Lecture 8:

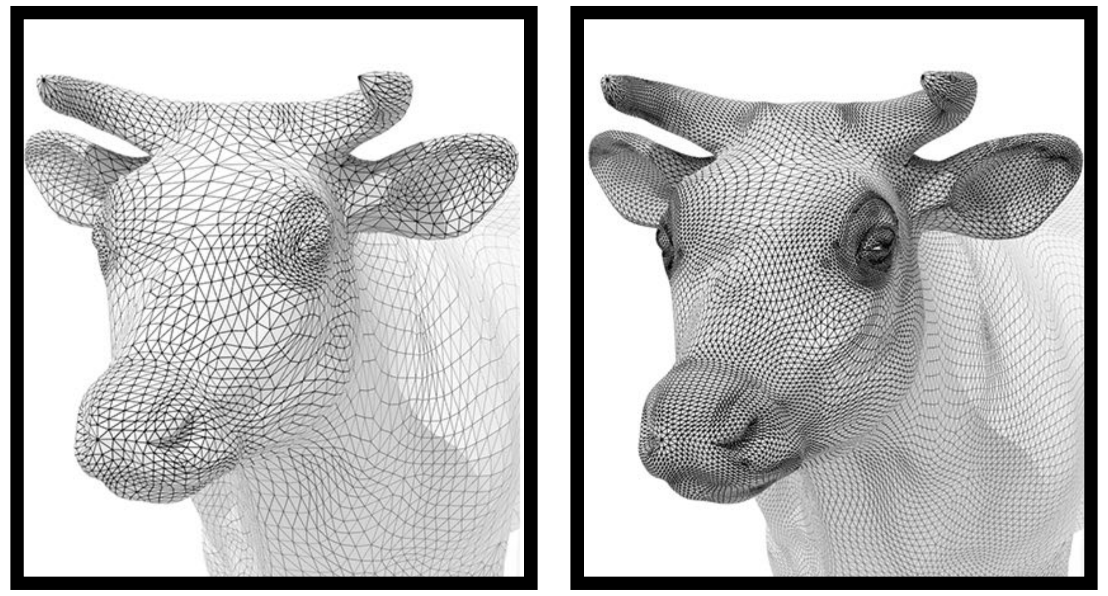
Geometric Queries

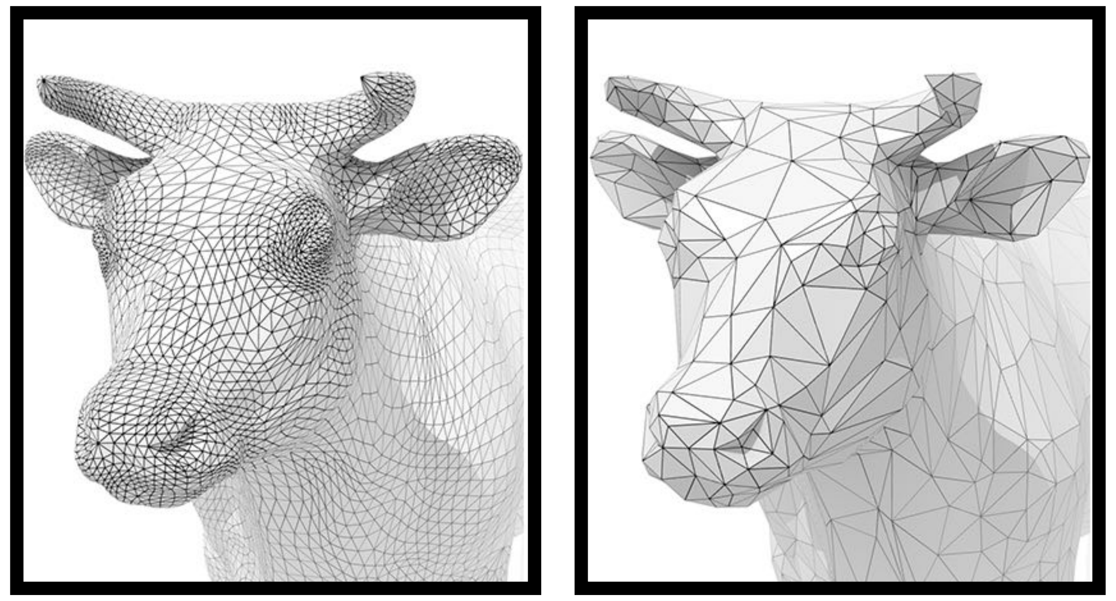
Computer Graphics: Rendering, Geometry, and Image Manipulation
Stanford CS248A, Winter 2025

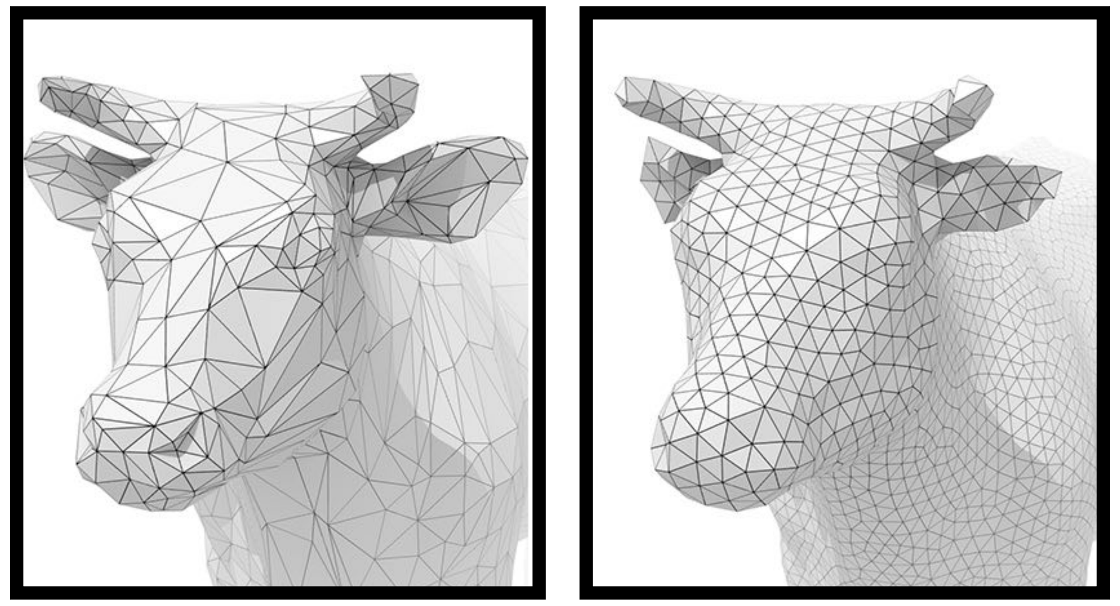
Last time

How to perform a number of basic mesh processing operations

- **Subdivision (upsampling)**
- **Mesh simplification (downsampling)**
  - **In supplemental video**
- **Mesh resampling**
  - **In supplemental video**

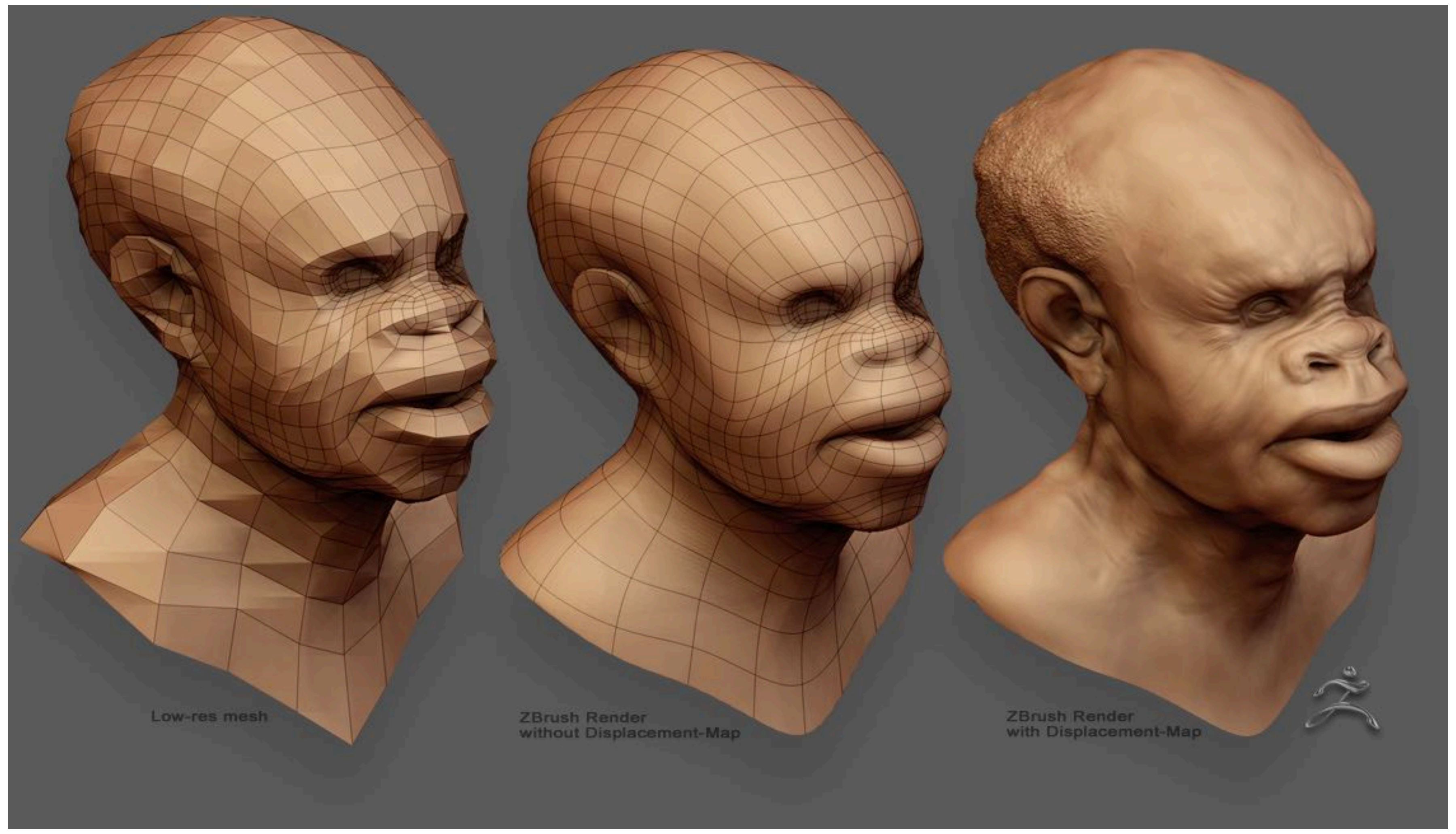






Geometric queries — motivation

Intersecting rays and triangles (ray tracing)



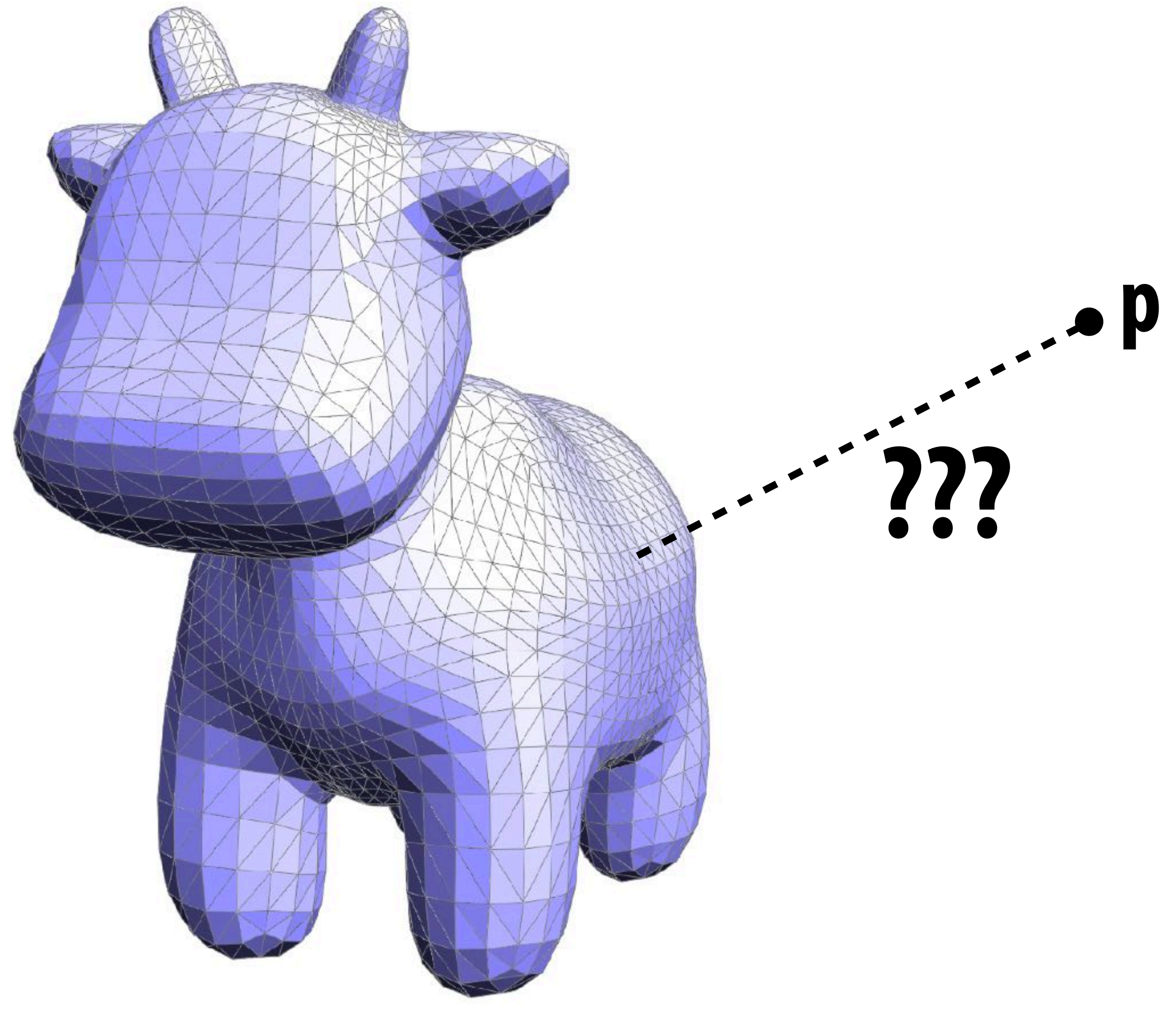
Closest point on surface queries

Intersecting triangles (collisions)

Closest point queries

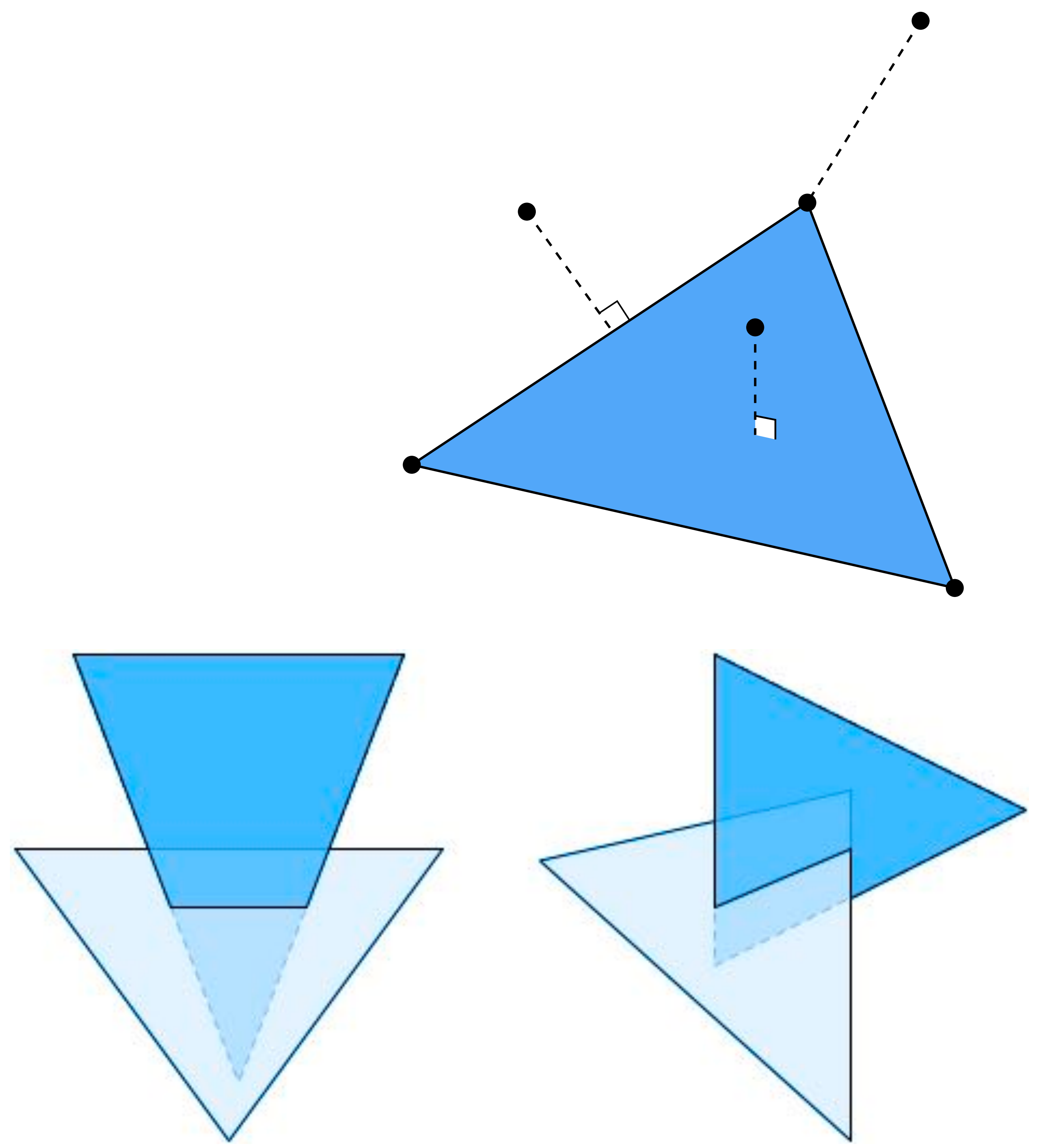
Given a point, in space (e.g., a new sample point), how do we find the closest point on a given surface?

- Q: Does implicit/explicit representation make this easier?
- Q: Does our half-edge data structure help?
- Q: What's the cost of the naïve algorithm?
- Q: How do we find the distance to a single triangle anyway?



Many types of geometric queries

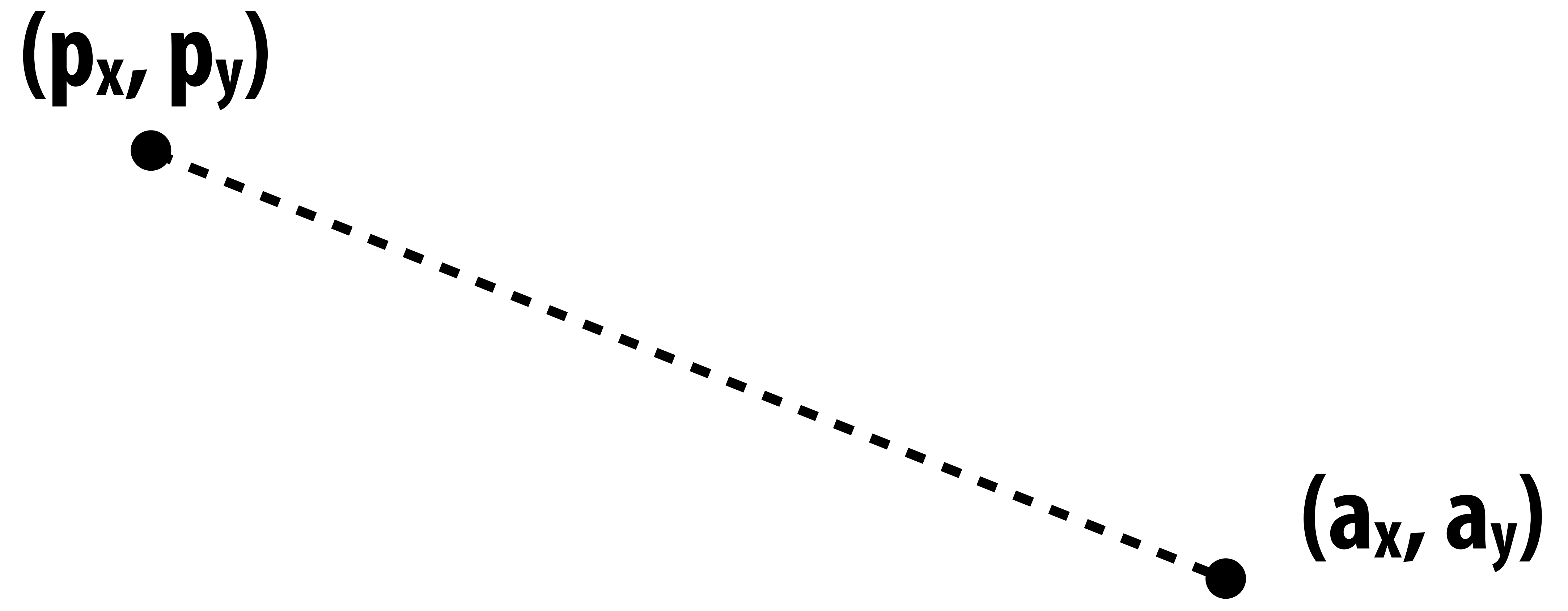
- Plenty of other things we might like to know:
  - Do two triangles intersect?
  - Are we inside or outside an object?
  - Does one object contain another?
  - ...



- Data structures we've seen so far not really designed for this...
- Need some new ideas!
- TODAY: come up with simple (aka: slow) algorithms
- NEXT TIME: intelligent ways to accelerate geometric queries

Warm up: closest point on point

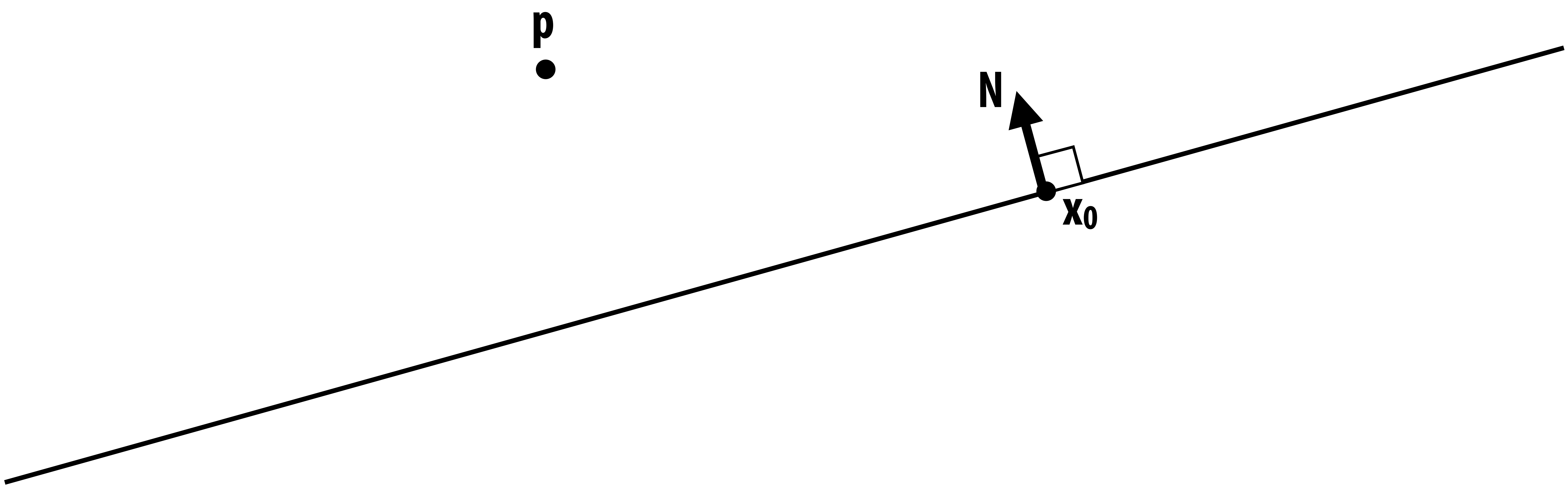
Given a query point (p_x,p_y), how do we find the closest point on the point (a_x,a_y)?



Bonus question: what's the distance?

Slightly harder: closest point on line

- Now suppose I have a line $N^T x = c$, where N is the unit normal
  - Remember: a line is all points x such that $N^T x = c$
- How do I find the point on the line closest to my query point p?



Review: matrix form of a line (and a plane)

Line is defined by:

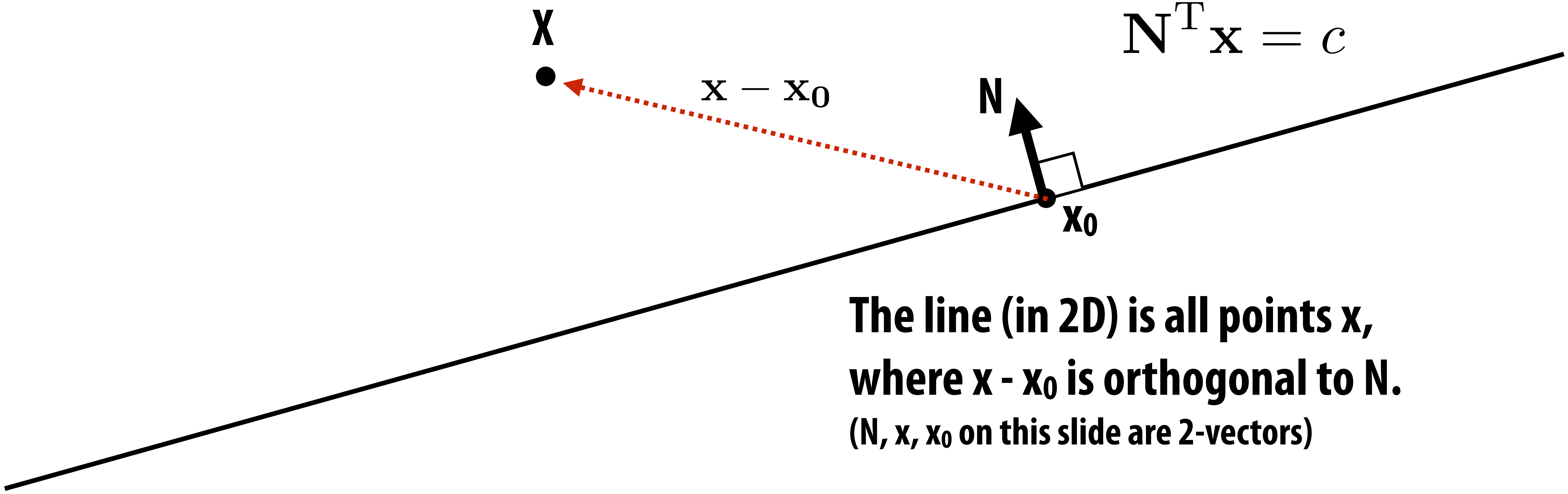
- Its normal: N
- A point x_0 on the line

$$\mathbf{N} \cdot (\mathbf{x} - \mathbf{x_0}) = 0$$

$$\mathbf{N}^{\mathrm{T}}(\mathbf{x} - \mathbf{x_0}) = 0$$

$$\mathbf{N}^{\mathrm{T}}\mathbf{x} = \mathbf{N}^{\mathbf{T}}\mathbf{x_0}$$

$$\mathbf{N}^{\mathrm{T}}\mathbf{x} = c$$



The line (in 2D) is all points x, where x - x_0 is orthogonal to N.

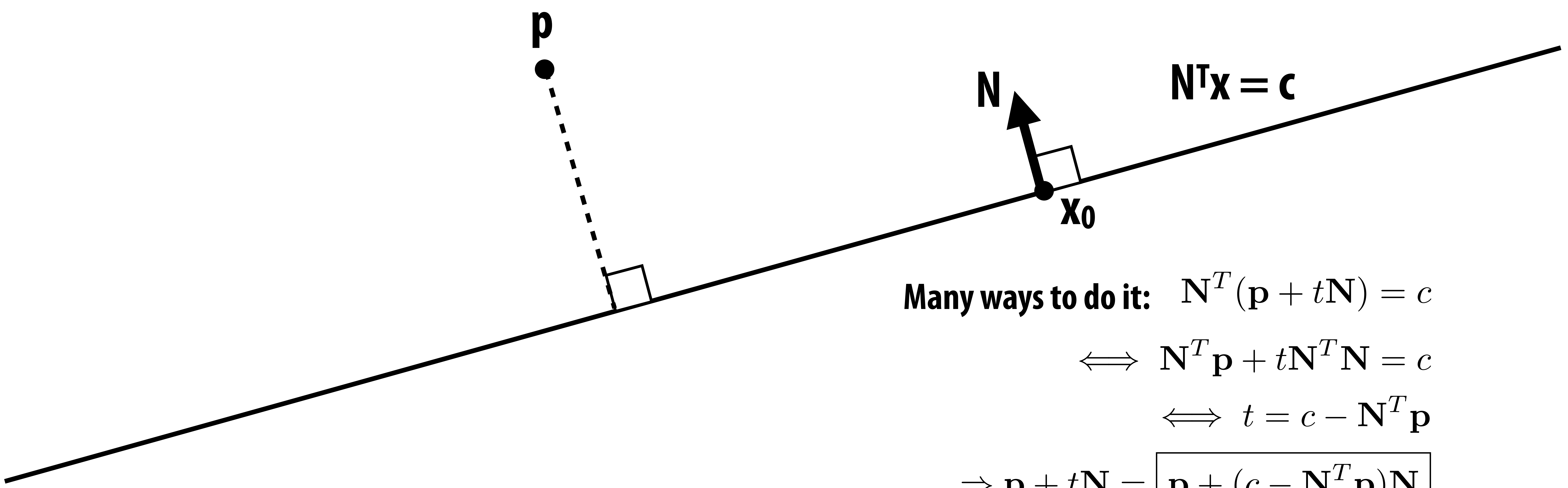
(N, x, x_0 on this slide are 2-vectors)

(And a plane (in 3D) is all points x where x - x_0 is orthogonal to N.)

(N, x, x_0 are 3-vectors)

Closest point on line

- Now suppose I have a line $N^T x = c$, where N is the unit normal
  - Remember: a line is all points x such that $N^T x = c$
- How do I find the point on line that is closest to my query point p?

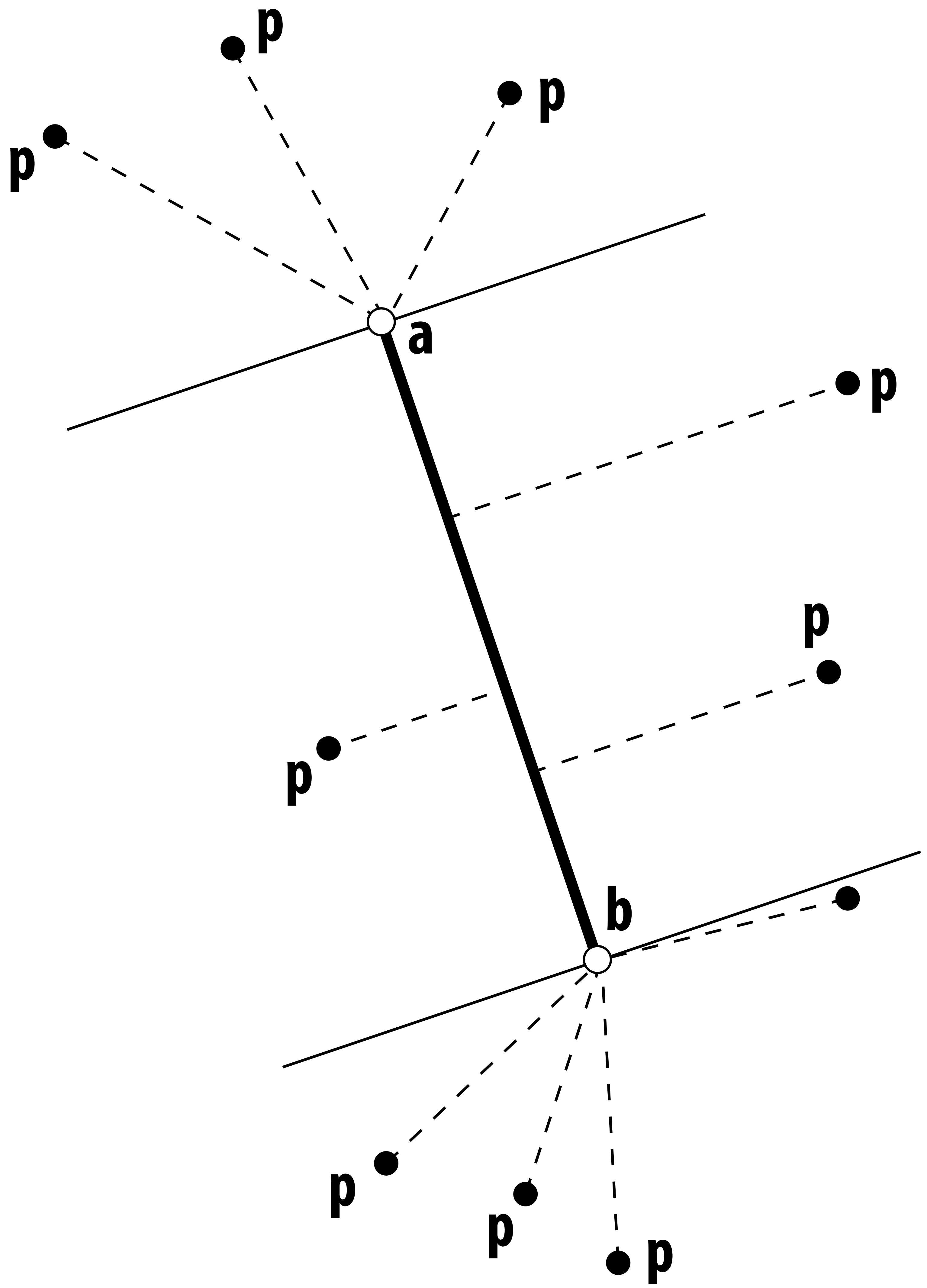


Many ways to do it:

$$\mathbf{N}^T(\mathbf{p} + t\mathbf{N}) = c$$
$$\iff \mathbf{N}^T\mathbf{p} + t\mathbf{N}^T\mathbf{N} = c$$
$$\iff t = c - \mathbf{N}^T\mathbf{p}$$
$$\Rightarrow \mathbf{p} + t\mathbf{N} = \boxed{\mathbf{p} + (c - \mathbf{N}^T\mathbf{p})\mathbf{N}}$$

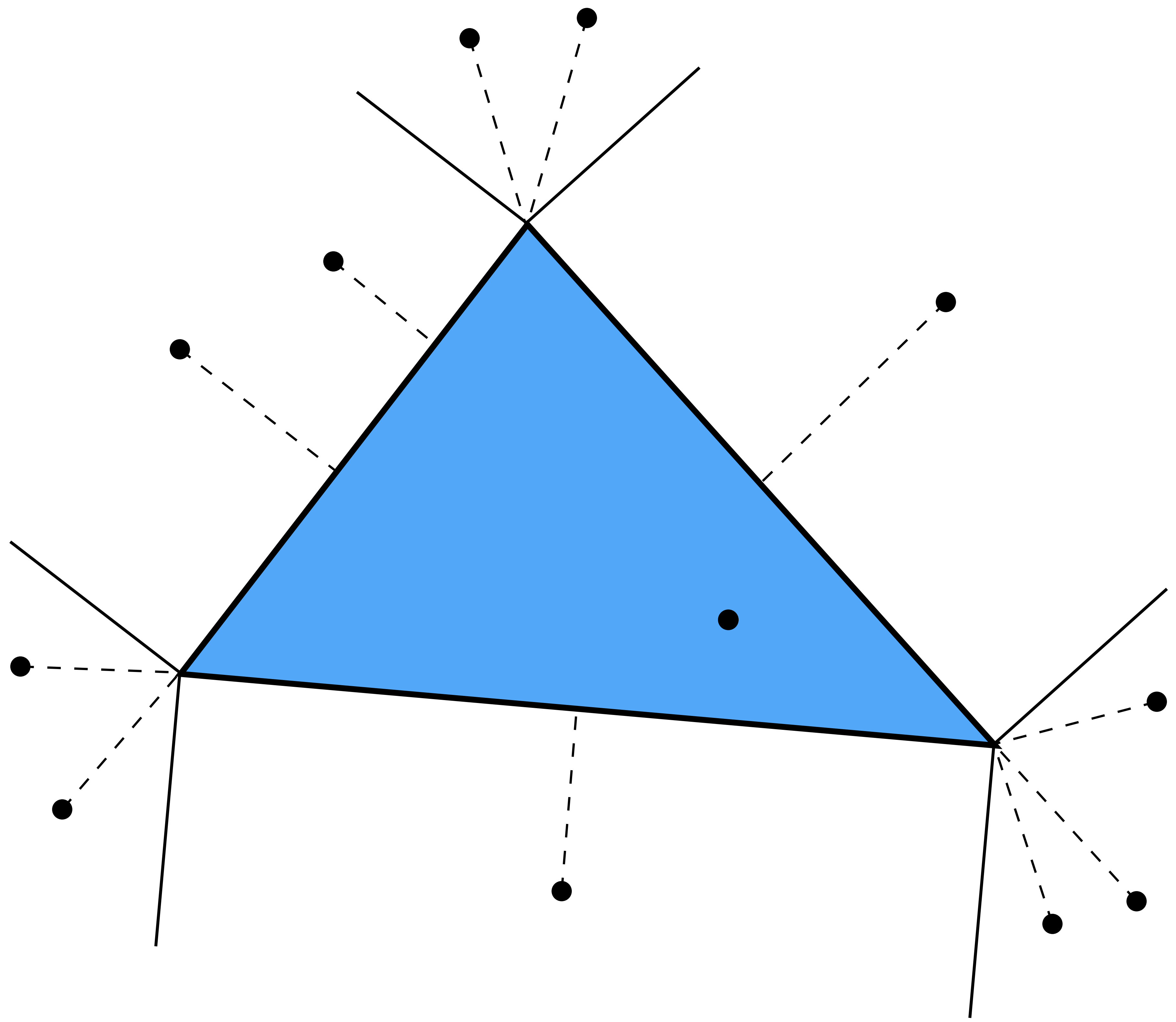
Harder: closest point on line segment

- Two cases: endpoint or interior
- Already have basic components:
  - point-to-point
  - point-to-line
- Algorithm?
  - find closest point on line
  - check if it is between endpoints
  - if not, take closest endpoint
- How do we know if it's between endpoints?
  - write closest point on line as a+t(b-a)
  - if t is between 0 and 1, it's inside the segment!



Even harder: closest point on triangle in 2D

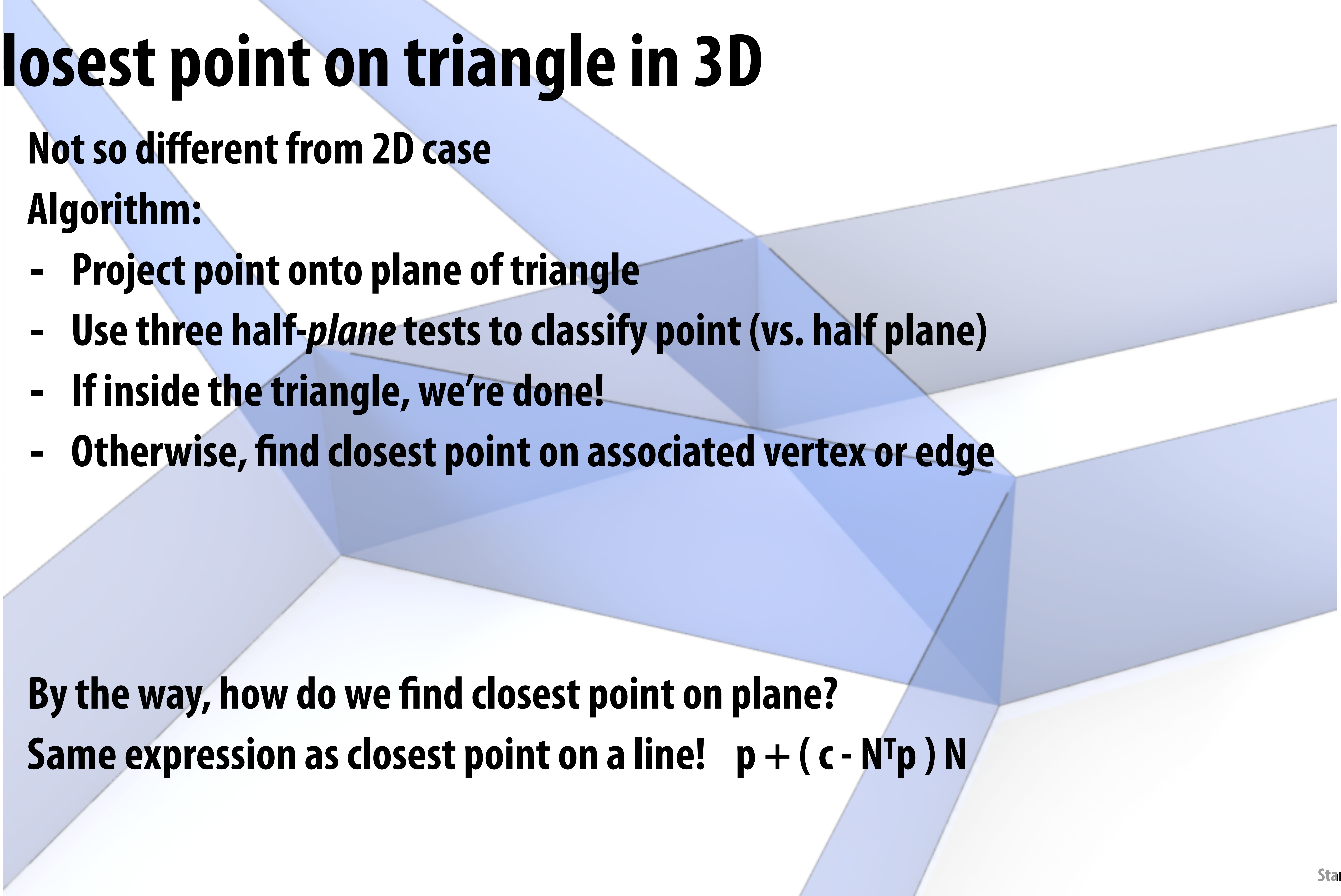
- What are all the possibilities for the closest point?
- Almost just minimum distance to three line segments:



Q: What about a point inside the triangle?

Closest point on triangle in 3D

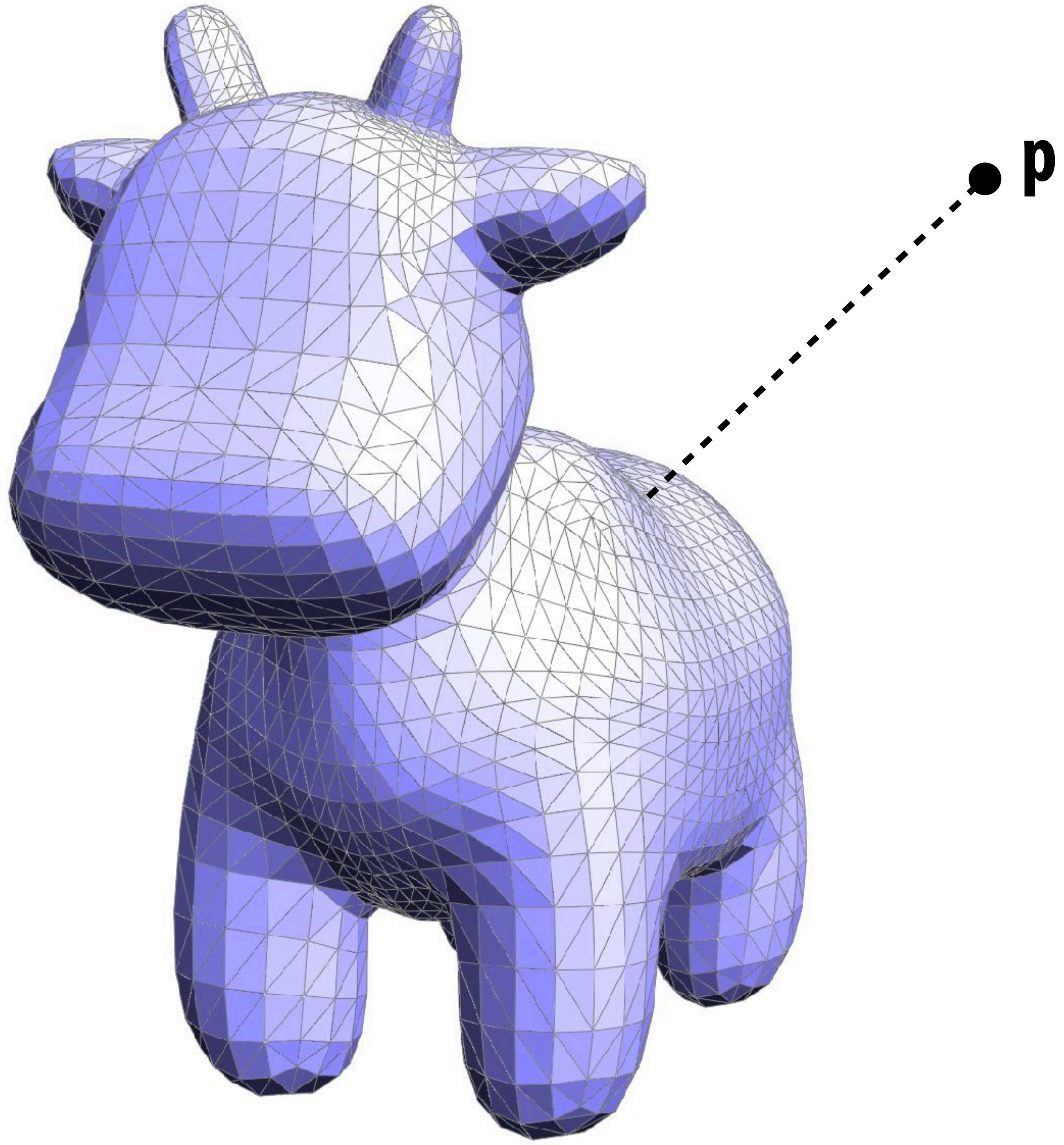
- Not so different from 2D case
- Algorithm:
    - Project point onto plane of triangle
    - Use three half-*plane* tests to classify point (vs. half plane)
    - If inside the triangle, we're done!
    - Otherwise, find closest point on associated vertex or edge

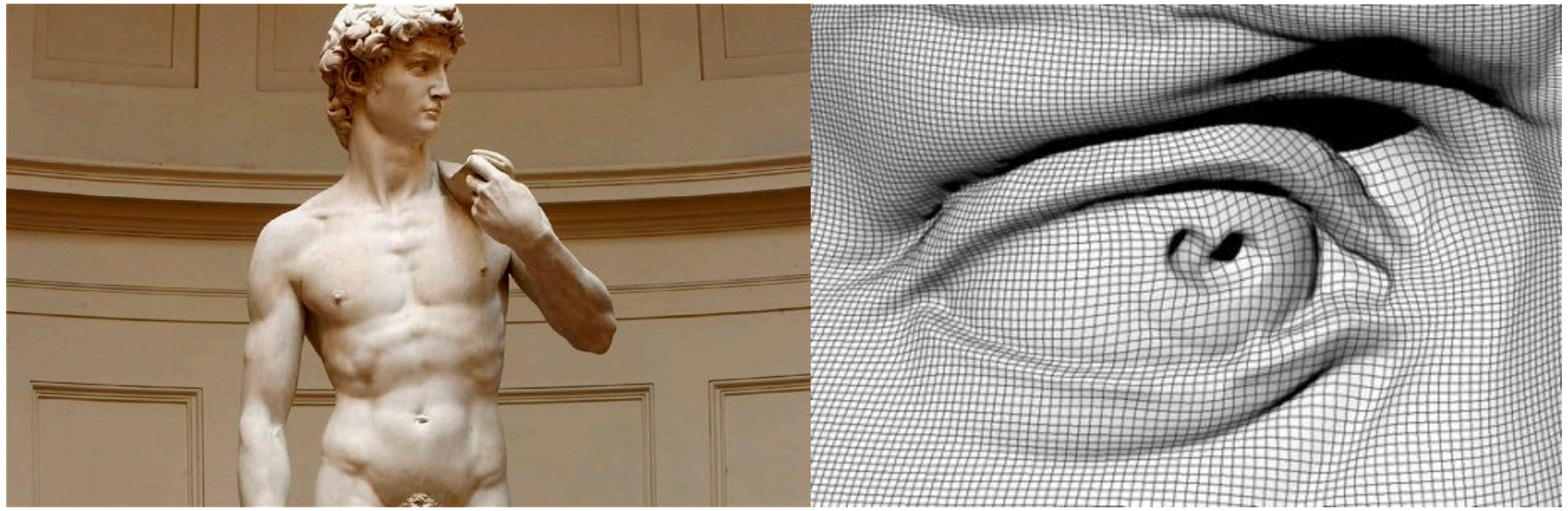


- By the way, how do we find closest point on plane?
- Same expression as closest point on a line! $p + (c - N^T p) N$

Closest point on triangle *mesh* in 3D?

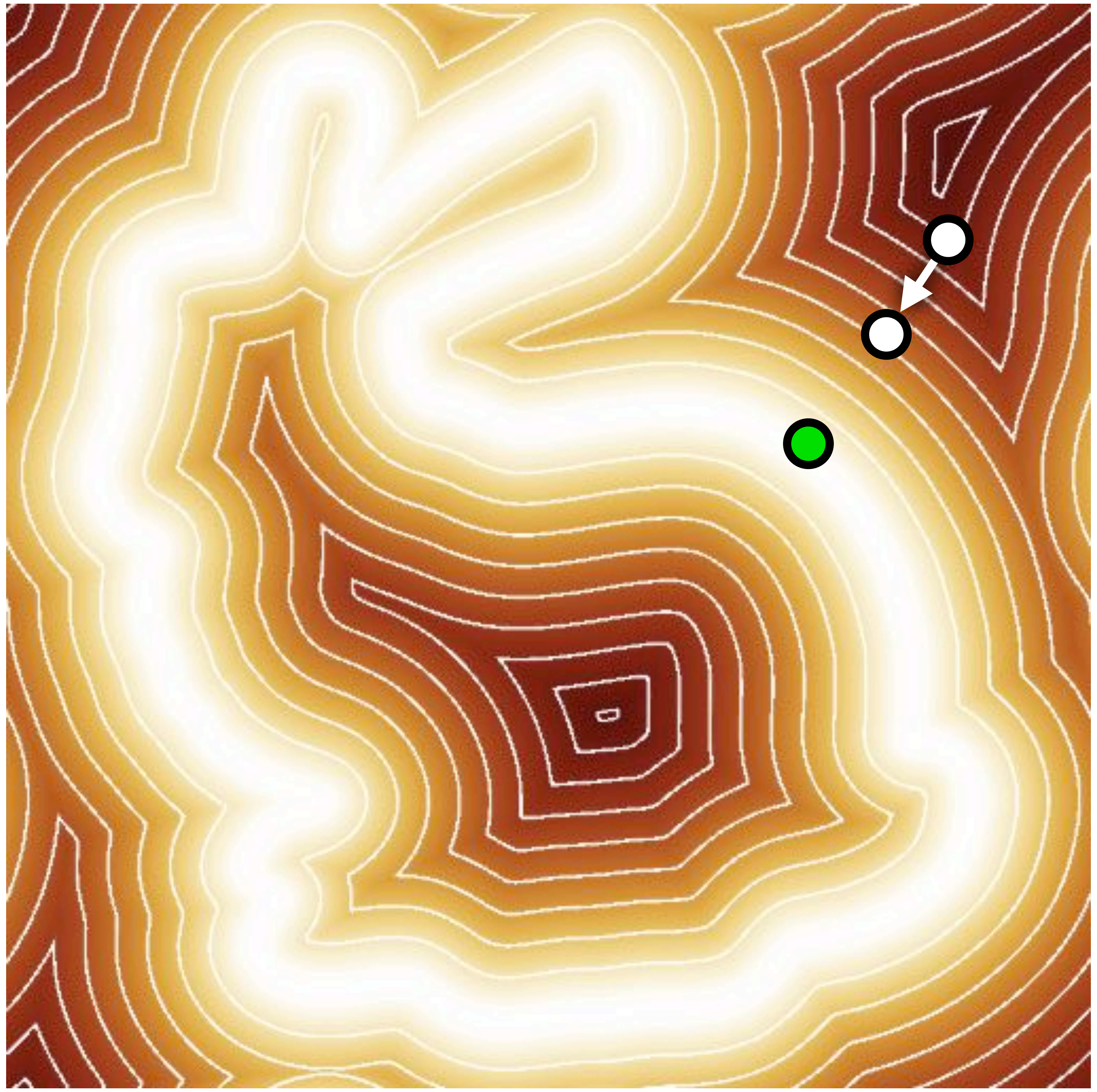
- **Conceptually easy:**
  - **loop over all triangles**
  - **compute closest point to current triangle**
  - **keep globally closest point**
- **Q: What's the cost?**
- **What if we have *billions* of faces?**
- **NEXT TIME: Better data structures!**





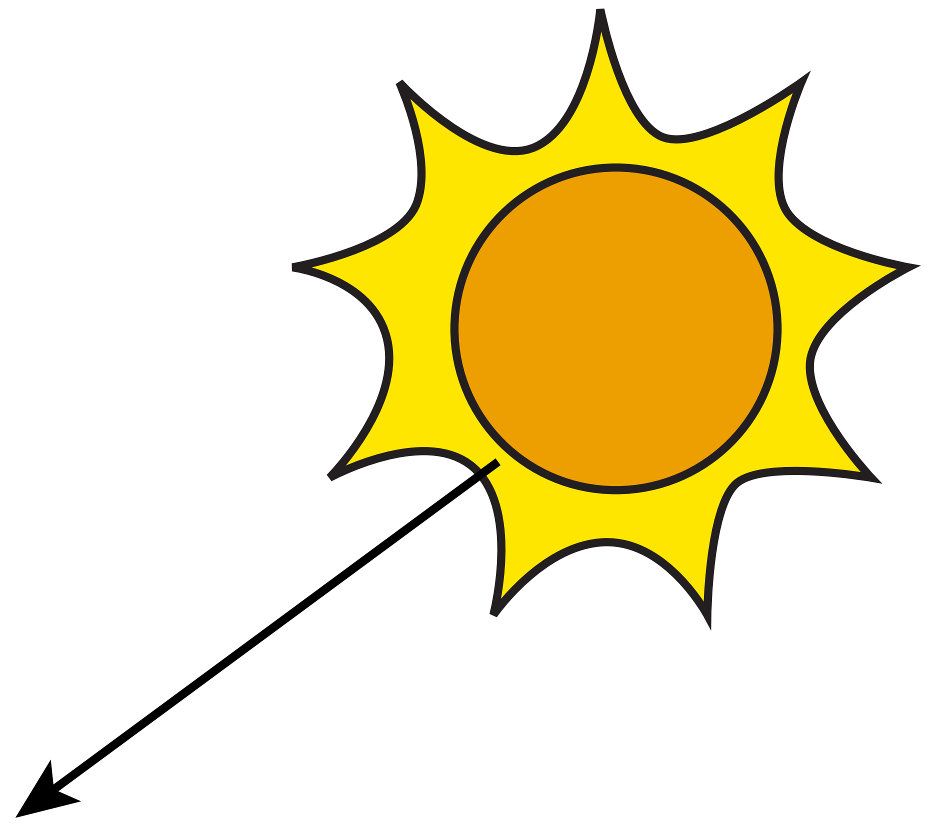
Closest point to *implicit* surface?

- If we change our representation of geometry, algorithms can change completely
- E.g., how might we compute the closest point on an implicit surface described via its distance function?

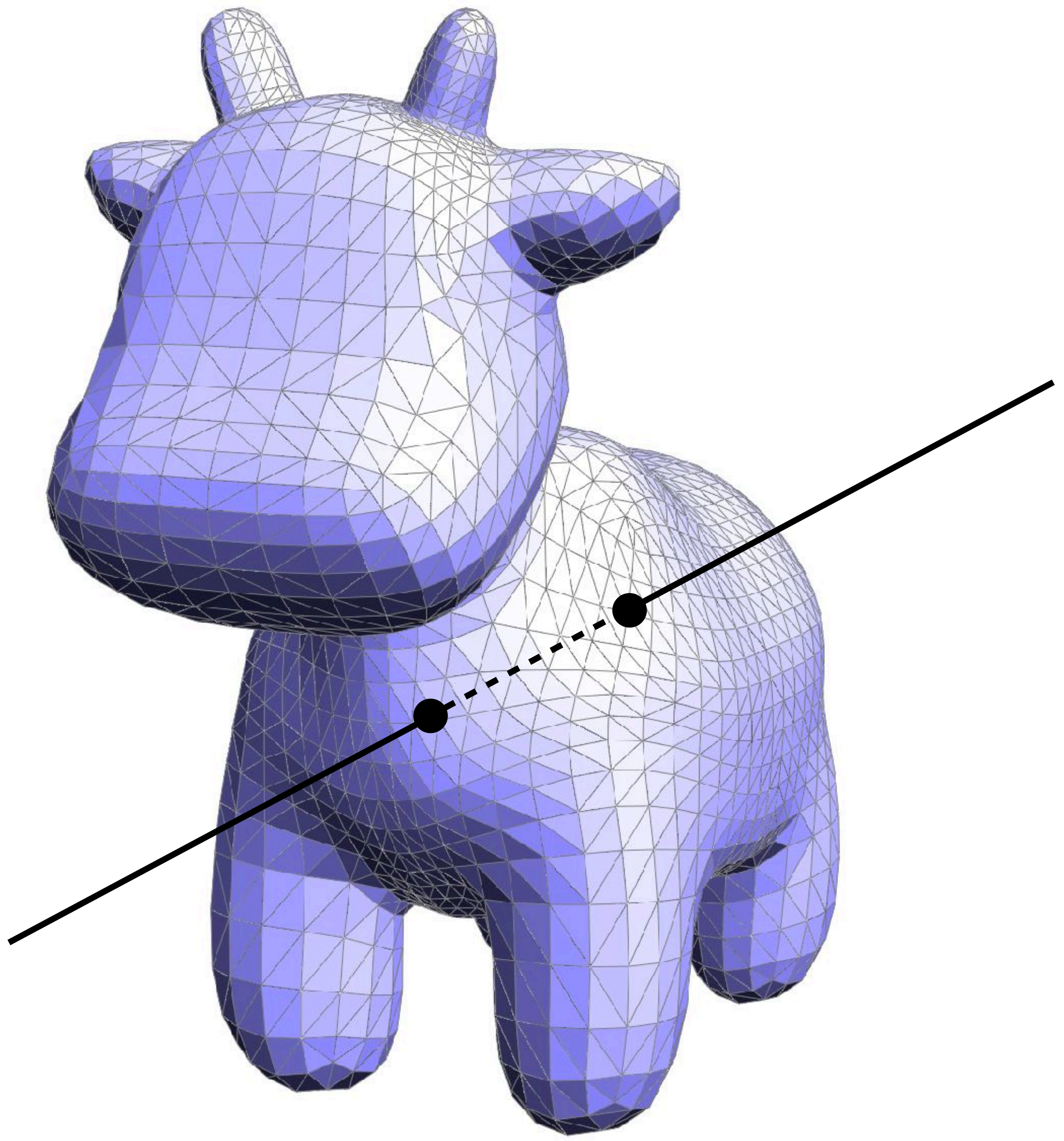


- One idea:
  - start at the query point
  - compute gradient of distance (using, e.g., finite differences)
  - take a little step (decrease distance)
  - repeat until we're at the surface (zero distance)

Different query: ray-mesh intersection



- A "ray" is an oriented line starting at a point
- Think about a ray of light traveling from the sun
- Want to know where a ray pierces a surface
  - Notice: this is a different query than finding the closest point on surface from ray's origin.
- Applications?
  - GEOMETRY: inside-outside test
  - RENDERING: visibility, ray tracing
  - ANIMATION: collision detection
- Ray might pierce surface in many places!



Ray equation

Can express ray as…

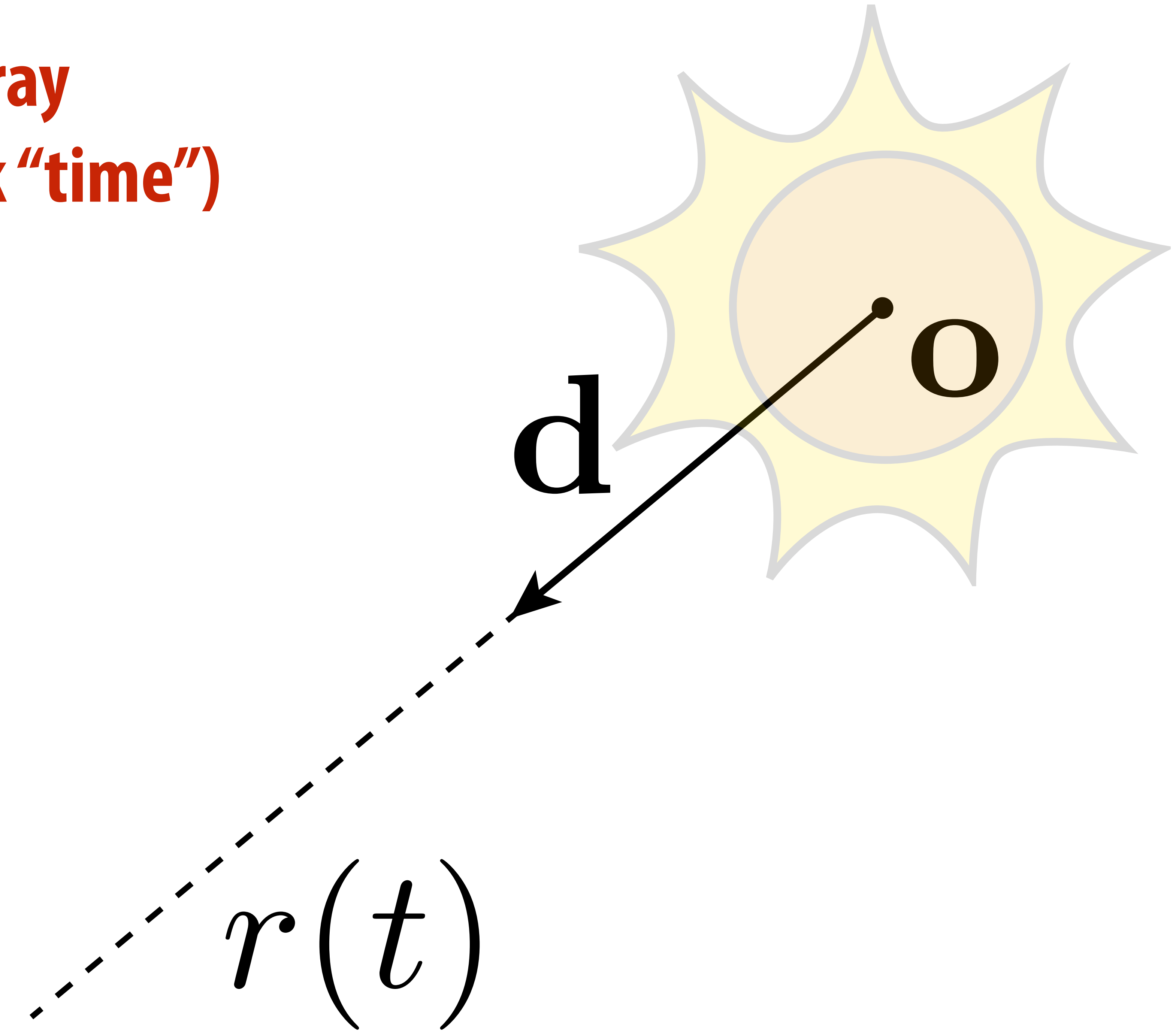
$$\mathbf{r}(t) = \mathbf{o} + t\mathbf{d}$$

origin

unit direction

point along ray

Position along ray
(some students think "time")



Intersecting a ray with an implicit surface

- Recall implicit surfaces: all points x such that f(x) = 0
- Q: How do we find points where a ray pierces this surface?
- Well, we know all points along the ray: r(t) = o + td
- Idea: replace "x" with "r(t)" in 1st equation, and solve for t
- Example: unit sphere

$$f(\mathbf{x}) = |\mathbf{x}|^2 - 1$$

$$\Rightarrow f(\mathbf{r}(t)) = |\mathbf{o} + t\mathbf{d}|^2 - 1$$

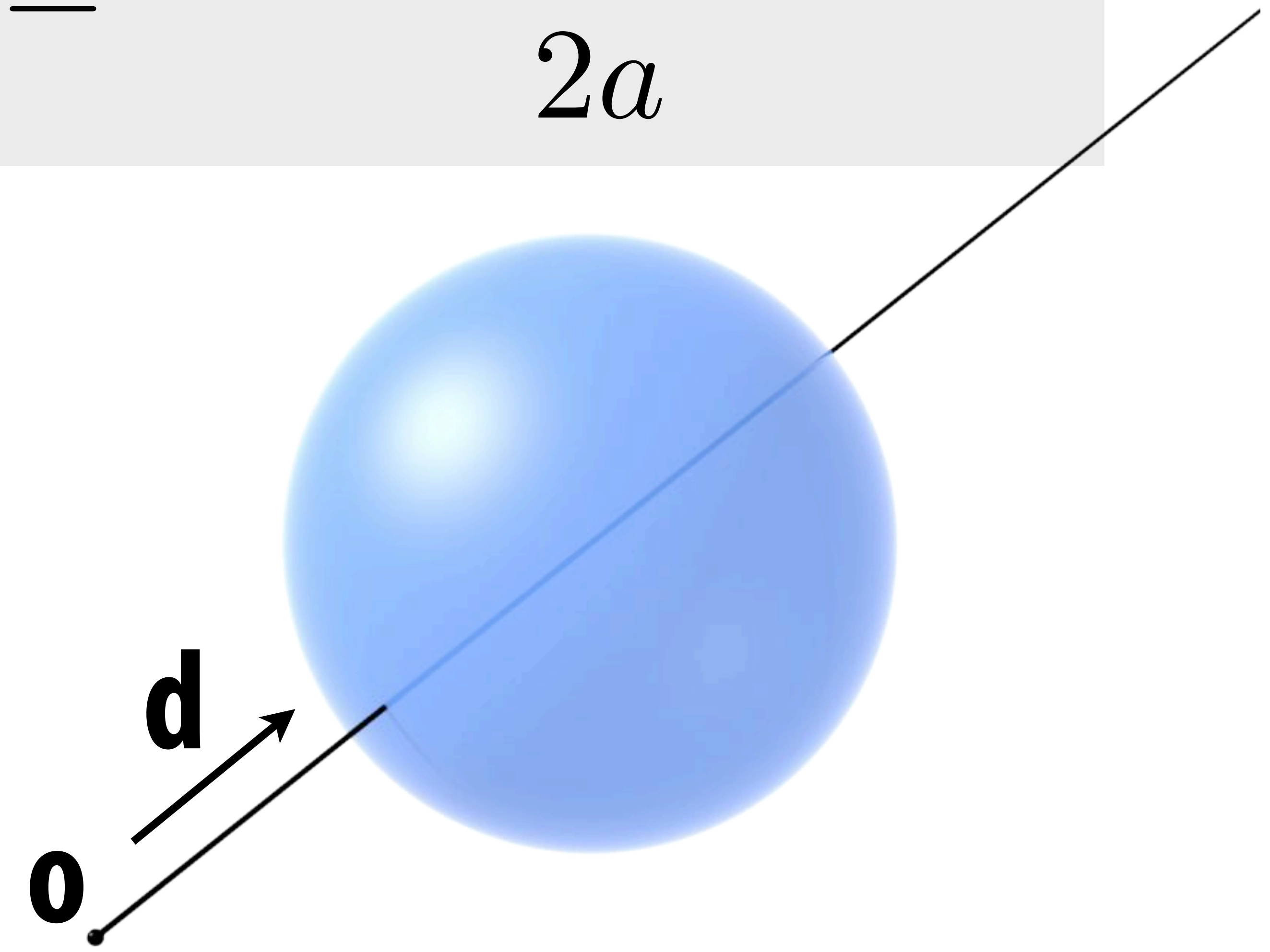
$$\underbrace{|\mathbf{d}|^2}_{a}\, t^2 + \underbrace{2(\mathbf{o}\cdot\mathbf{d})}_{b}\, t + \underbrace{|\mathbf{o}|^2 - 1}_{c} = 0$$

Note: $|\mathrm{d}|^2 = 1$ since d is a unit vector

$$t = \boxed{-\mathbf{o}\cdot\mathbf{d} \pm \sqrt{(\mathbf{o}\cdot\mathbf{d})^2 - |\mathbf{o}|^2 + 1}}$$

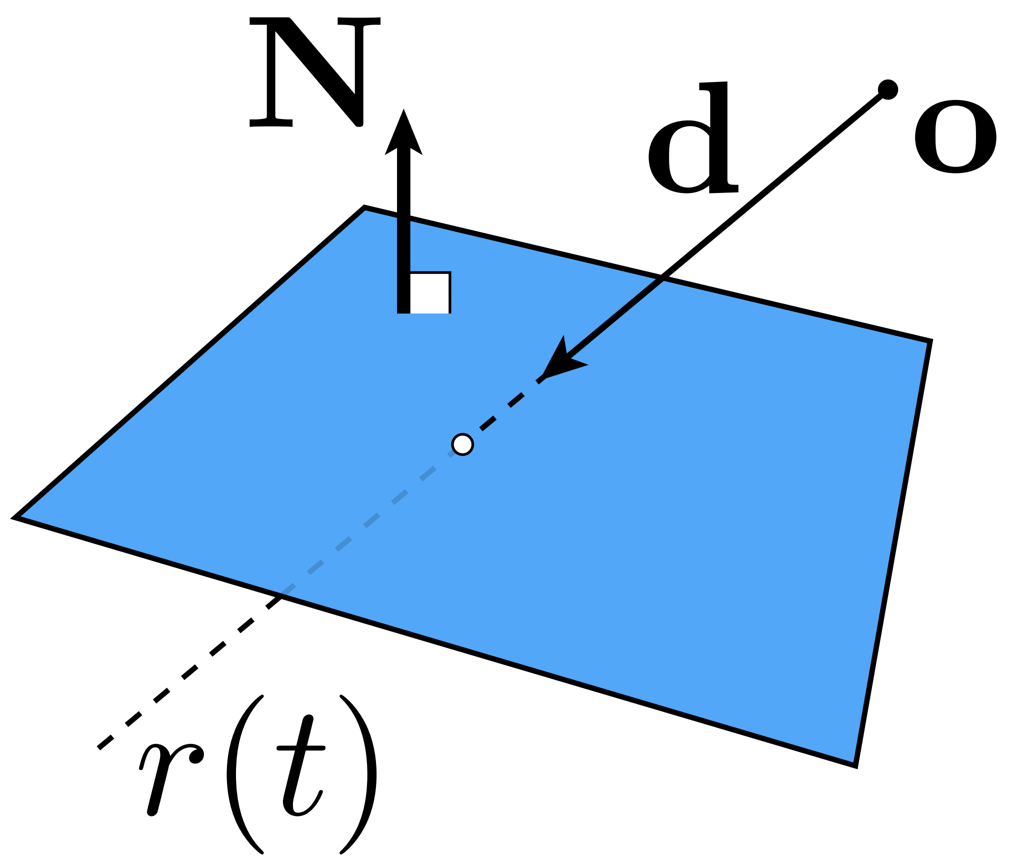
quadratic formula:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



Why two solutions?

Ray-plane intersection



- Suppose we have a plane $\mathbf{N}^\mathsf{T}\mathbf{x} = c$
  - N - unit normal
  - c - offset
- How do we find intersection with ray $\mathbf{r}(t) = \mathbf{o} + t\mathbf{d}$?

- *Key idea:* again, replace the point x with the ray equation t:

$$\mathbf{N}^\mathsf{T}\mathbf{r}(t) = c$$

- Now solve for t:

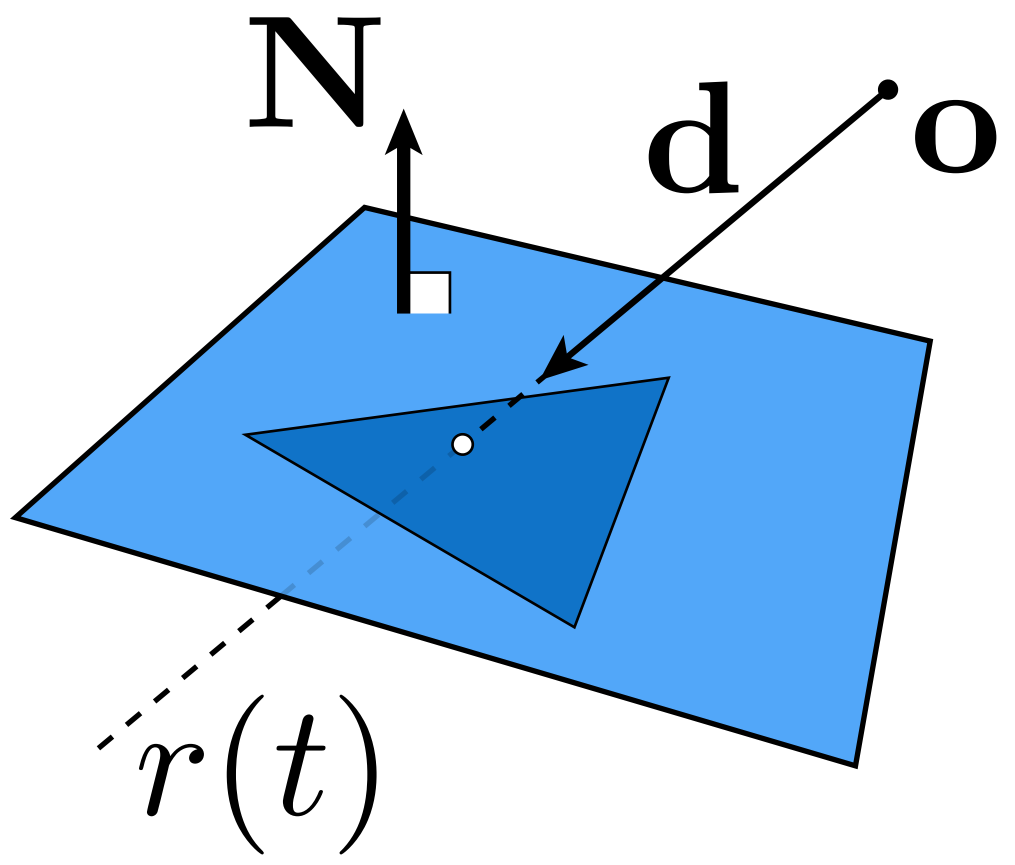
$$\mathbf{N}^\mathsf{T}(\mathbf{o} + t\mathbf{d}) = c \qquad \Rightarrow t = \frac{c - \mathbf{N}^\mathsf{T}\mathbf{o}}{\mathbf{N}^\mathsf{T}\mathbf{d}}$$

- And plug t back into ray equation:

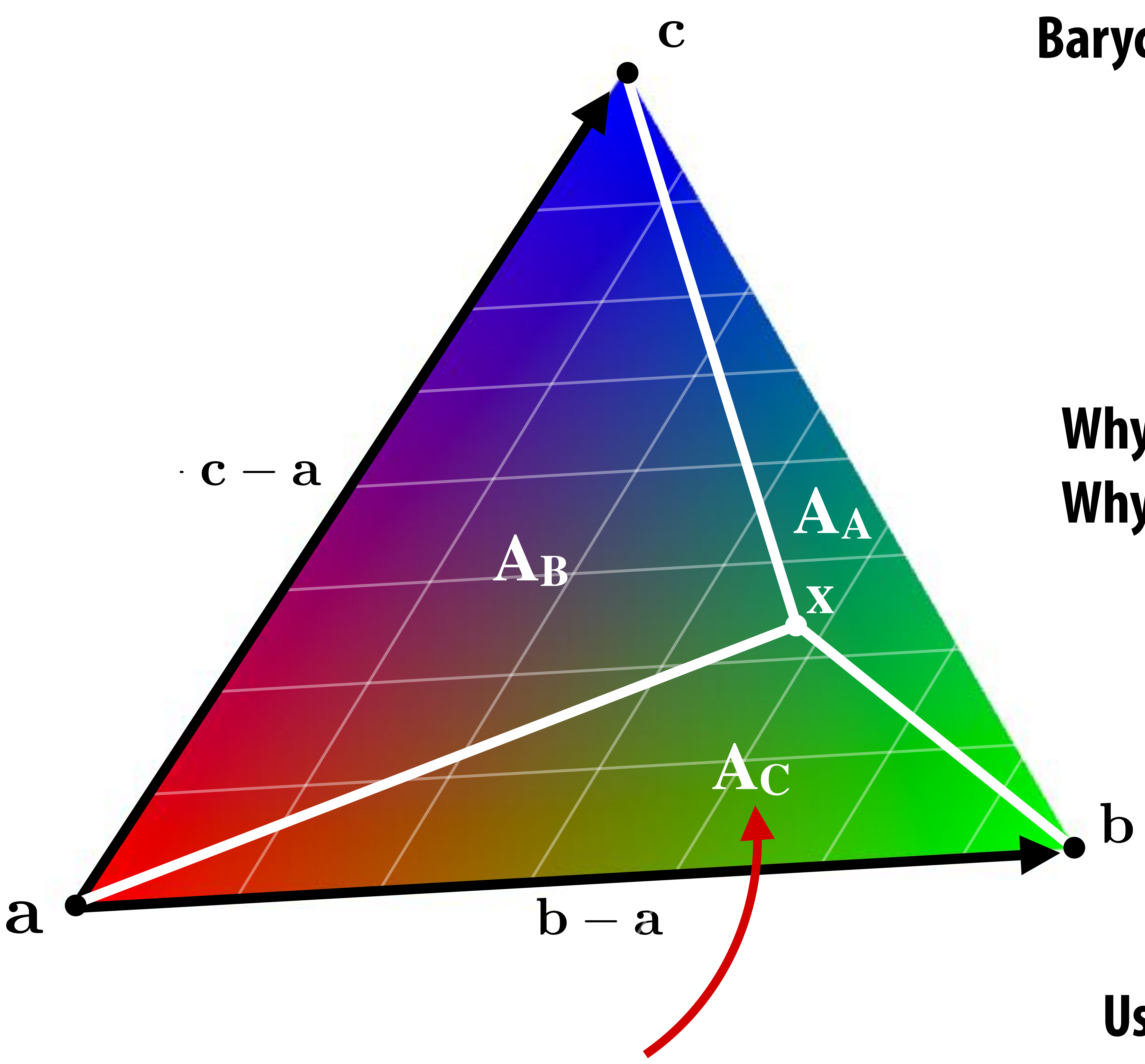
$$r(t) = \mathbf{o} + \frac{c - \mathbf{N}^\mathsf{T}\mathbf{o}}{\mathbf{N}^\mathsf{T}\mathbf{d}}\mathbf{d}$$

Ray-triangle intersection

- Triangle is in a plane...
- Algorithm:
  - Compute ray-plane intersection
  - Q: What do we do now?



Barycentric coordinates (as ratio of areas)



Area of triangle formed by points: a, b, x

Barycentric coords are *signed* areas:

$$\alpha = A_A / A$$

$$\beta = A_B / A$$

$$\gamma = A_C / A$$

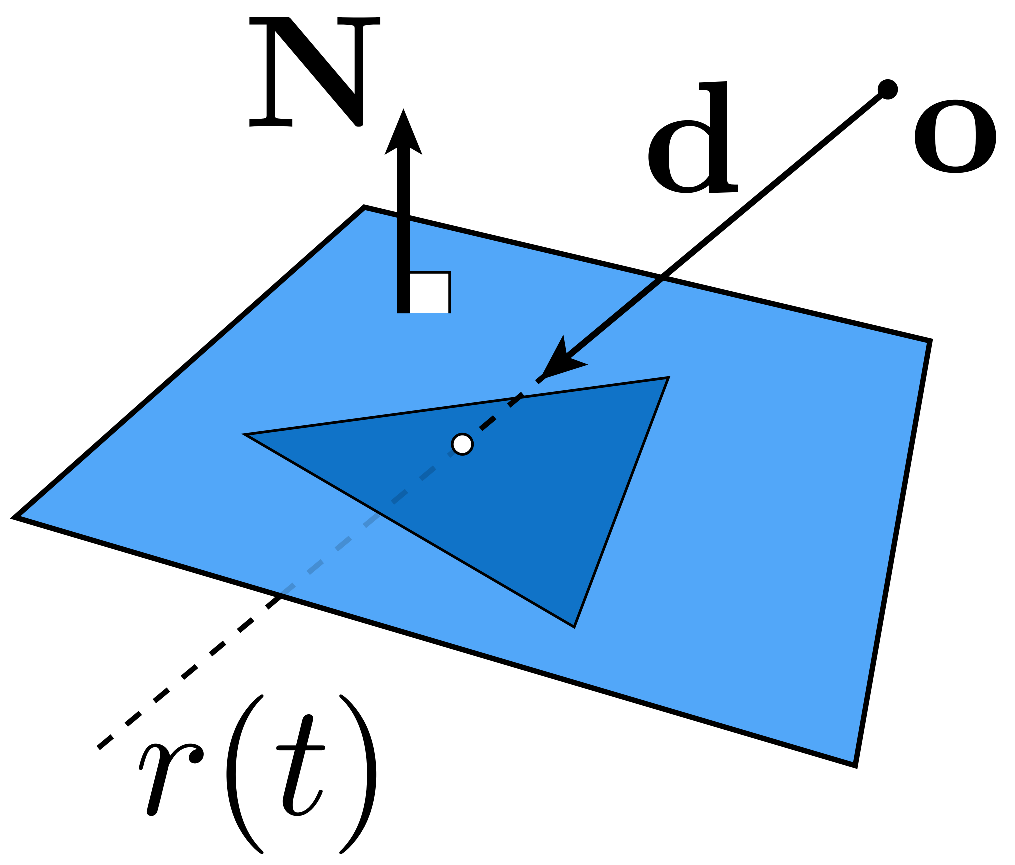
Why must coordinates sum to one?

Why must coordinates be between 0 and 1?

Useful: Heron's formula:

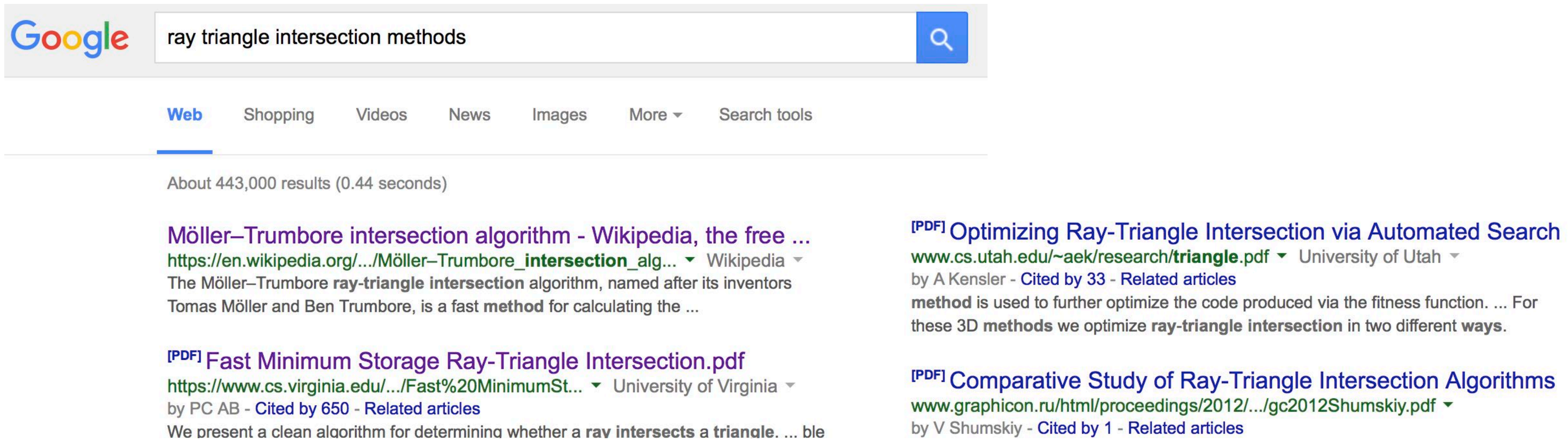
$$A_C = \frac{1}{2}(\mathbf{b} - \mathbf{a}) \times (\mathbf{x} - \mathbf{a})$$

Ray-triangle intersection



- **Algorithm:**
  - Compute ray-plane intersection
  - Compute barycentric coordinates of hit point
  - If barycentric coordinates are all positive, point is in triangle

- **Many different techniques if you care about efficiency**

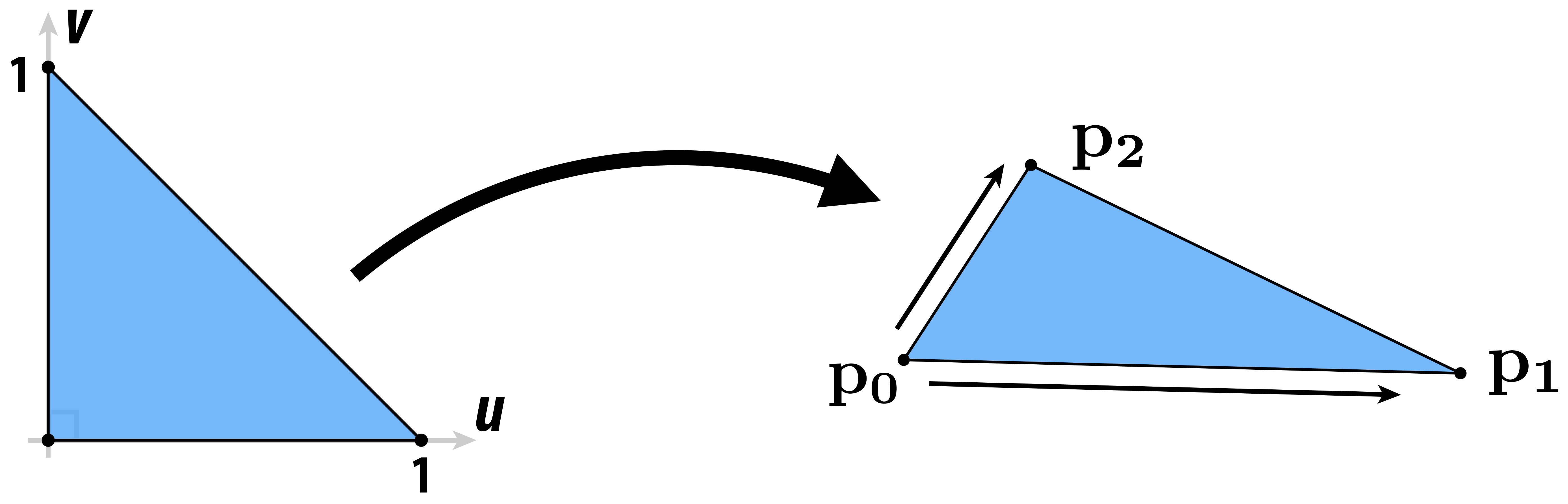


Ray-triangle intersection (another way)

- **Parameterize triangle with vertices $\mathbf{p_0}, \mathbf{p_1}, \mathbf{p_2}$ using barycentric coordinates ***

$$f(u,v) = \mathbf{p_0} + u(\mathbf{p_1} - \mathbf{p_0}) + v(\mathbf{p_2} - \mathbf{p_0}) = (1-u-v)\mathbf{p_0} + u\mathbf{p_1} + v\mathbf{p_2}$$

- **Can think of a triangle as an affine map of the unit triangle**



* I'm writing u,v instead of beta, gamma to make explicit representation of triangle very clear.

Another way: ray-triangle intersection

Plug parametric ray equation directly into equation for points on triangle:

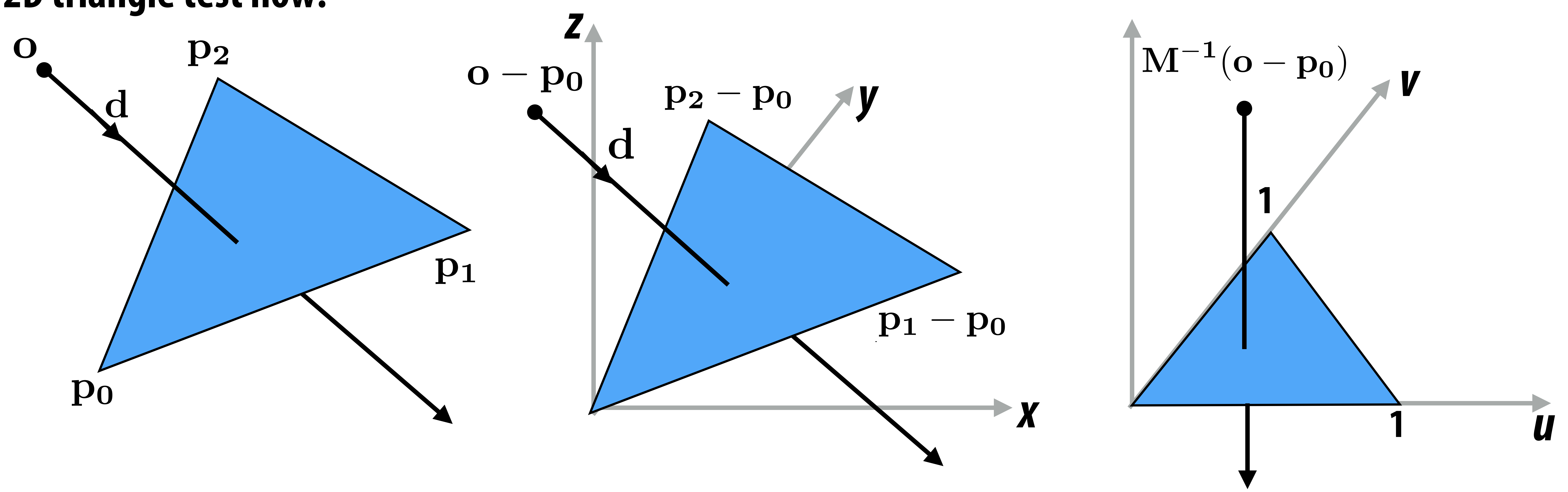
$$\mathbf{p_0} + u(\mathbf{p_1} - \mathbf{p_0}) + v(\mathbf{p_2} - \mathbf{p_0}) = \mathbf{o} + t\mathbf{d}$$

Solve for u, v, t:

$$\underbrace{\begin{bmatrix}\mathbf{p_1} - \mathbf{p_0} & \mathbf{p_2} - \mathbf{p_0} & -\mathbf{d}\end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} u \\ v \\ t \end{bmatrix} = \mathbf{o} - \mathbf{p_0}$$

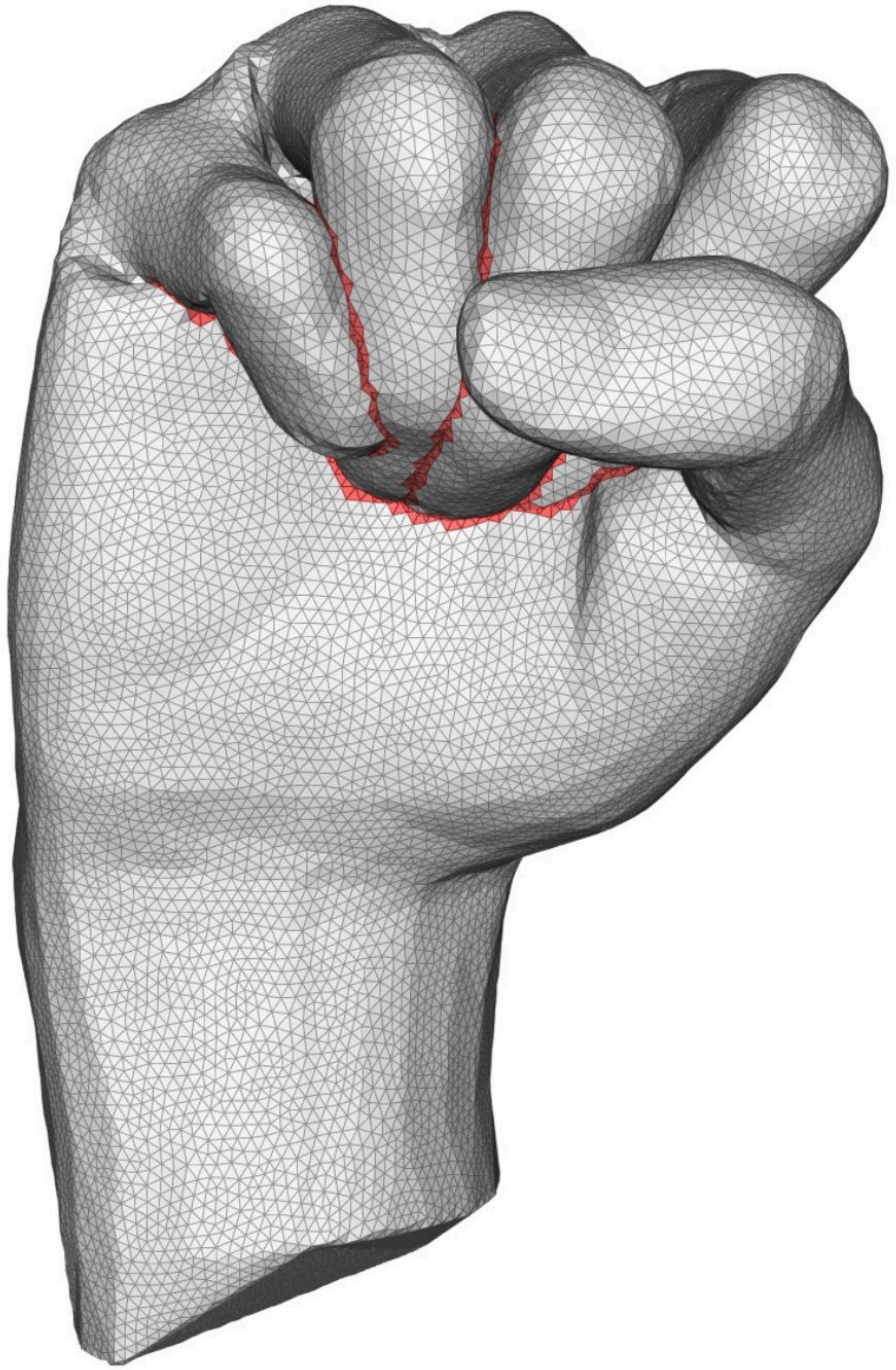
$\mathbf{M}^{-1}$ transforms triangle back to unit triangle in u,v plane, and transforms ray's direction to be orthogonal to plane.

It's a point in 2D triangle test now!



One more query: mesh-mesh intersection

- GEOMETRY: How do we know if a mesh intersects itself?
- ANIMATION: How do we know if a collision occurred?

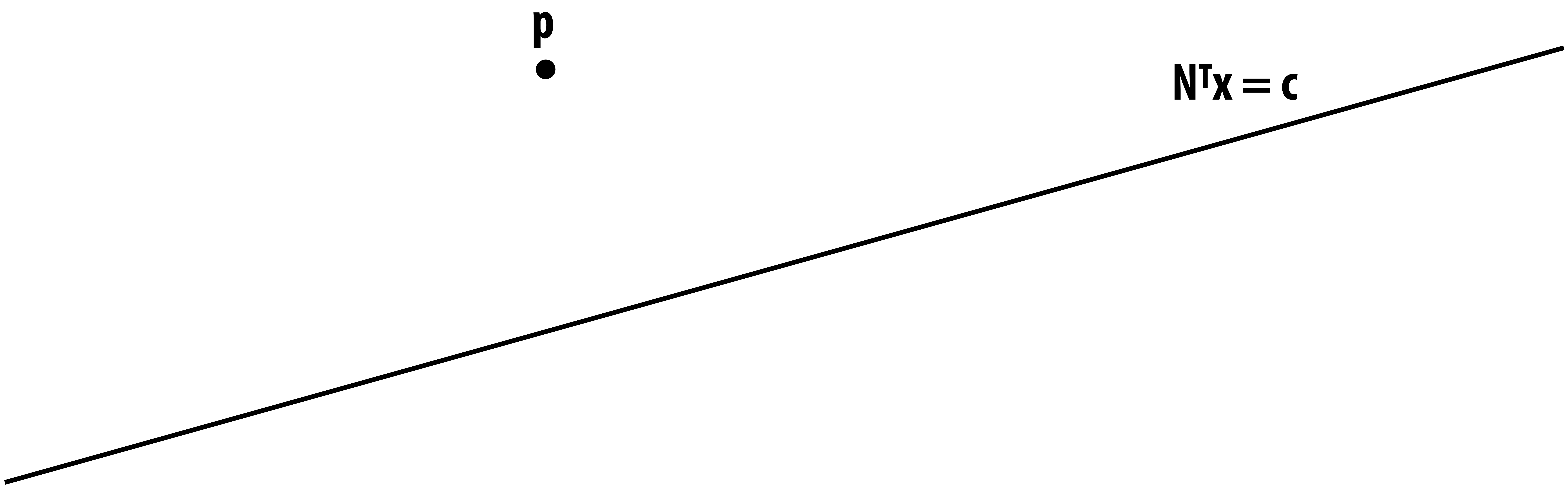


Warm up: point-point intersection

- **Q: How do we know if p intersects a?**
- **A: ...check if they're the same point!**

Slightly harder: point-line intersection

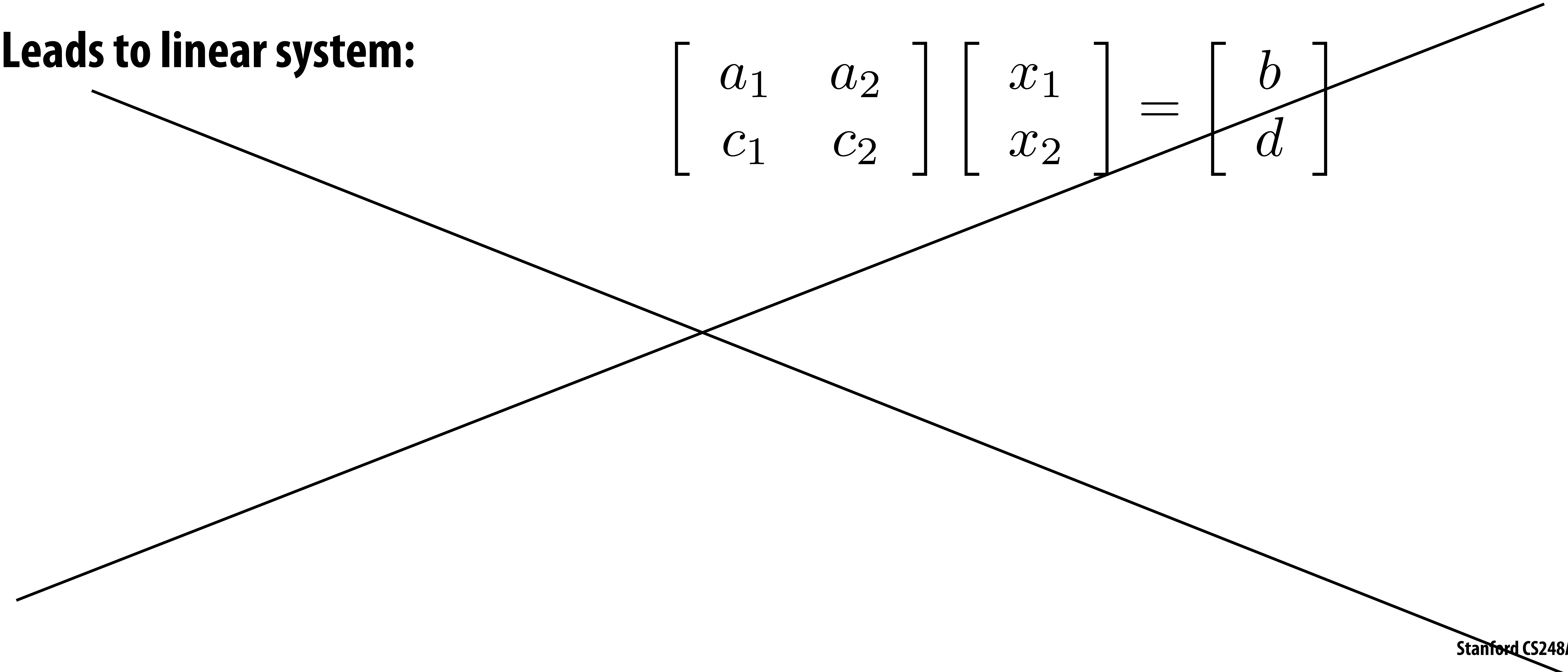
- Q: How do we know if a point intersects a given line?
- A: ...plug it into the line equation!



Line-line intersection

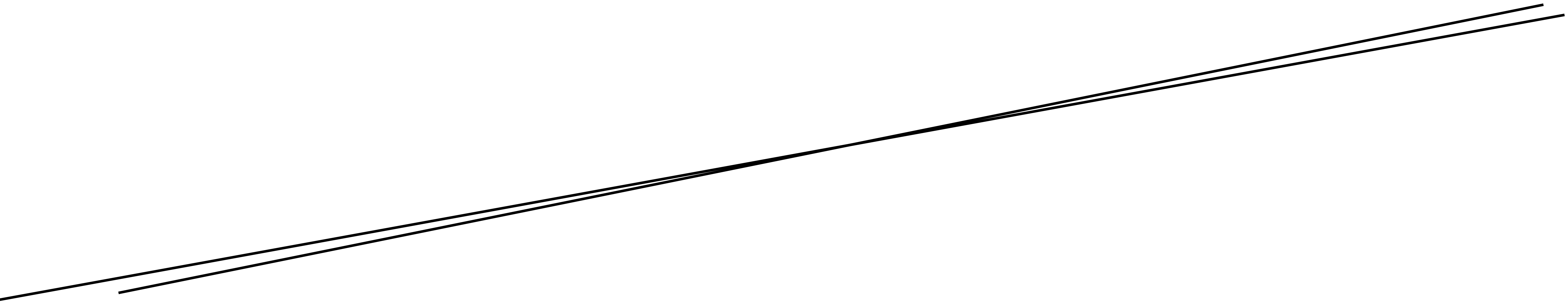
- **Two lines: ax=b and cx=d**
- **Q: How do we find the intersection?**
- **A: See if there is a simultaneous solution**
- **Leads to linear system:**

$$\begin{bmatrix} a_1 & a_2 \\ c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$



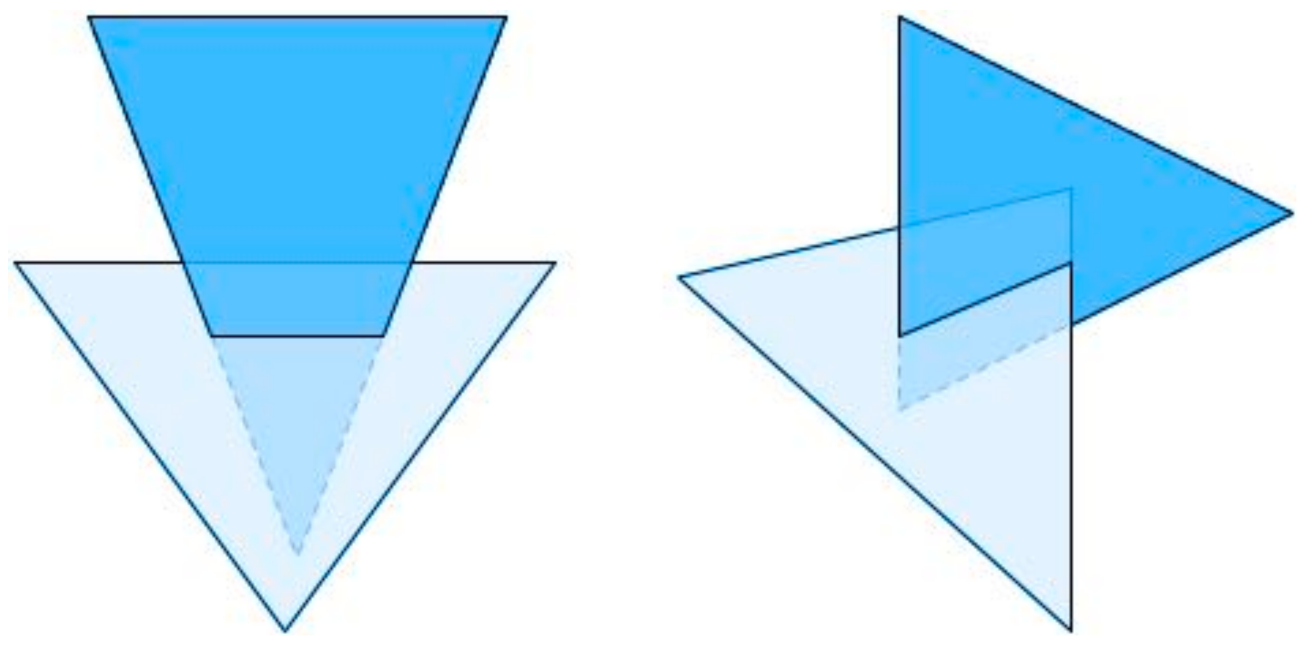
Degenerate line-line intersection?

- **What if lines are almost parallel?**
- **Small change in normal can lead to big change in intersection!**
- **Instability very common, very important with geometric predicates. Demands special care (e.g., analysis of matrix).**

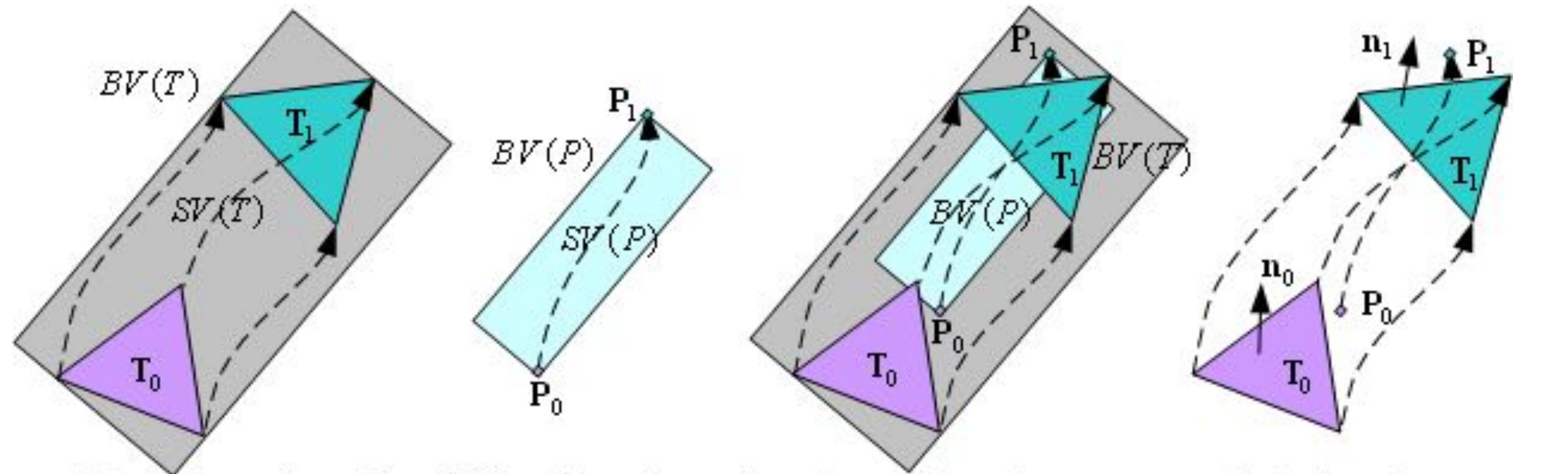


See for example Shewchuk, "*Adaptive Precision Floating-Point Arithmetic and Fast Robust Geometric Predicates*"

Triangle-triangle intersection?



- Lots of ways to do it
- Basic idea:
  - Q: Any ideas?
  - One way: reduce to edge-triangle intersection
  - Check if each line passes through plane (ray-triangle)
  - Then do interval test
- What if triangle is *moving*?
  - Important case for animation
  - Can think of triangles as *prisms* in time
  - Turns dynamic problem (in nD + time) into purely geometric problem in (n+1)-dimensions



(a) Bounding volume of a deforming triangle (b) Bounding volume of a deforming vertex (c) Bounding volume test (d) Coplanarity test

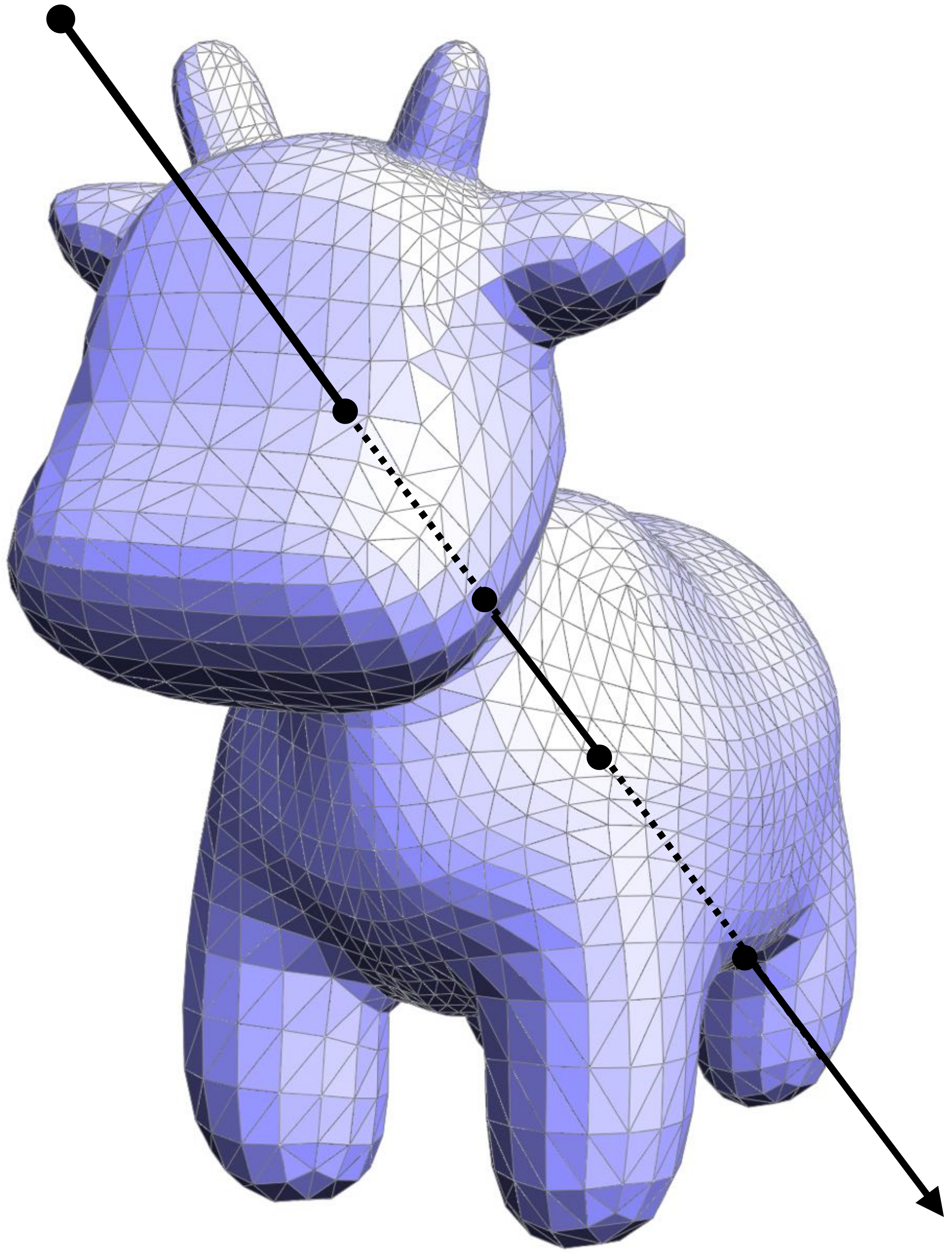
Ray-scene intersection

Given a scene defined by a set of *N* primitives and a ray *r*, find the closest point of intersection of *r* with the scene

```
t_closest = inf
for each primitive p in scene:
   t = p.intersect(r)
   if t >= 0 && t < t_closest:
      t_closest = t

// closest hit is:
// r.o + t_closest * r.d
```

(Assume p.intersect(r) returns value of *t* corresponding to the point of intersection with ray *r*)



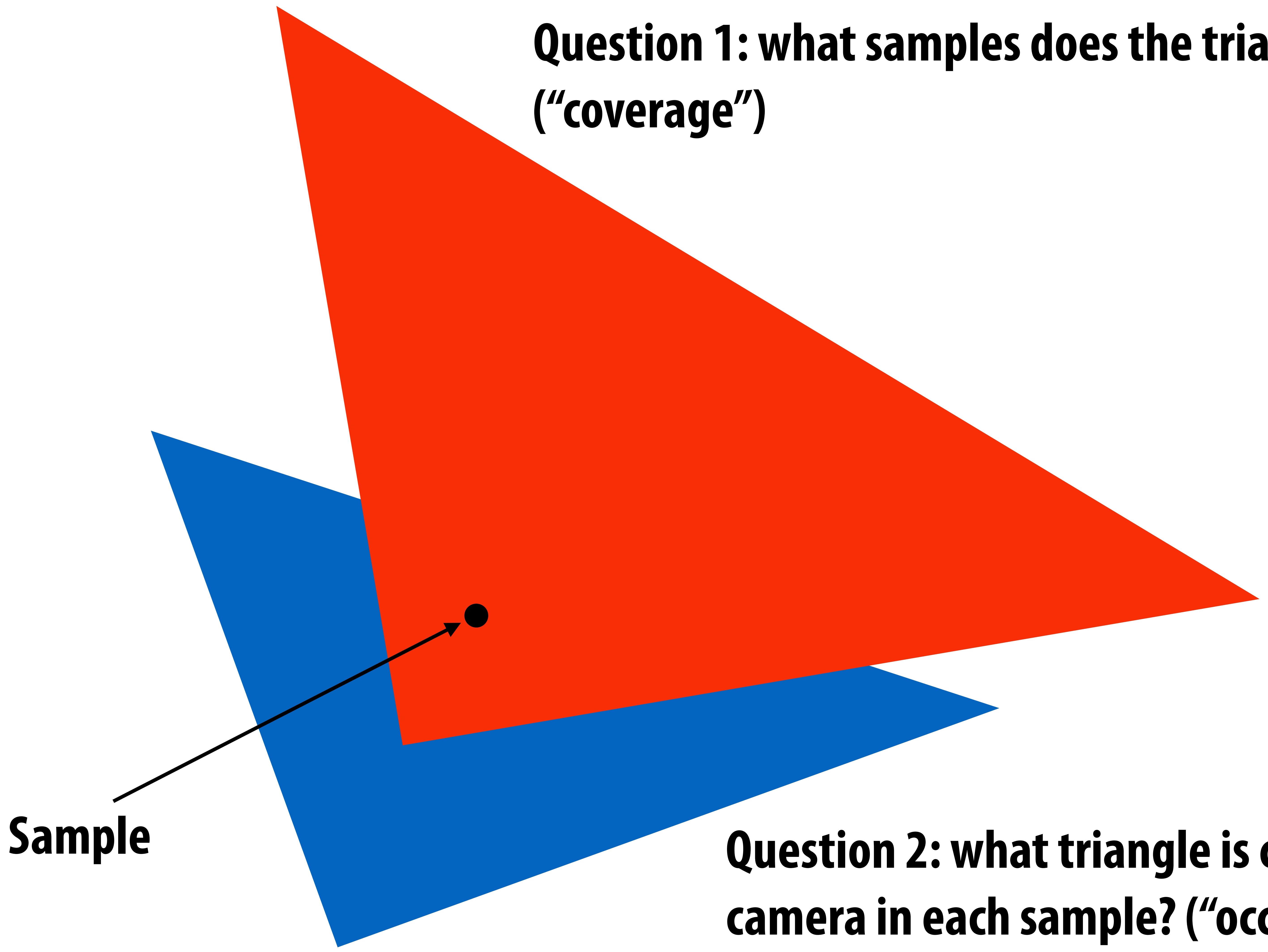
Complexity? $O(N)$

Can we do better? Of course… but you'll have to wait until next class

Rendering via ray casting:
(one common use of ray-scene intersection tests)

Rasterization and ray casting are two algorithms for solving the same problem: determining surface "visibility" from a virtual camera

Recall triangle visibility problem:

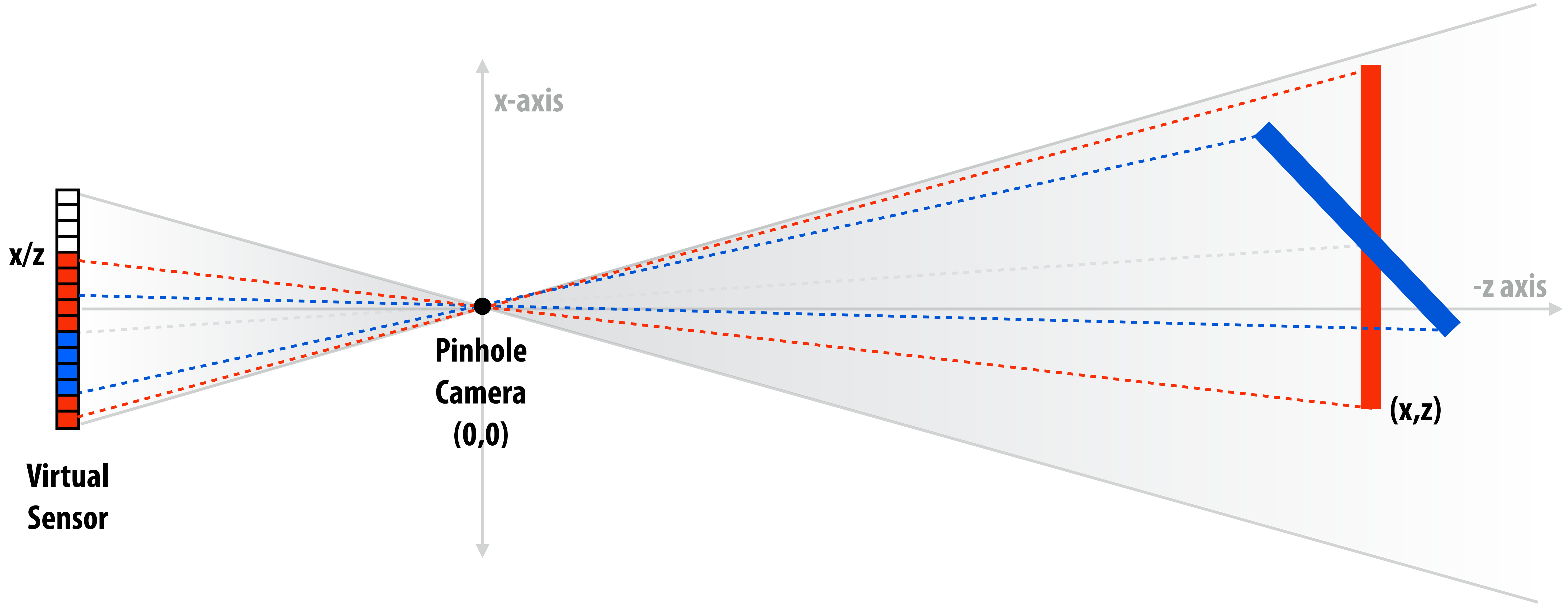


Question 1: what samples does the triangle overlap? ("coverage")

Question 2: what triangle is closest to the camera in each sample? ("occlusion")

The visibility problem (rasterization perspective)

- **What scene geometry is visible at each screen sample?**
  - What scene geometry *projects* onto screen sample points? (coverage)
  - Which geometry is visible from the camera at each sample? (occlusion)



Basic rasterization algorithm

Sample = 2D point

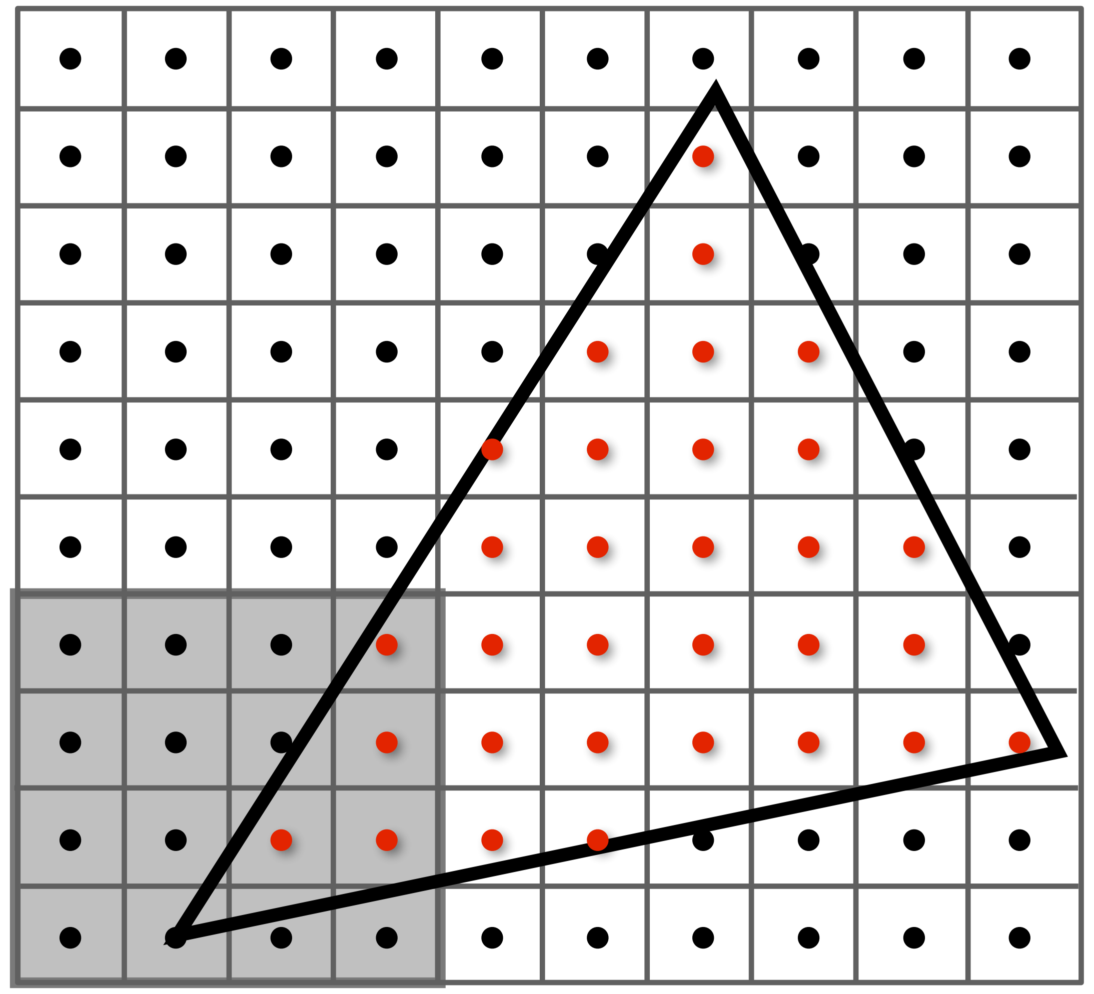
Coverage: 2D triangle/sample tests (does projected triangle cover 2D sample point)

Occlusion: depth buffer

```
initialize z_closest[] to INFINITY              // store closest-surface-so-far for all samples
initialize color[]                              // store scene color for all samples
for each triangle t in scene:                   // loop 1: over triangles
    t_proj = project_triangle(t)
    for each 2D sample s in frame buffer:       // loop 2: over visibility samples
        if (t_proj covers s)
            compute color of triangle at sample
            if (depth of t at s is closer than z_closest[s])
                update z_closest[s] and color[s]
```

"Given a triangle, find the samples it covers"

(finding the samples is relatively easy since they are distributed uniformly on screen)



More efficient hierarchical rasterization:

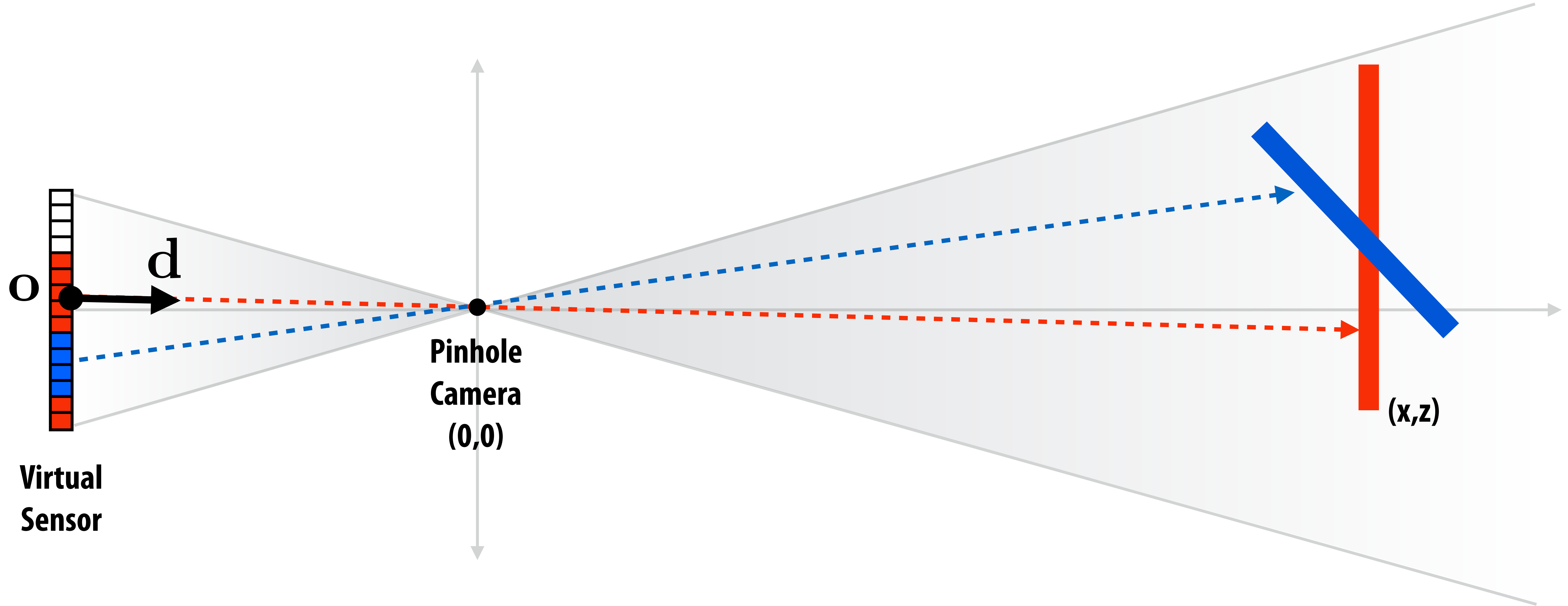
For each TILE of image

If triangle overlaps tile

test all samples in tile

The visibility problem (described differently)

- In terms of casting rays from the camera:
  - Is a scene primitive hit by a ray originating from a point on the virtual sensor and traveling through the opening of a pinhole camera? (coverage)
  - What primitive is the first hit along that ray? (occlusion)



Basic ray casting algorithm

Sample = a ray in 3D

Coverage: 3D ray-triangle intersection tests (does ray "hit" triangle)

Occlusion: closest intersection along ray

```
initialize color[]                                    // store scene color for all samples
for each sample s in frame buffer:                    // loop 1: over visibility samples (rays)
    r = ray from s on sensor through pinhole aperture
    r.min_t = INFINITY                                // only store closest-so-far for current ray
    r.tri = NULL;
    for each triangle tri in scene:                   // loop 2: over triangles
        if (intersects(r, tri)) {                     // 3D ray-triangle intersection test
            if (intersection distance along ray is closer than r.min_t)
                update r.min_t and r.tri = tri;
        }
    color[s] = compute surface color of triangle r.tri at hit point
```

Compared to rasterization approach: just a reordering of the loops!

"Given a ray, find the closest triangle it hits."

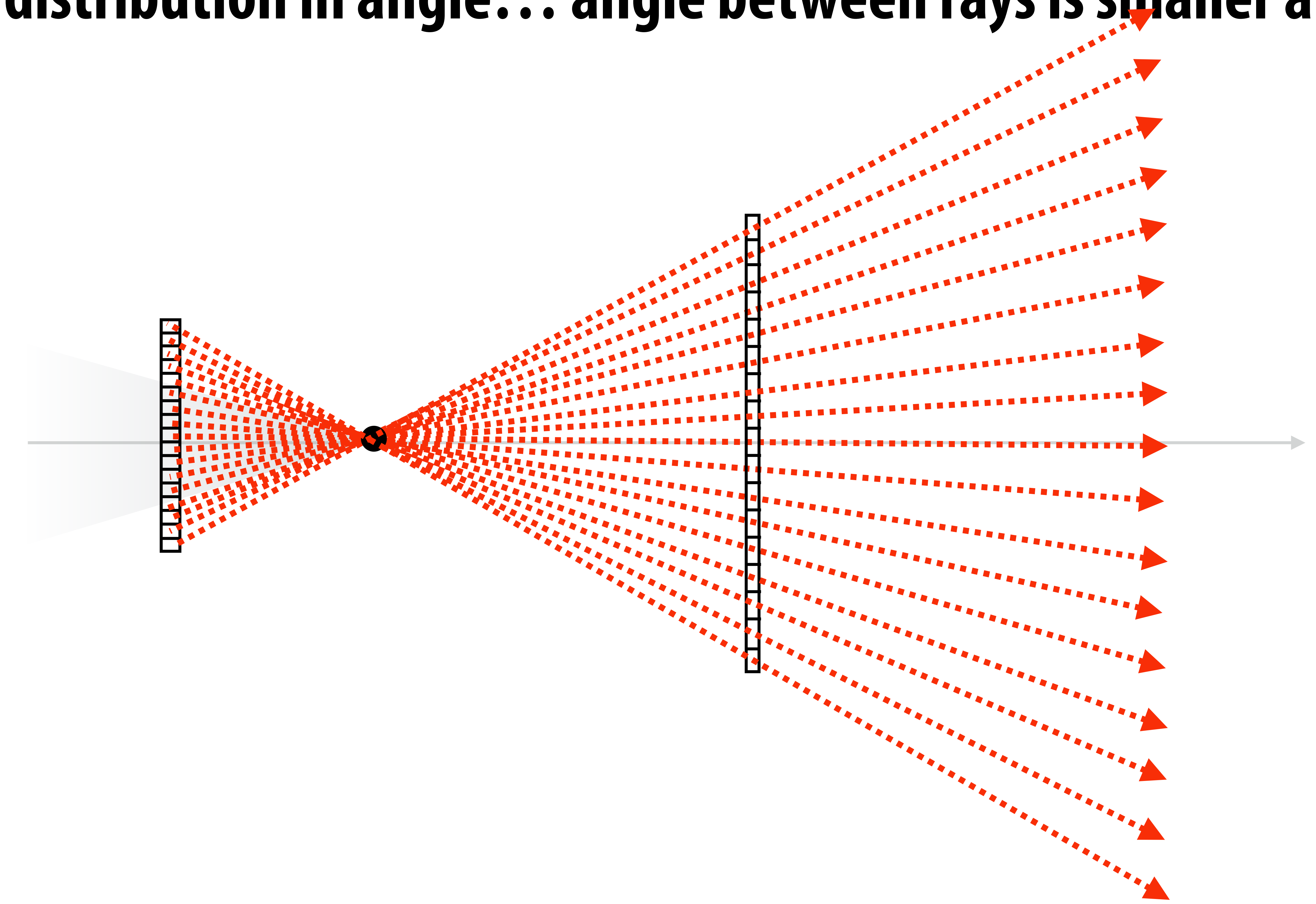
Basic rasterization vs. ray casting

- **Rasterization:**
  - Outer loop: iterate over all triangles ("for all triangles")
  - Store closest surface seen so far for all screen samples
    - Done via the depth buffer (requires constant-time access to 2D array of fixed size)
  - Do not have to store entire scene geometry in memory
    - Naturally supports unbounded size scenes (since algorithm iterates over the triangles)

- **Ray casting:**
  - Outer loop: iterate over all screen samples ("for all rays")
    - Do not have to store closest surface seen so far for the entire screen (just the current ray)
    - Easy solution for rendering transparent surfaces: Process surfaces in the order they are encountered along the ray: front-to-back (find first "hit", then "second", etc)
  - Must store entire scene geometry in a manner that allows fast access
    - Can be challenging for very large scenes

In other words…

- Rasterization is a efficient implementation of ray casting where:
  - Ray-scene intersection is computed for a batch of rays
  - All rays in the batch originate from same origin
  - Rays are distributed uniformly in (x,y) in plane of projection
    (Note: not uniform distribution in angle… angle between rays is smaller away from view direction)

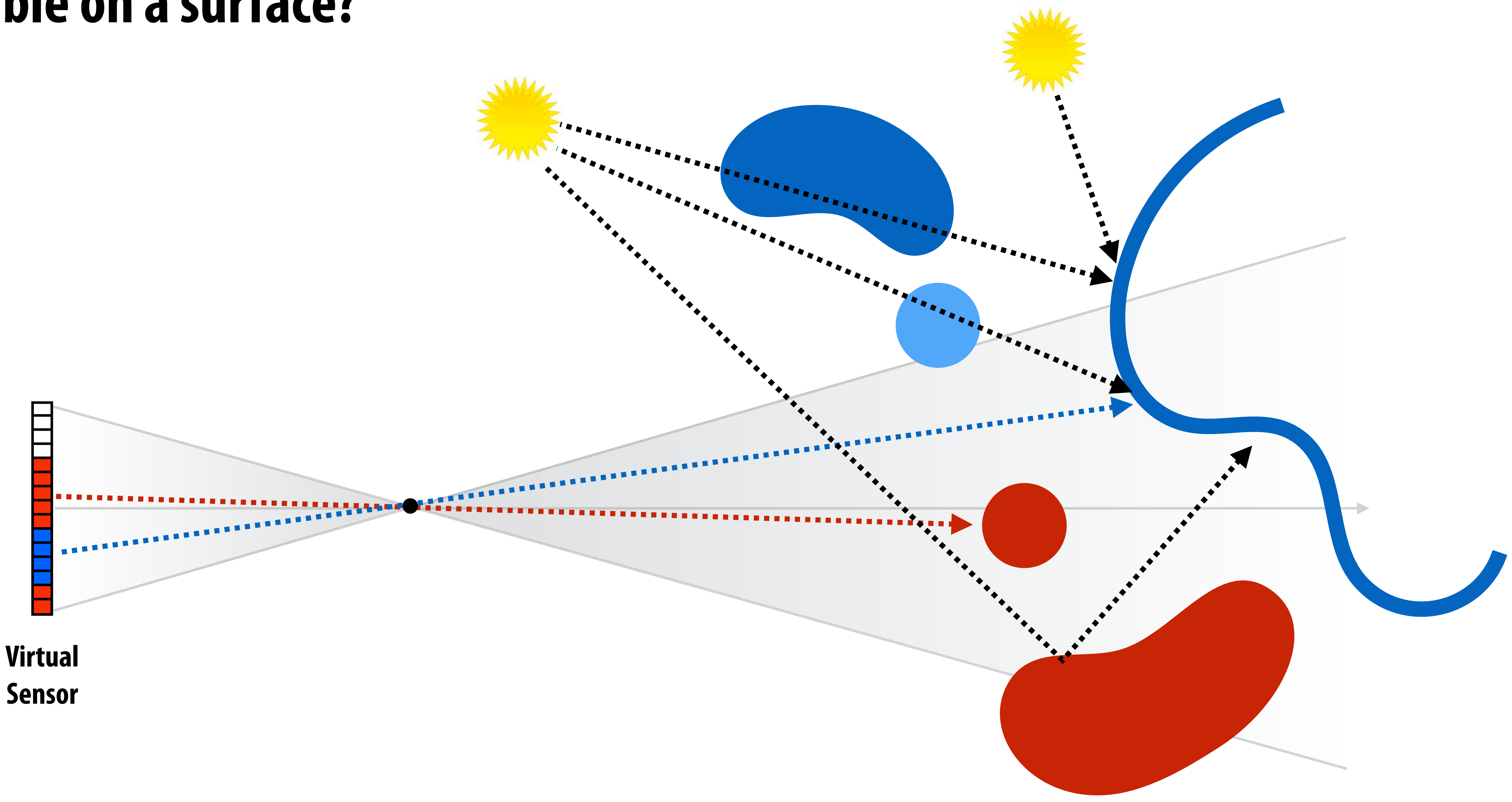


Generality of ray-scene queries

What object is visible to the camera?

What light sources are visible from a point on a surface (is a surface in shadow?)

What reflection is visible on a surface?



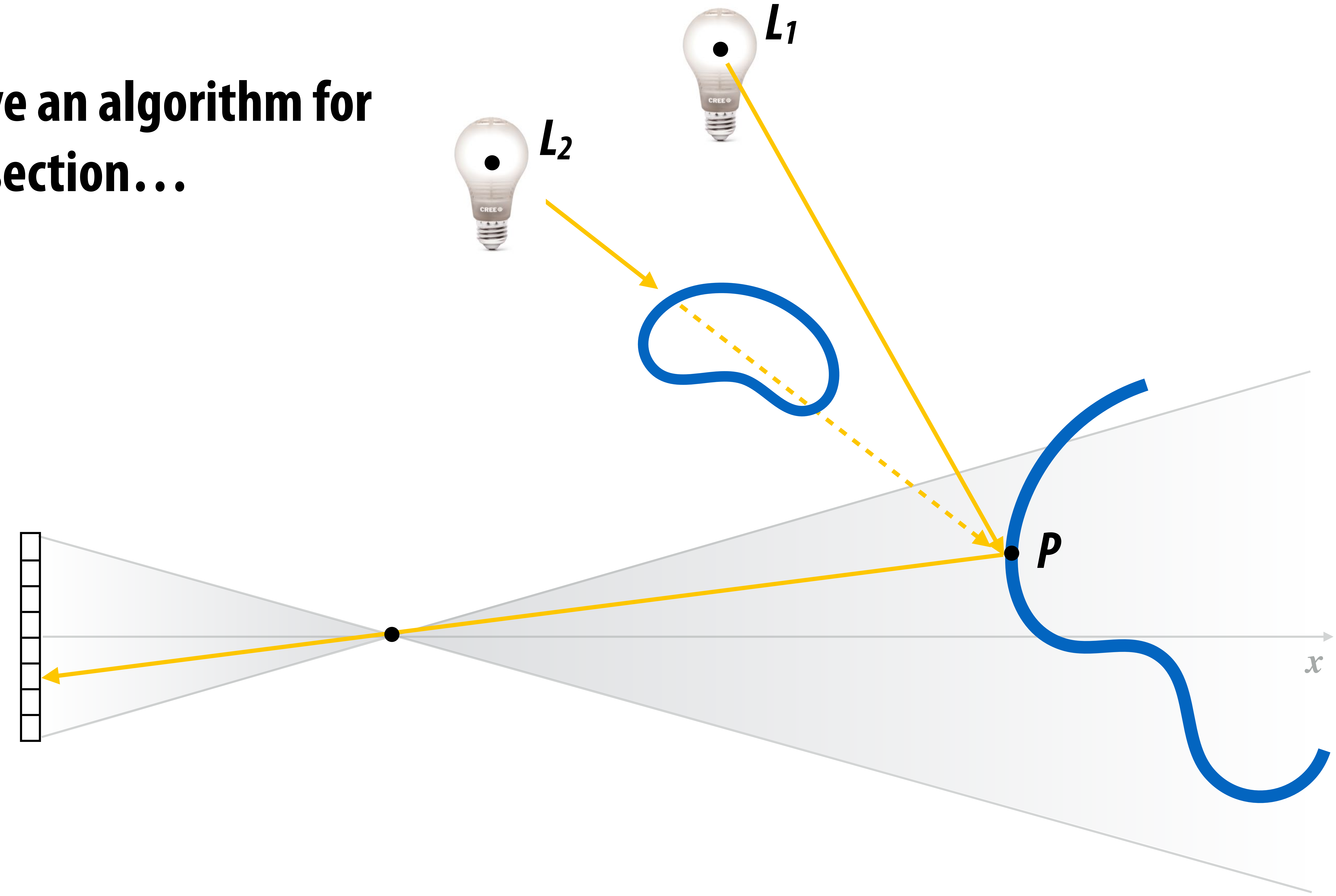
In contrast, rasterization is a highly-specialized solution for computing visibility for a set of uniformly distributed rays originating from the same point (most often: the camera)

Shadows

Image credit: Grand Theft Auto V

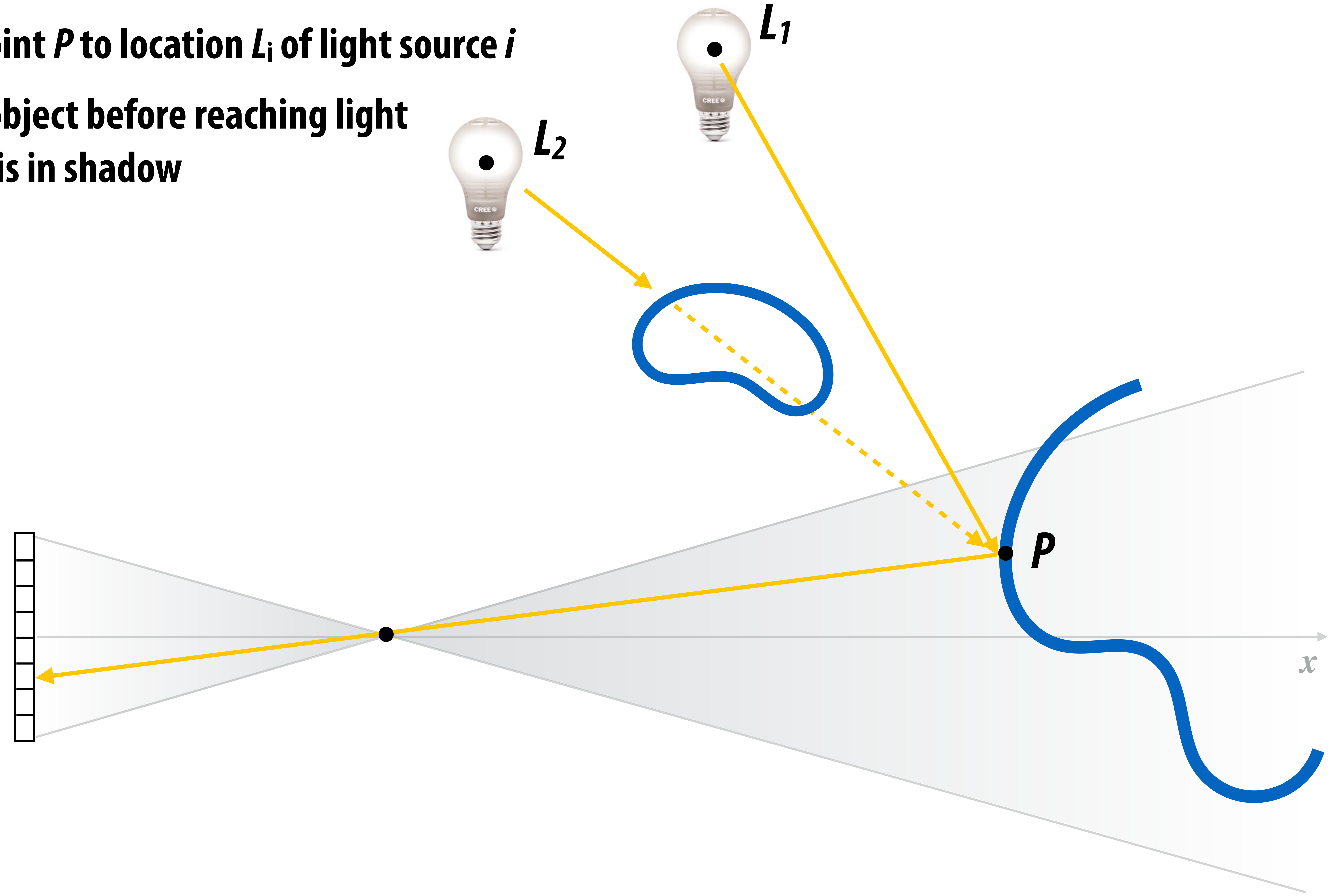
How to compute if a surface point is in shadow?

Assume you have an algorithm for ray-scene intersection…



A simple shadow computation algorithm

- Trace ray from point *P* to location L_i of light source *i*
- If ray hits scene object before reaching light source… then *P* is in shadow



Direct illumination + reflection + transparency

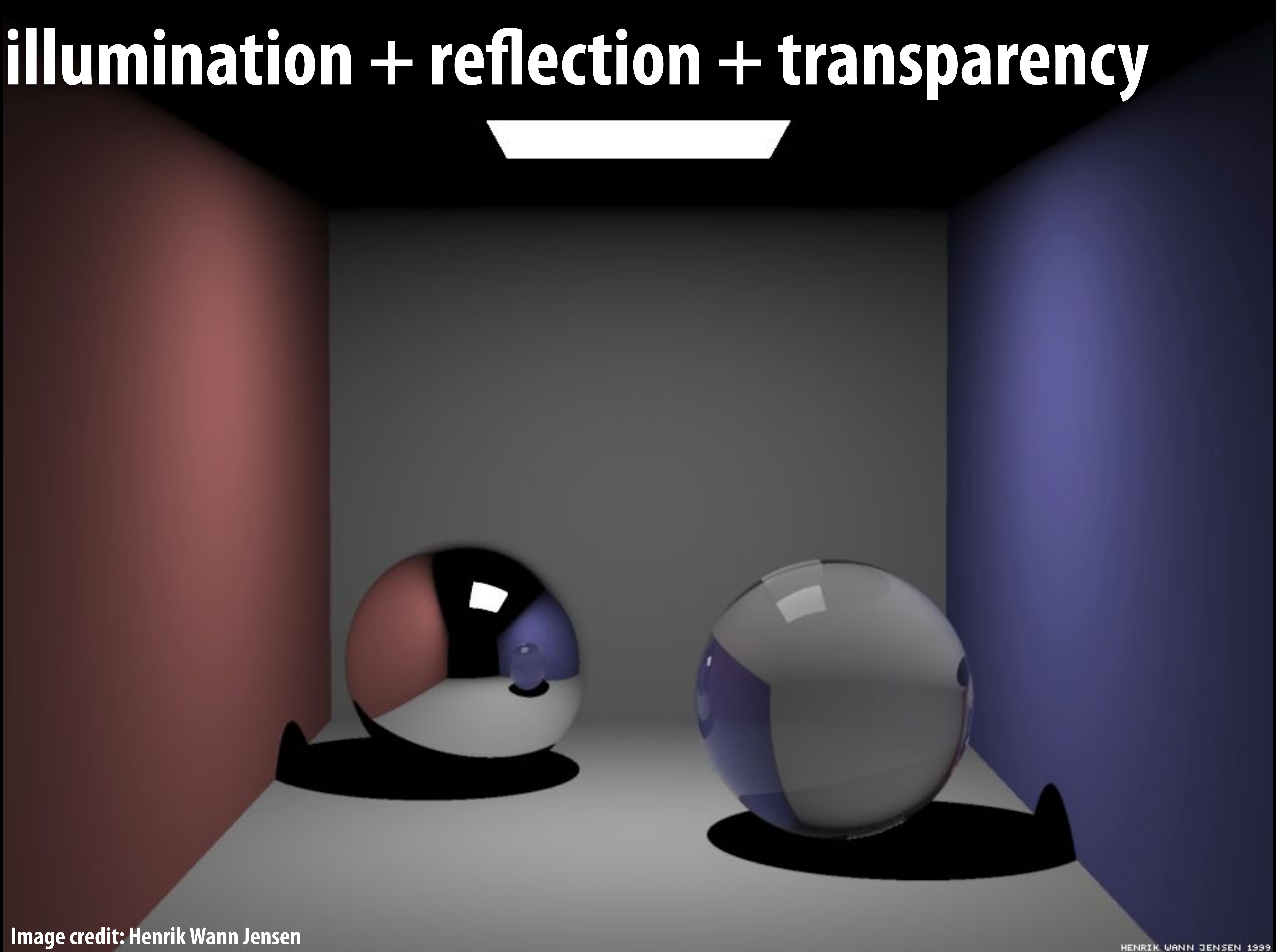


Image credit: Henrik Wann Jensen

Global illumination solution

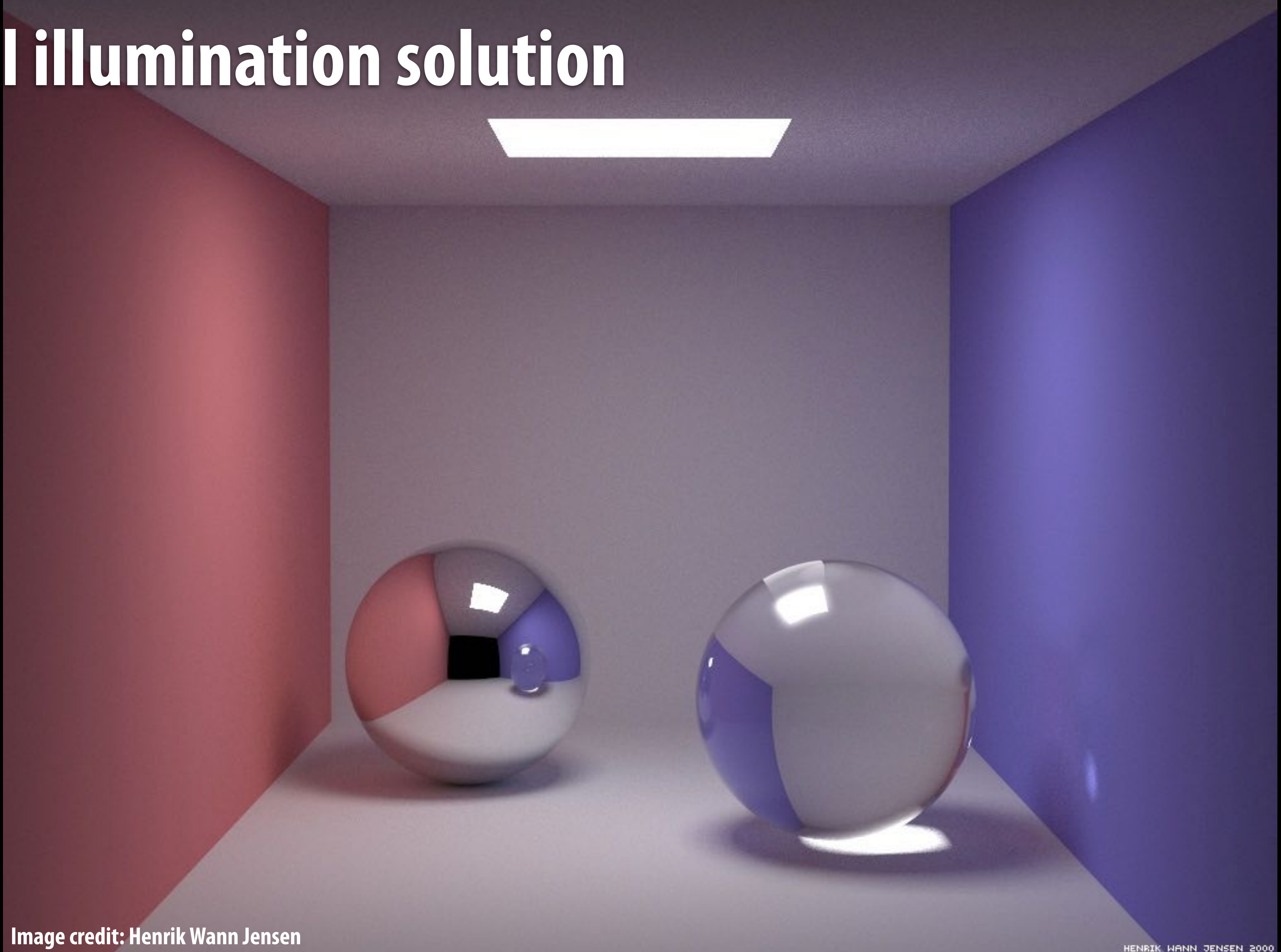


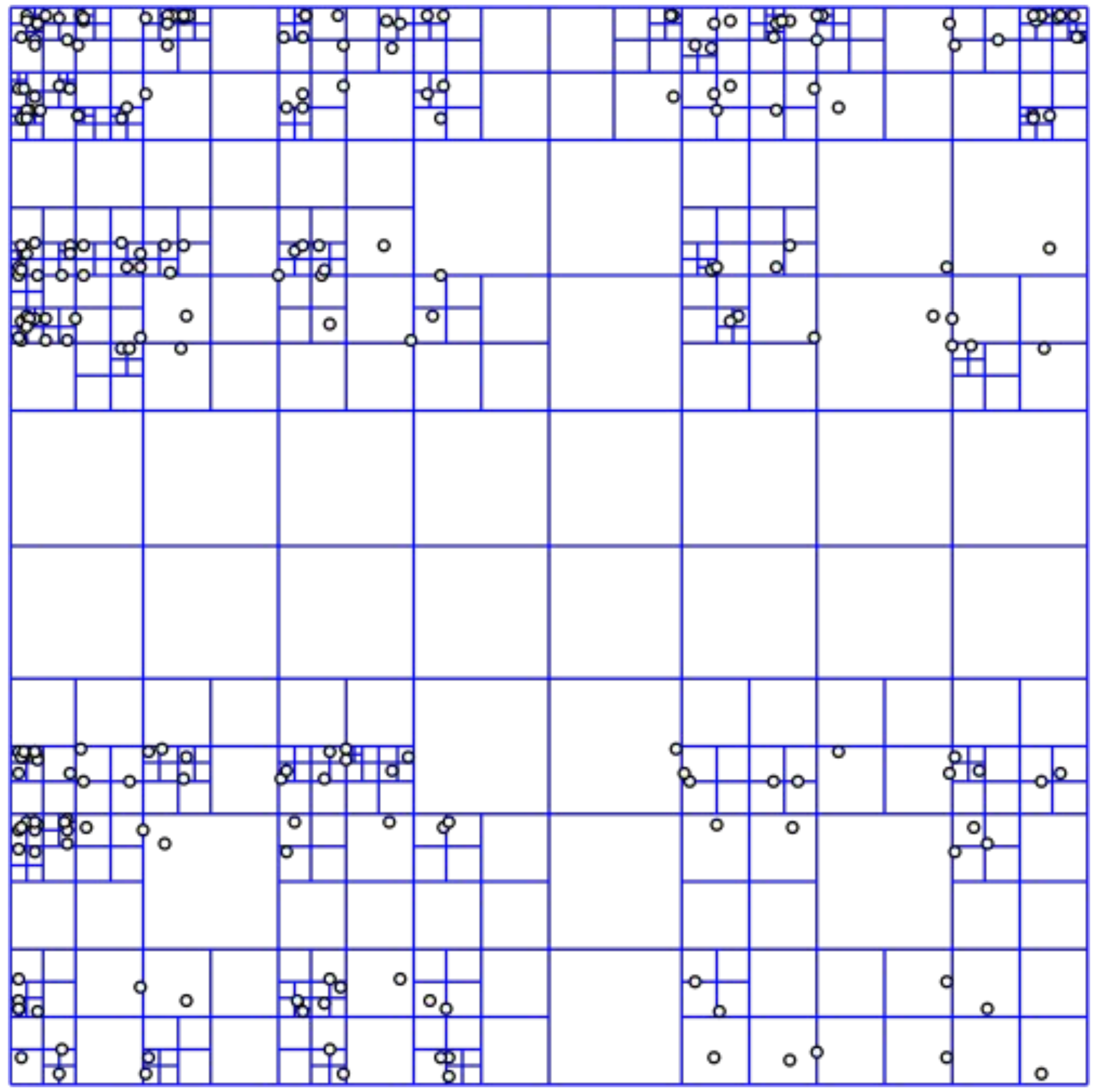
Image credit: Henrik Wann Jensen

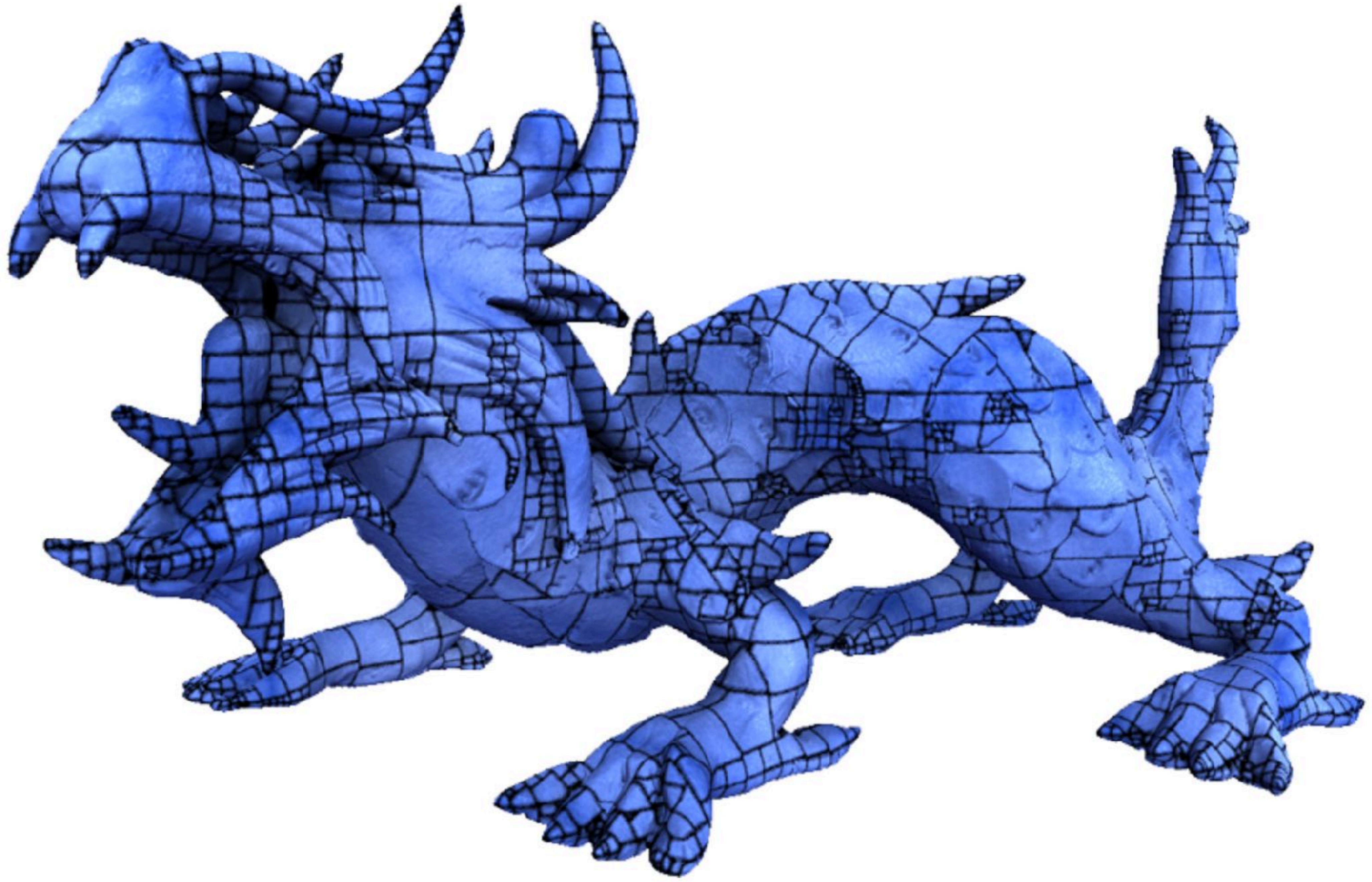
Direct illumination

Sixteen-bounce global illumination

Next time: spatial acceleration data structures

- Testing every primitive in the scene to find ray-scene intersection is *slow!*
- Consider accelerating a linear scan through an array with binary search
  - We can apply a similar type of thinking to accelerating 3D geometric queries





Acknowledgements

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