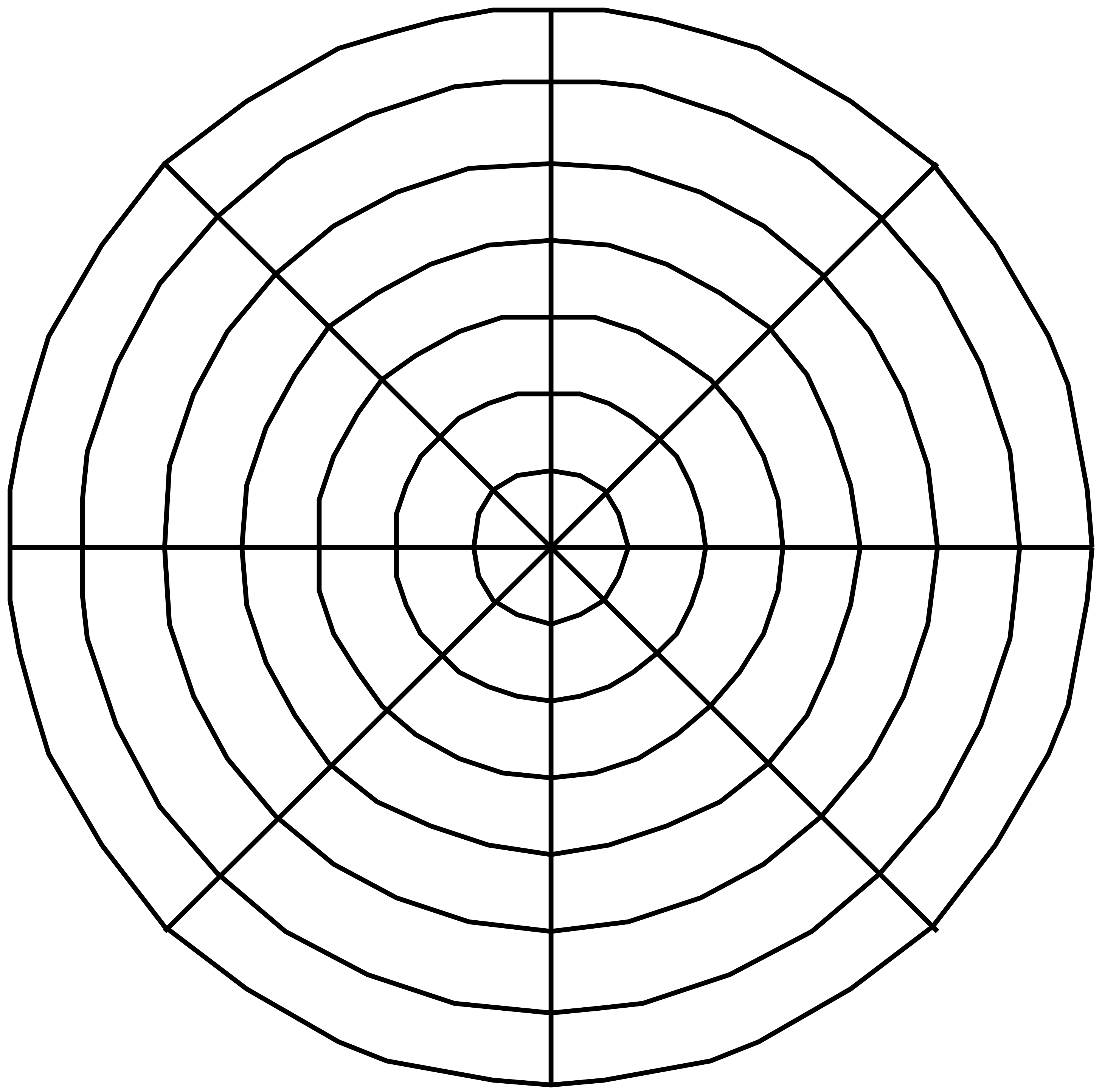
This algorithm <u>does not</u> produce the desired uniform sampling of the area of a circle.

Why?

Because sampling is not uniform in area!

Points farther from center of circle are less likely to be chosen



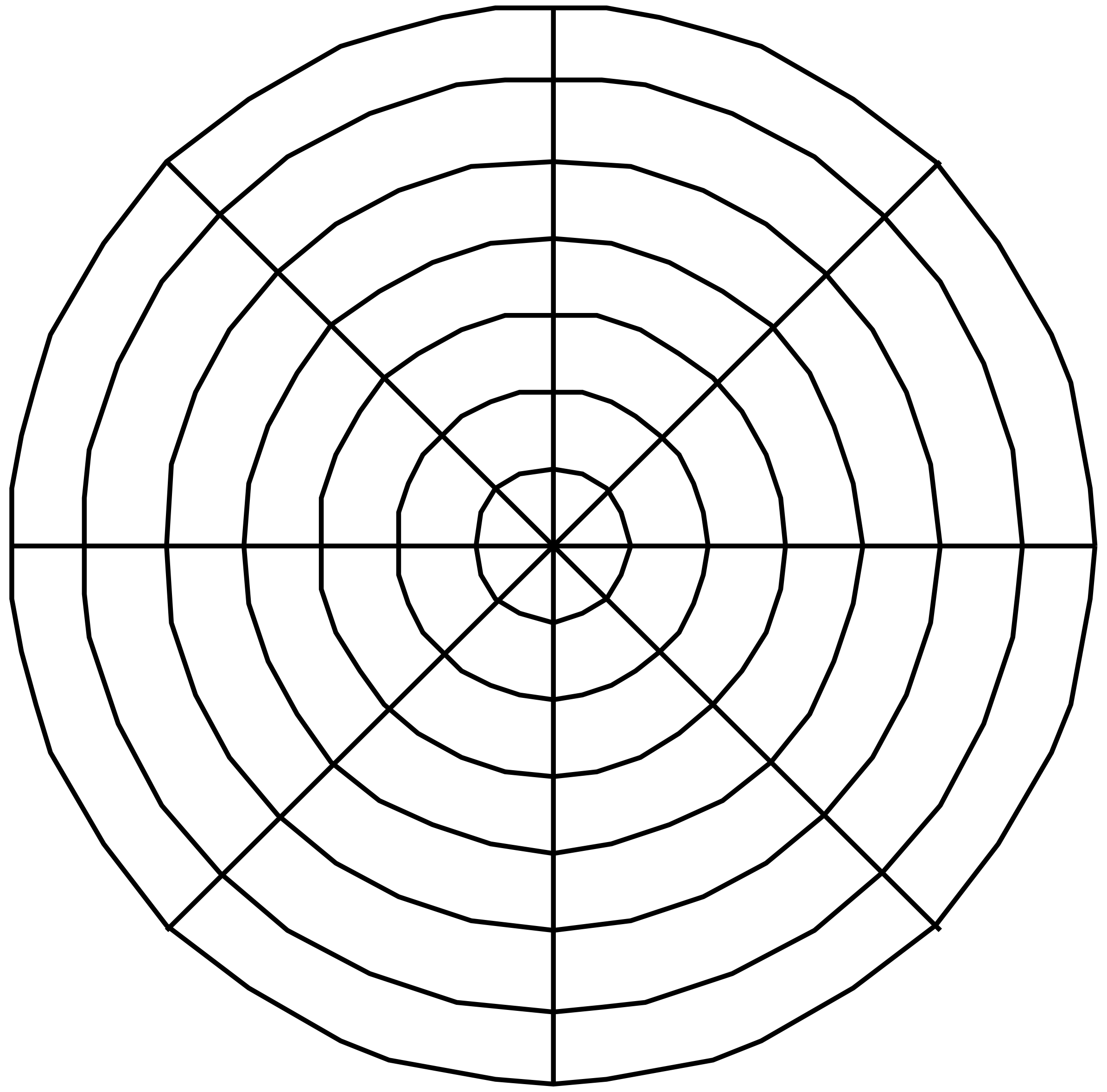


$$\theta = 2\pi\xi_1 \qquad r = \xi_2$$

$$p(r,\theta)drd\theta \sim rdrd\theta$$
$$p(r,\theta) \sim r$$

Uniform area sampling of a circle

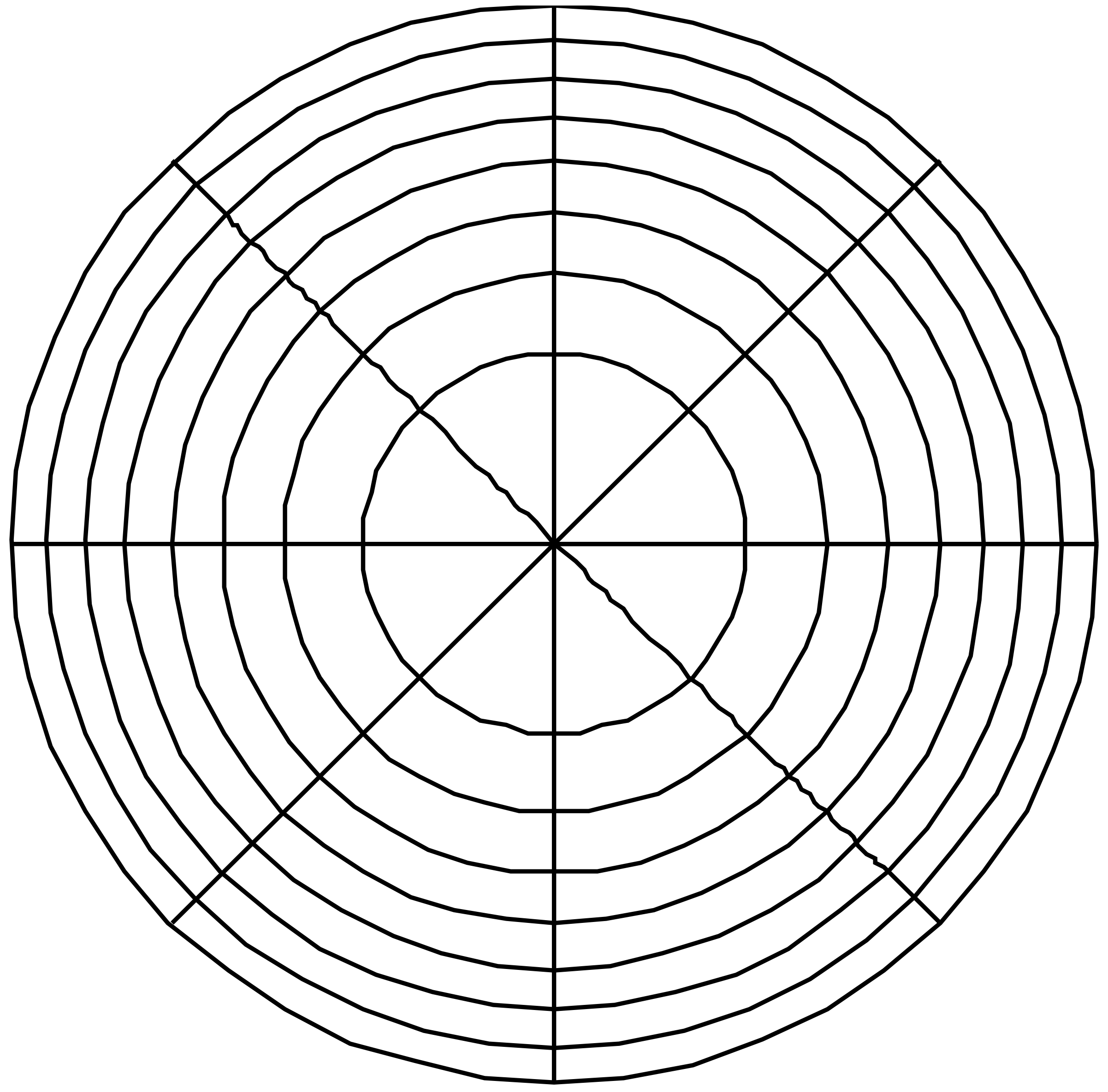
WRONG
Not Equi-areal



$$\theta = 2\pi\xi_1$$

$$r = \xi_2$$

RIGHT
Equi-areal



$$\theta = 2\pi\xi_1$$

$$r = \sqrt{\xi_2}$$

Sampling a circle (via inversion in 2D)

$$A = \int_0^{2\pi} \int_0^1 r \, \mathrm{d}r \, \mathrm{d}\theta = \int_0^1 r \, \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta = \left(\frac{r^2}{2} \right) \Big|_0^1 \theta \Big|_0^{2\pi} = \pi$$

$$p(r, \theta) \, \mathrm{d}r \, \mathrm{d}\theta = \frac{1}{\pi} r \, \mathrm{d}r \, \mathrm{d}\theta \rightarrow p(r, \theta) = \frac{r}{\pi}$$

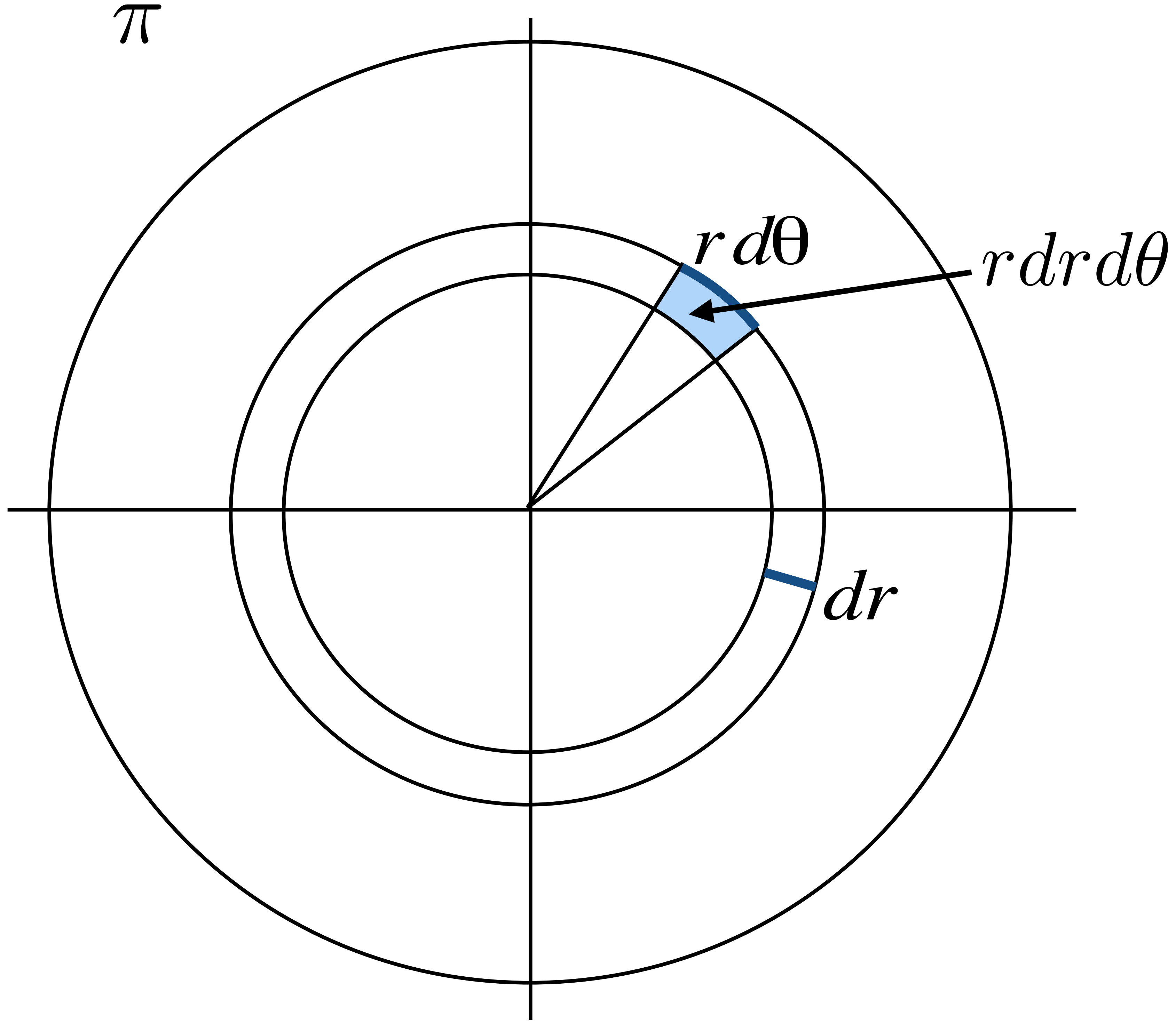
$$p(r, \theta) = p(r)p(\theta)$$ ⟵ r, θ **independent**

$$p(\theta) = \frac{1}{2\pi}$$

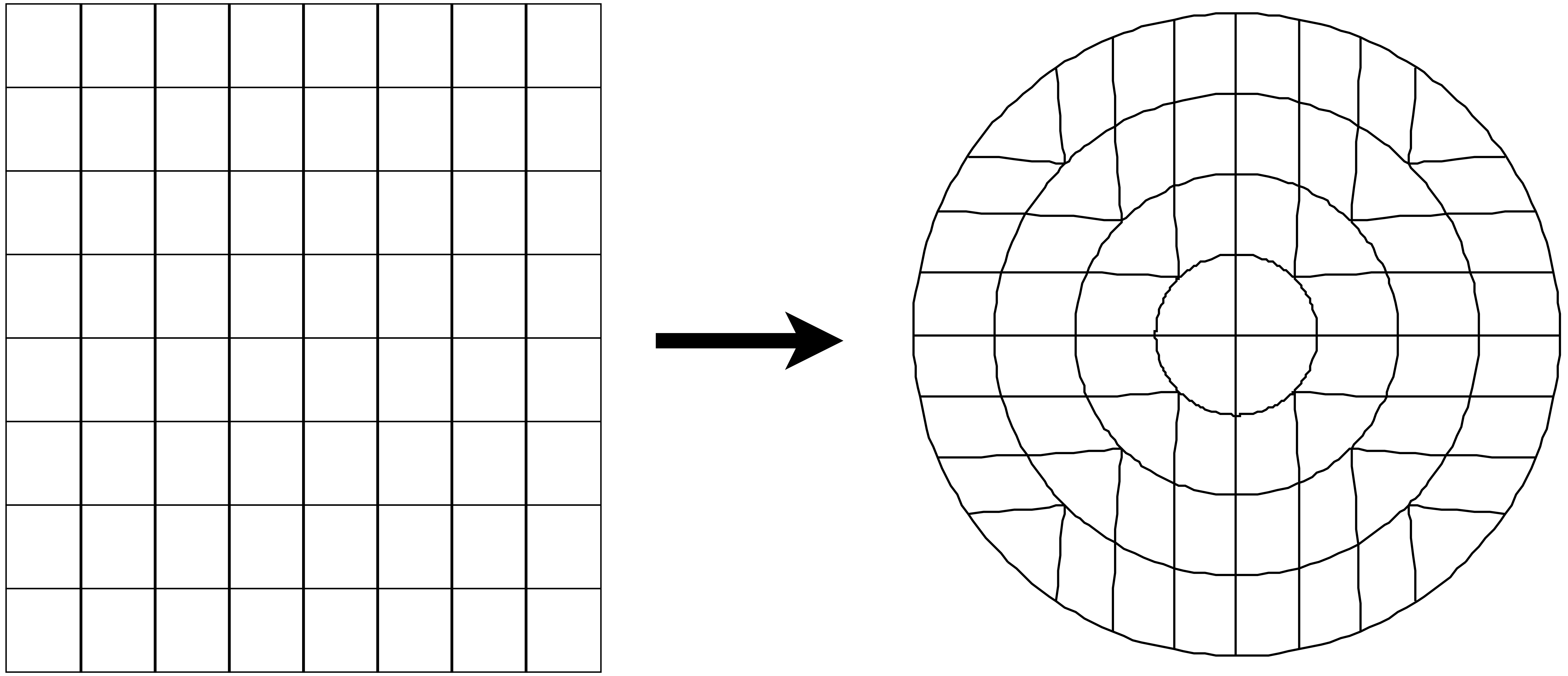
$$P(\theta) = \frac{1}{2\pi}\theta \qquad \theta = 2\pi\xi_1$$

$$p(r) = 2r$$

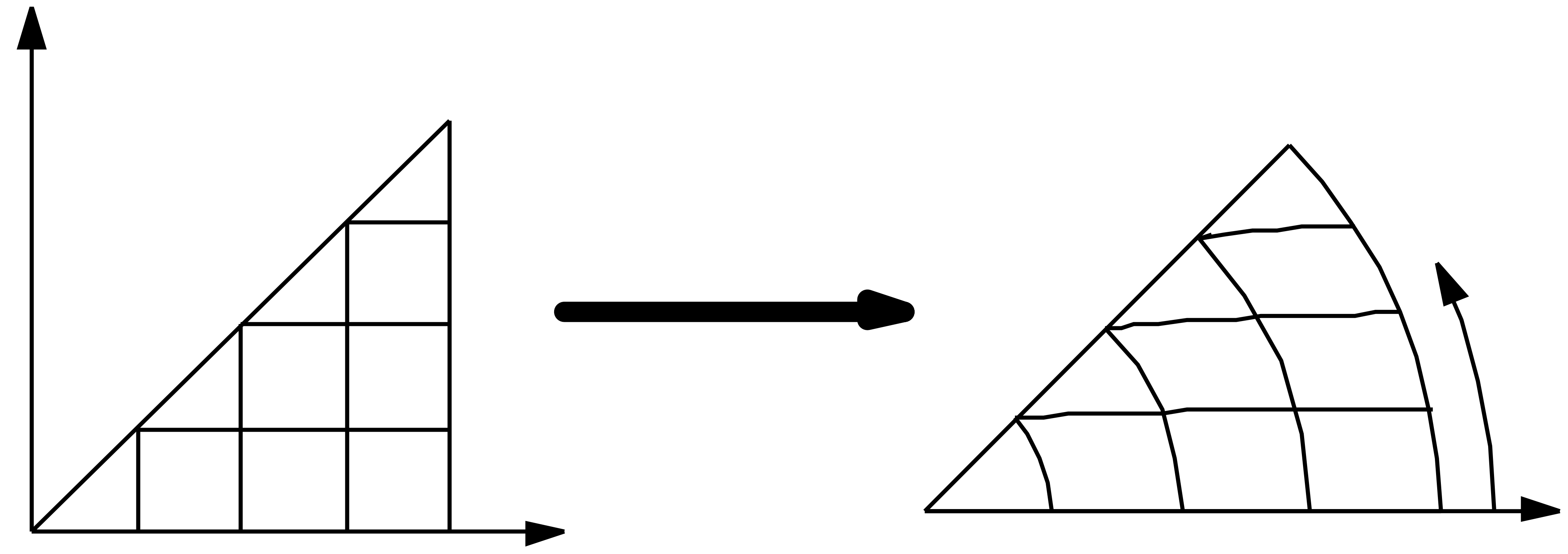
$$P(r) = r^2 \qquad r = \sqrt{\xi_2}$$



Shirley's mapping



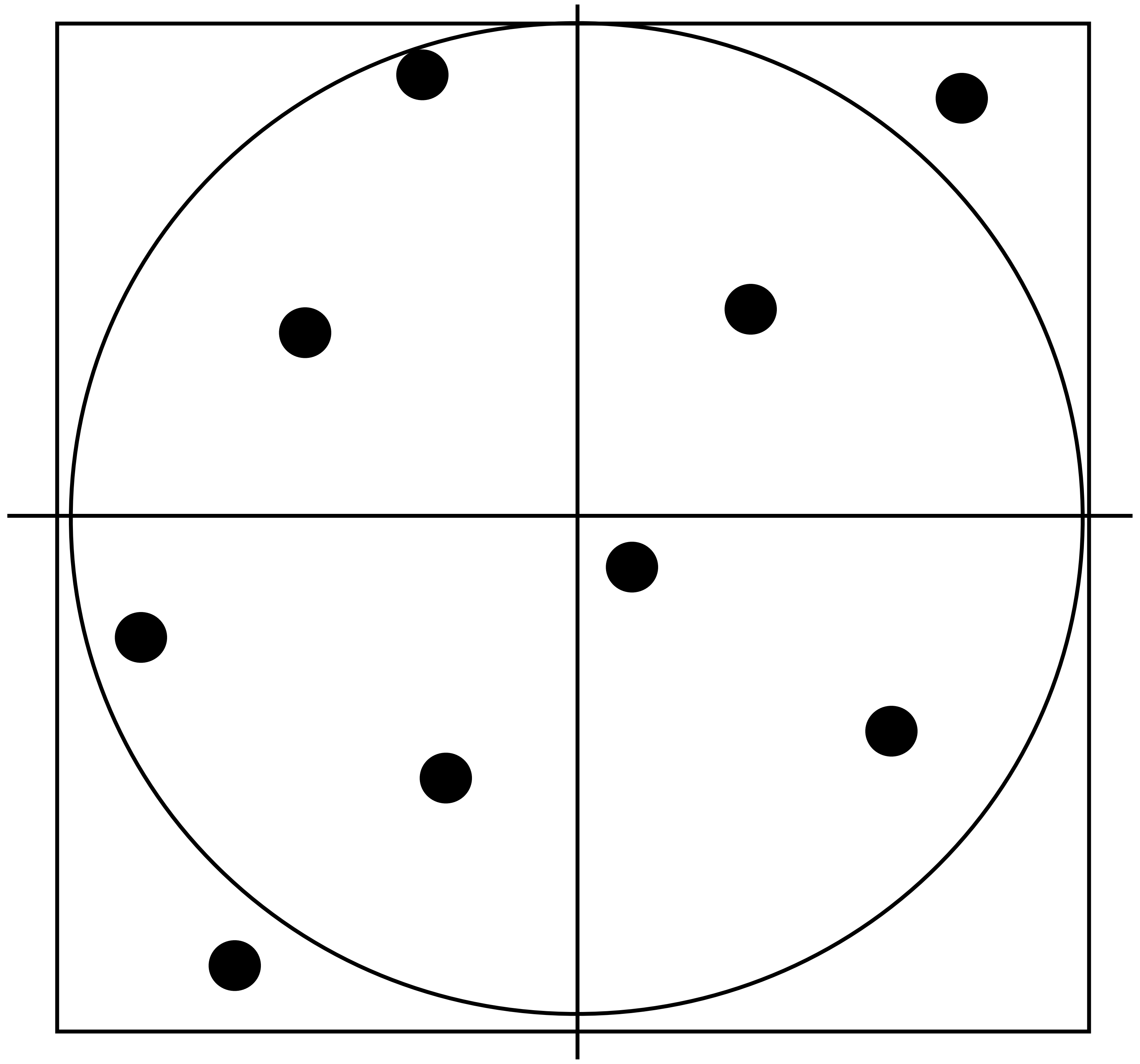
Distinct cases for eight octants



$$r = \xi_1$$

$$\theta = \frac{\pi \xi_2}{4r}$$

Uniform sampling via rejection sampling

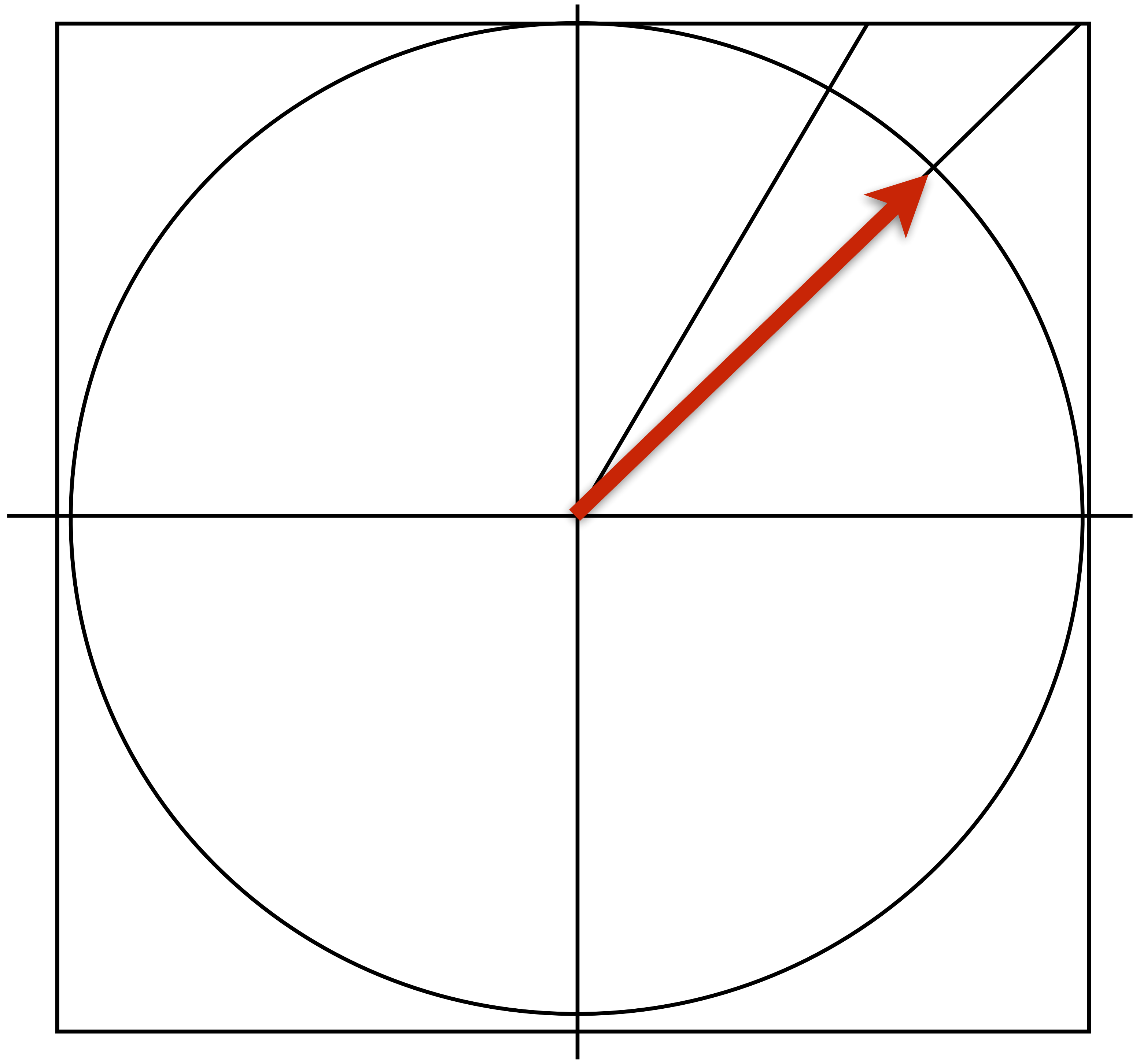


Generate random point within unit circle

```
do {
  x = uniform(-1,1);
  y = uniform(-1,1);
} while (x*x + y*y > 1.);
```

Efficiency of technique: area of circle / area of square

Rejection sampling to generate 2D directions



Goal: generate random directions in 2D with uniform probability

```
x = uniform(-1,1);
y = uniform(-1,1);

r = sqrt(x*x+y*y);
x_dir = x/r;
y_dir = y/r;
```

This algorithm is not correct! What is wrong?
What's a better algorithm?

Monte Carlo integration

- **Definite integral**

  What we seek to estimate

$$\int_a^b f(x)dx$$

- **Random variables**

  X_i **is the value of a random sample drawn from the distribution** $p(x)$
  Y_i **is also a random variable.**

$$X_i \sim p(x)$$
$$Y_i = f(X_i)$$

- **Expectation of *f***

$$E[Y_i] = E[f(X_i)] = \int_a^b f(x)\,p(x)\,\mathrm{d}x$$

- **Estimator**

  Monte Carlo estimate of $\int_a^b f(x)dx$

  Assuming samples X_i **drawn from uniform pdf. I will provide estimator for arbitrary PDFs later in lecture.**

$$F_N = \frac{b-a}{N}\sum_{i=1}^{N} Y_i$$

Basic unbiased Monte Carlo estimator

Unbiased estimator: Expected value of estimator is the integral we wish to evaluate.

$$E[F_N] = E\left[\frac{b-a}{N}\sum_{i=1}^{N} Y_i\right]$$

$$= \frac{b-a}{N}\sum_{i=1}^{N} E[Y_i] = \frac{b-a}{N}\sum_{i=1}^{N} E[f(X_i)]$$

$$= \frac{b-a}{N}\sum_{i=1}^{N} \int_a^b f(x)\, p(x) \mathrm{d}x$$

$$= \frac{1}{N}\sum_{i=1}^{N} \int_a^b f(x)\, \mathrm{d}x$$

$$= \int_a^b f(x)\, \mathrm{d}x$$

Assume uniform probability density for now

$$X_i \sim U(a,b)$$

$$p(x) = \frac{1}{b-a}$$

Properties of expectation:

$$E\left[\sum_i Y_i\right] = \sum_i E[Y_i]$$

$$E[aY] = aE[Y]$$

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