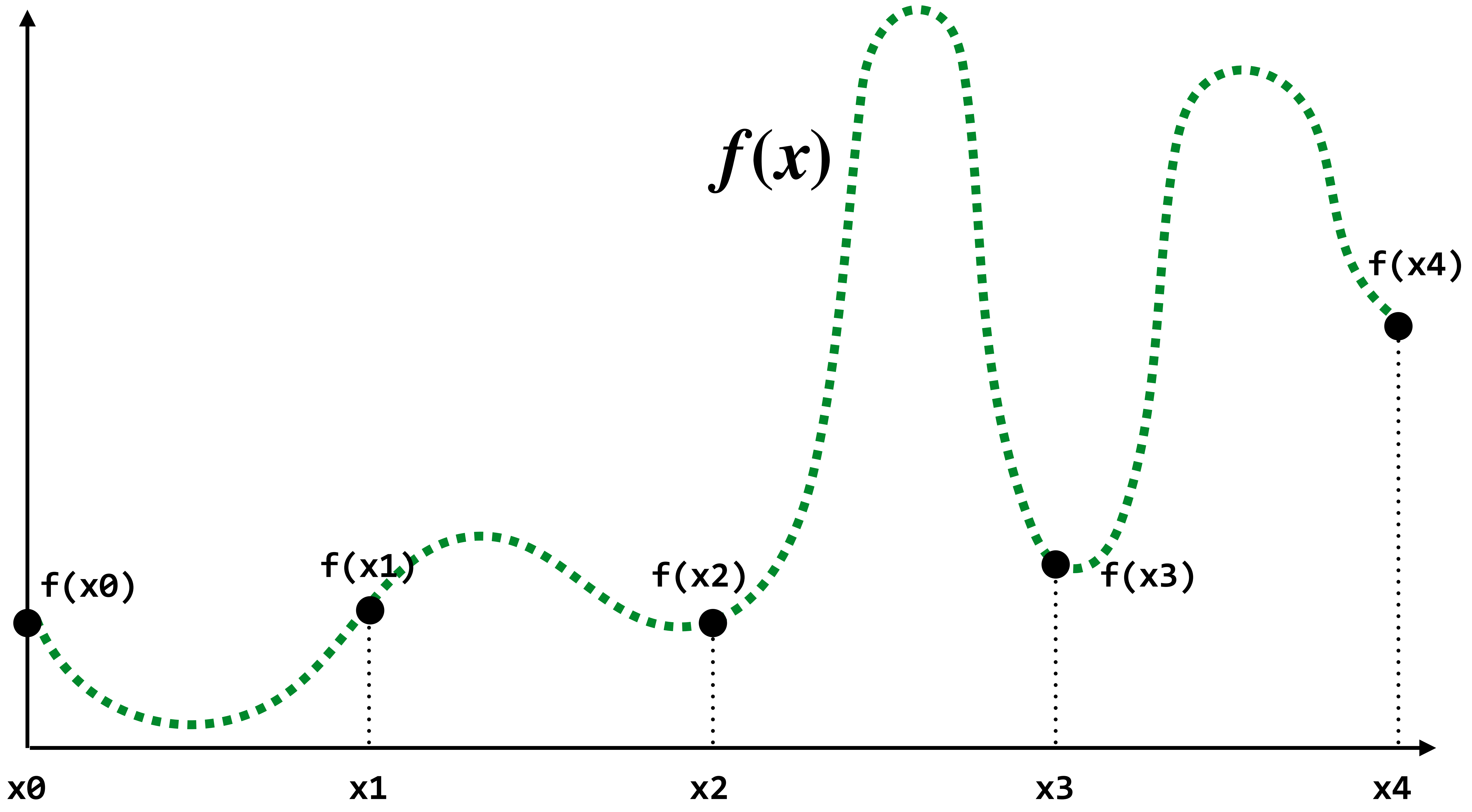
Lecture 3:

Coordinate Spaces and Transformations

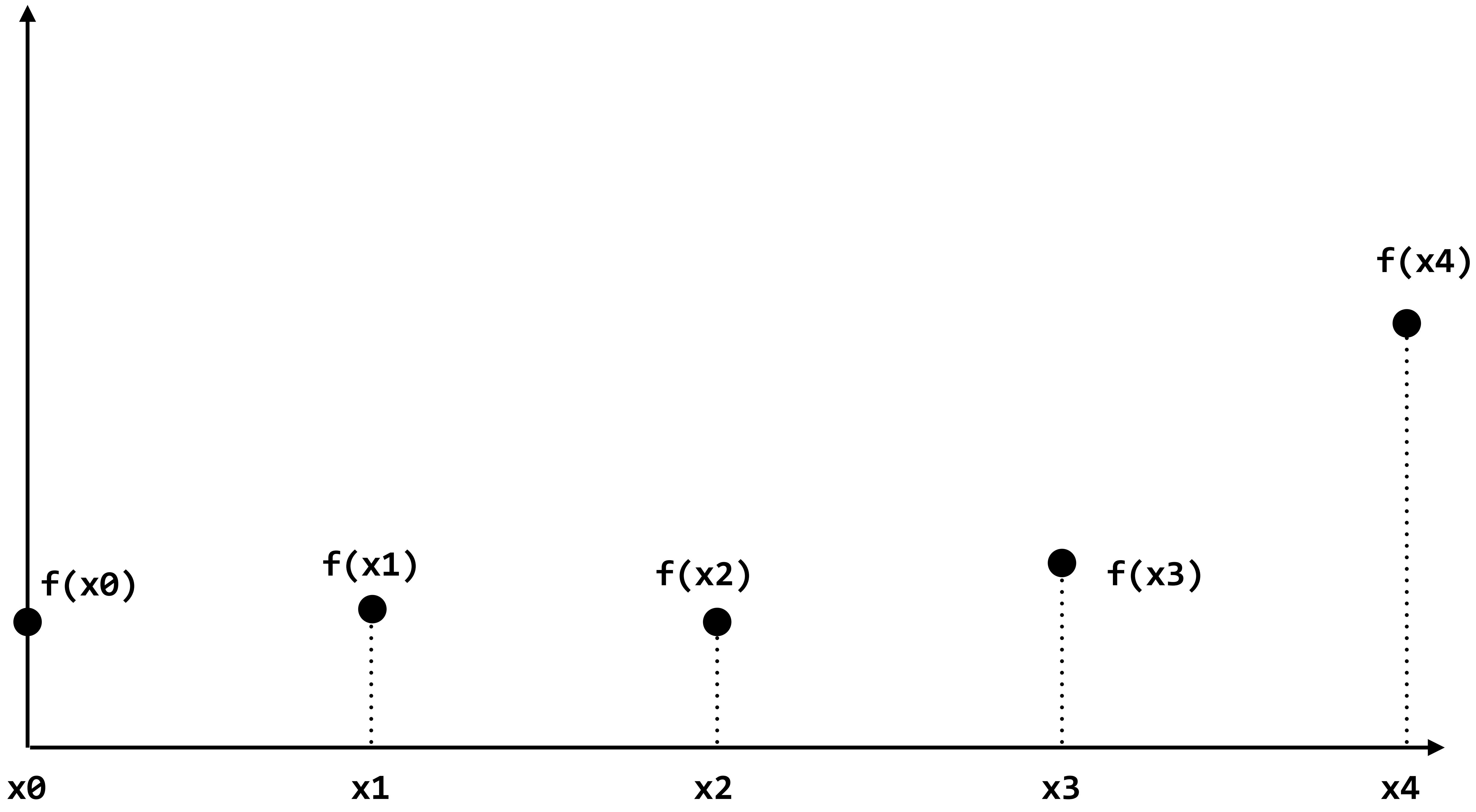
Computer Graphics: Rendering, Geometry, and Image Manipulation
Stanford CS248A, Winter 2025

Review:
Summarizing what we learned last time

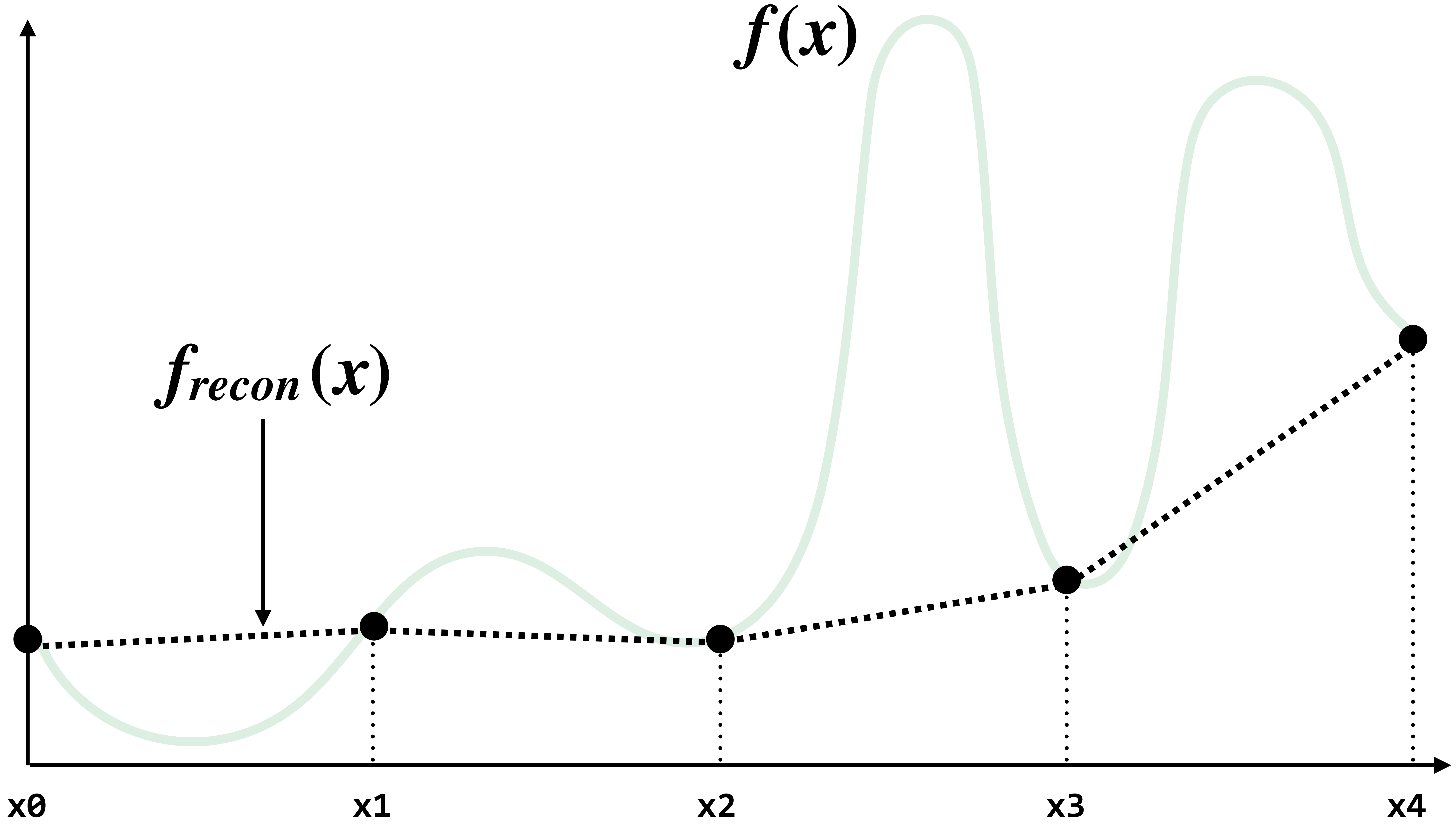
Sampling: taking measurements of a signal



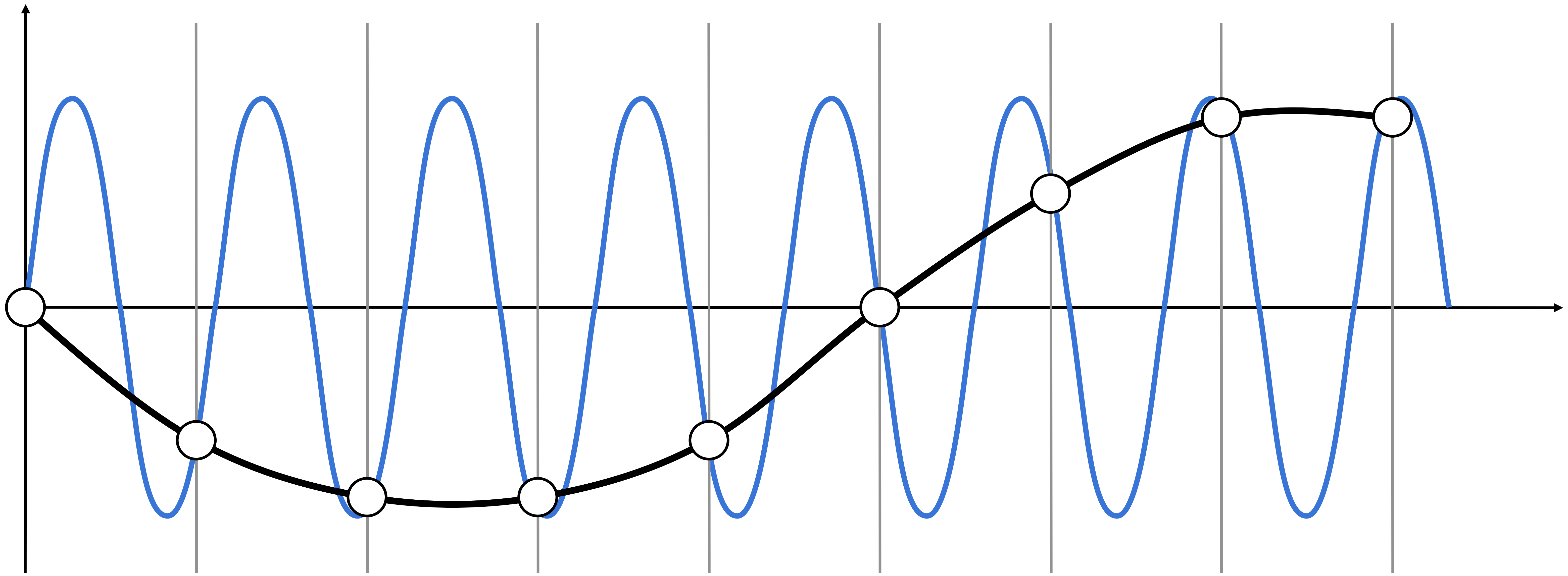
Sampling: taking measurements of a signal



Reconstruction: approximating continuous signal from the discrete set of measurements



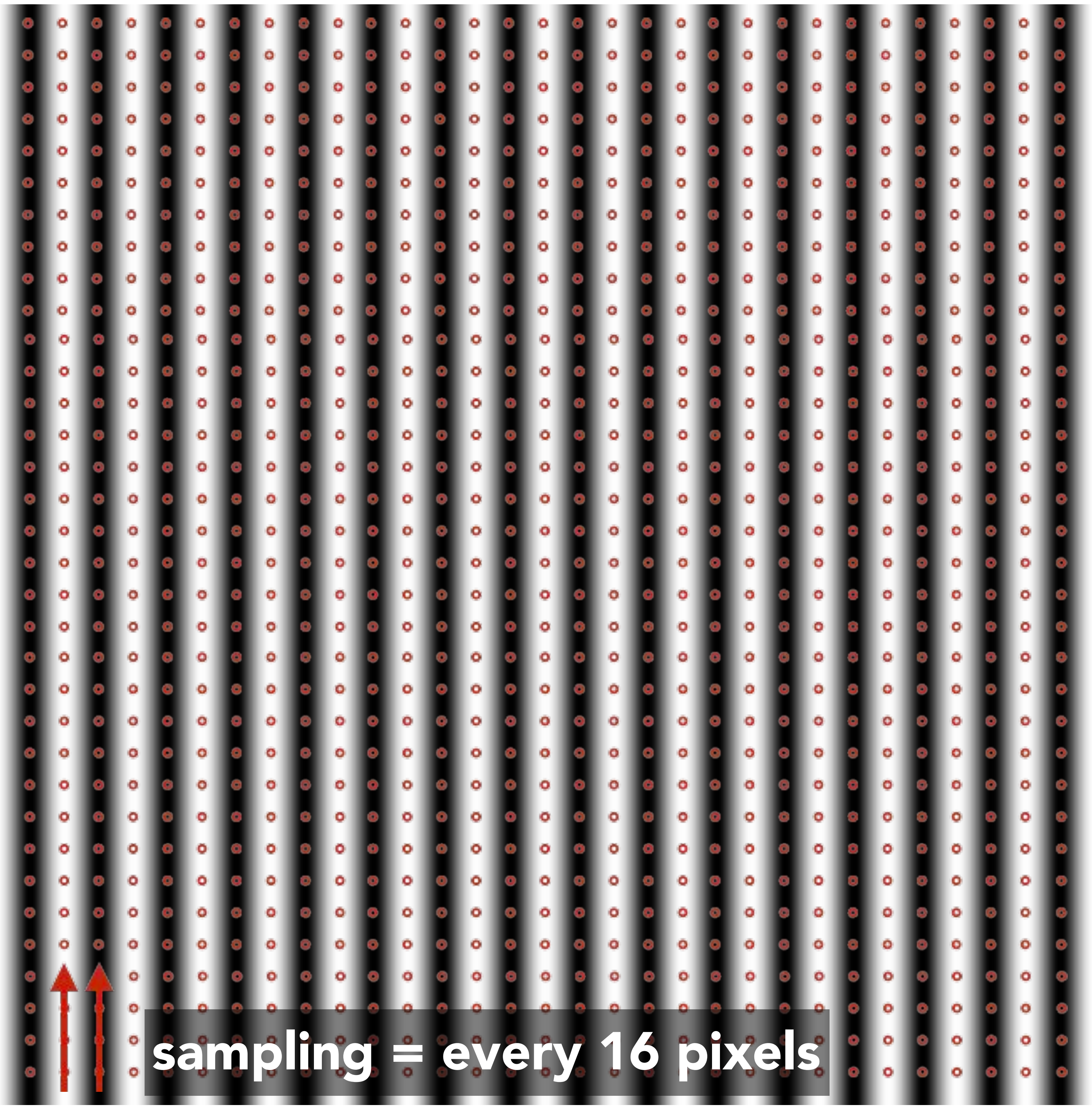
Sampling a signal too sparsely can result in aliasing



High-frequency signal is insufficiently sampled: samples erroneously appear to be from a low-frequency signal

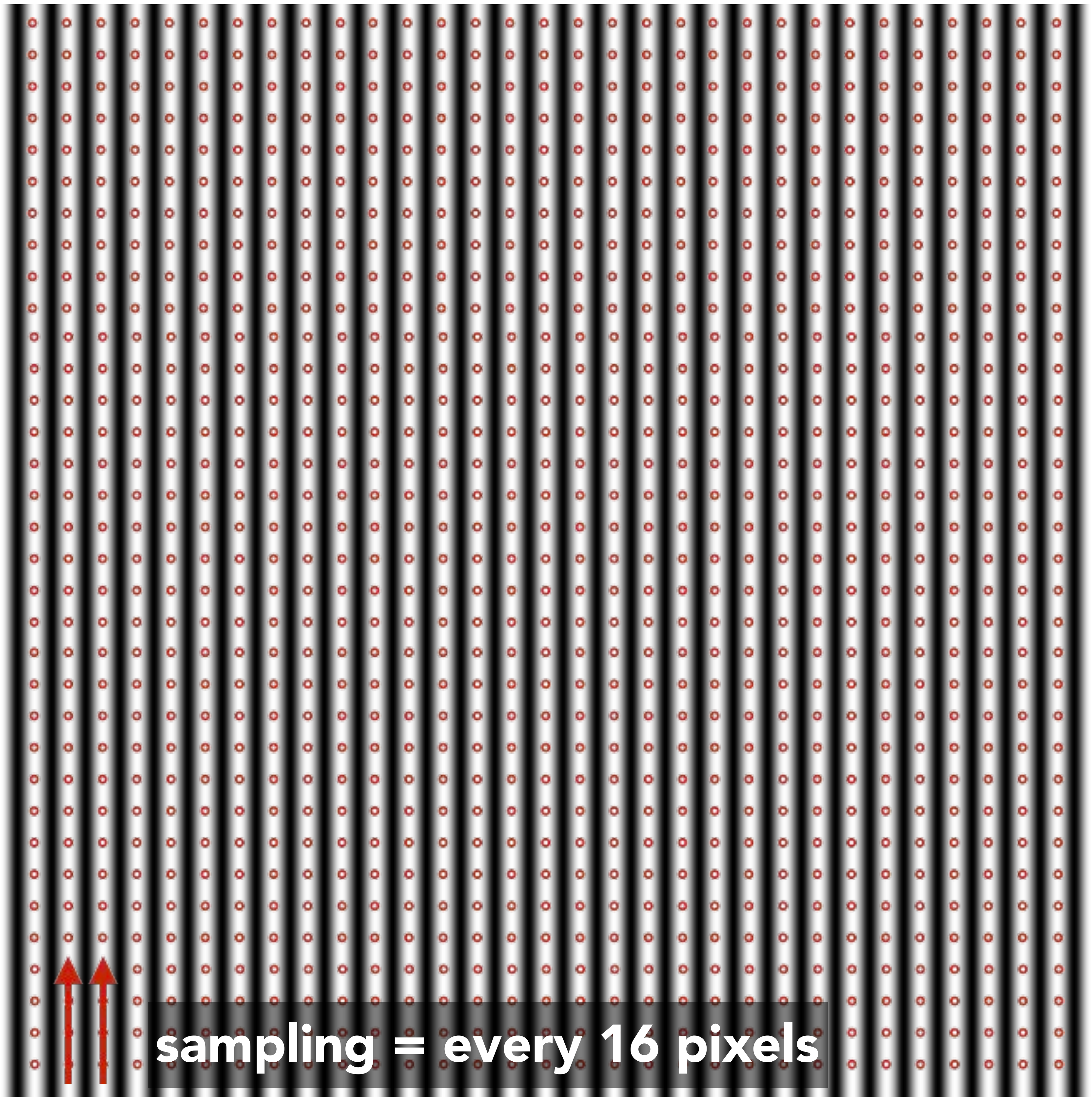
Two frequencies that are indistinguishable at a given sampling rate are called "aliases"

Sampling a signal too sparsely can result in aliasing



$$\sin(2\pi/32)x$$

frequency 1/32; 32 pixels per cycle

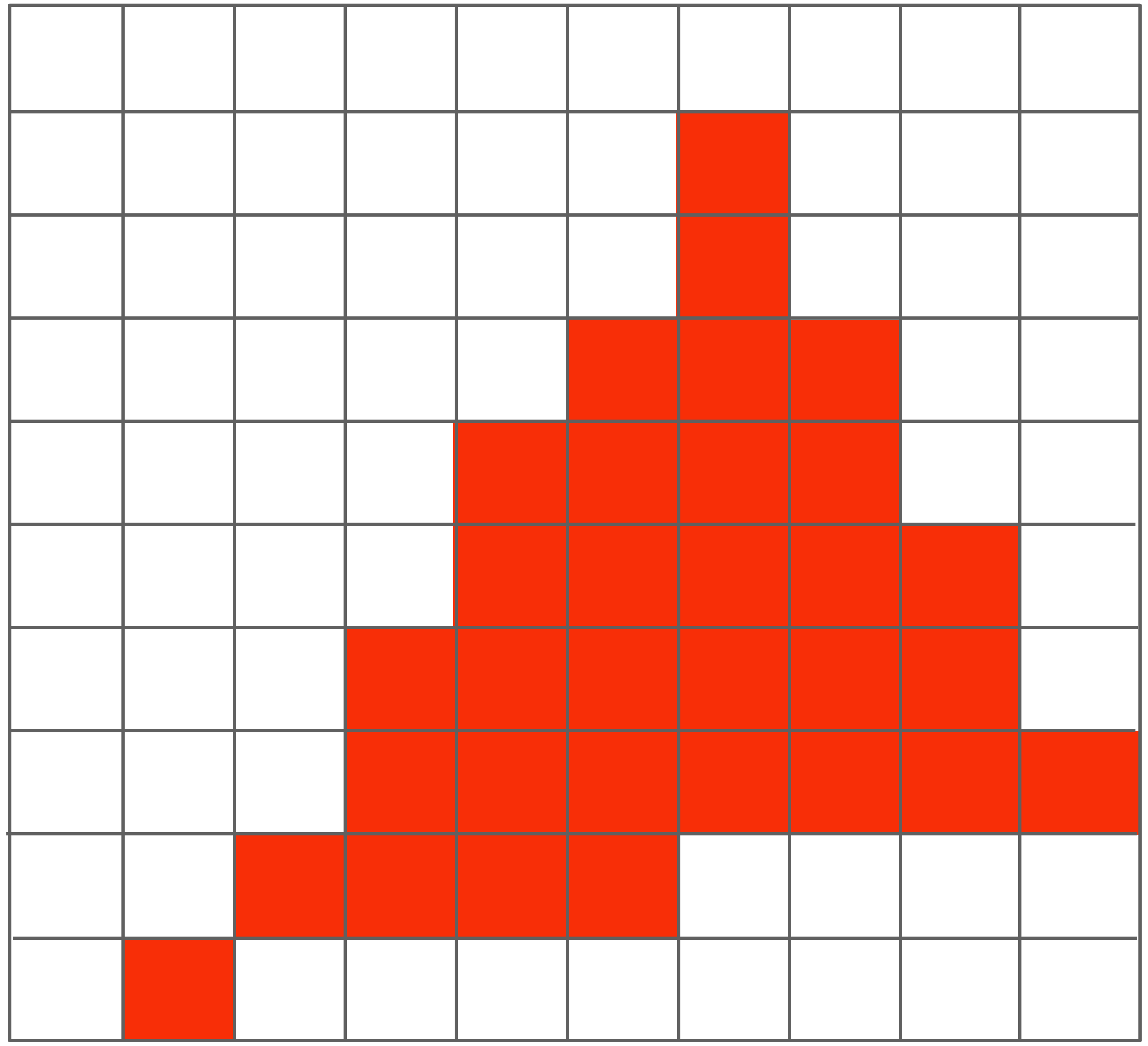


$$\sin(2\pi/16)x$$

frequency 1/16; 16 pixels per cycle

Sampling a signal too sparsely can result in aliasing

Sampling a signal too sparsely can result in aliasing

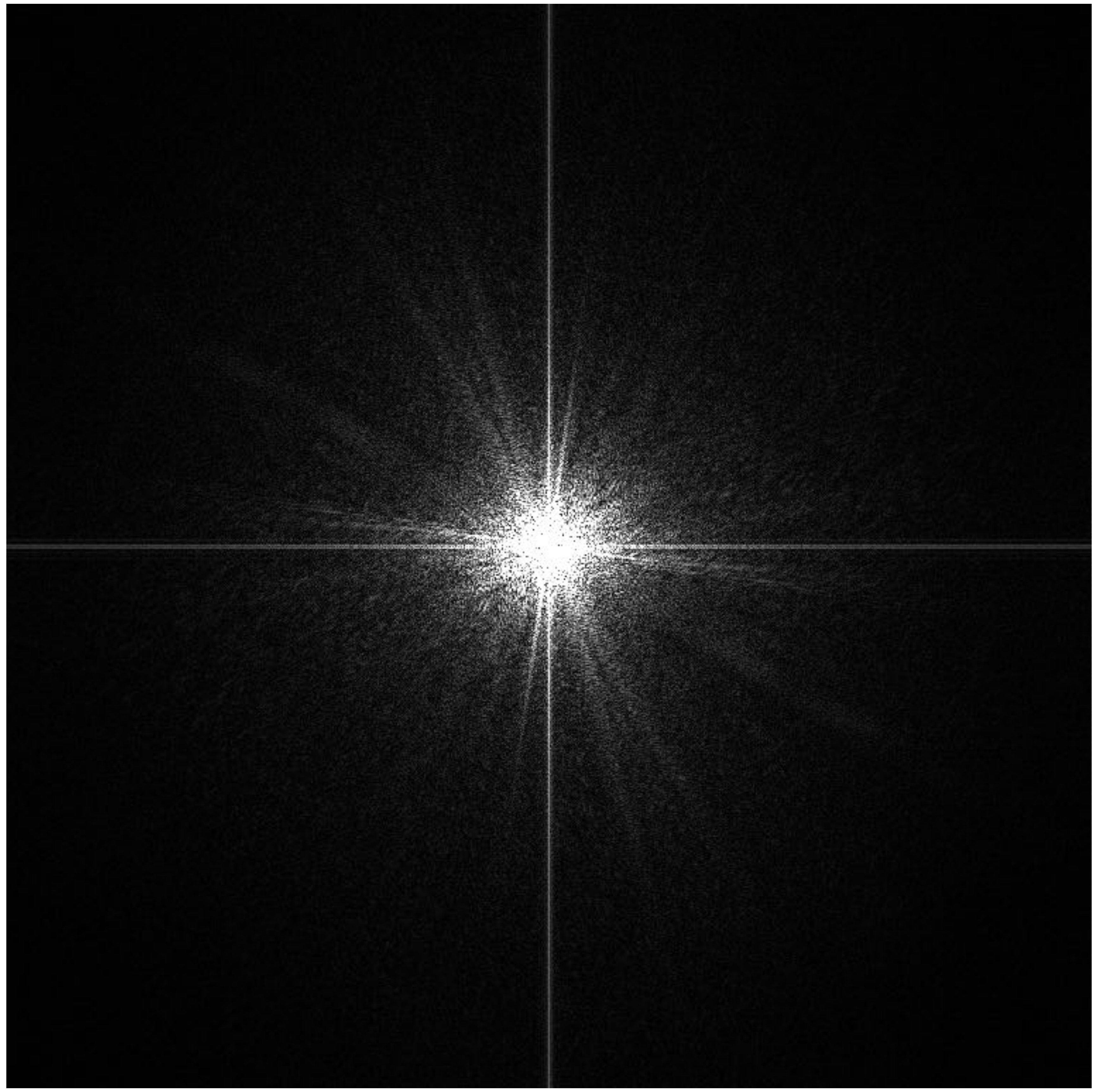


Jaggies!

What does high frequency detail in an image “look like”?

Visualizing the frequency content of images

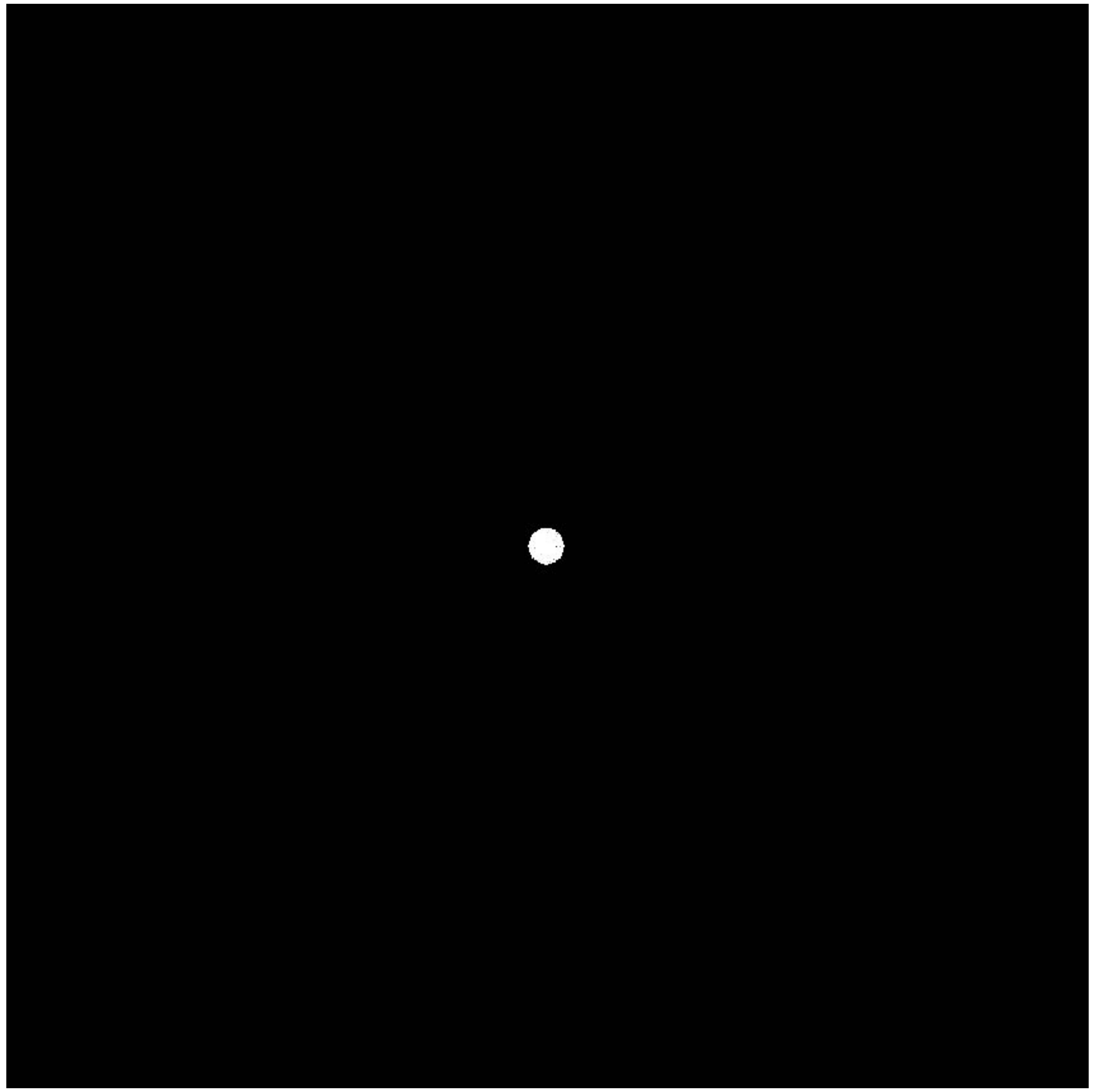
Spatial domain



Frequency domain

Retain low frequencies only (smooth gradients)

Spatial domain

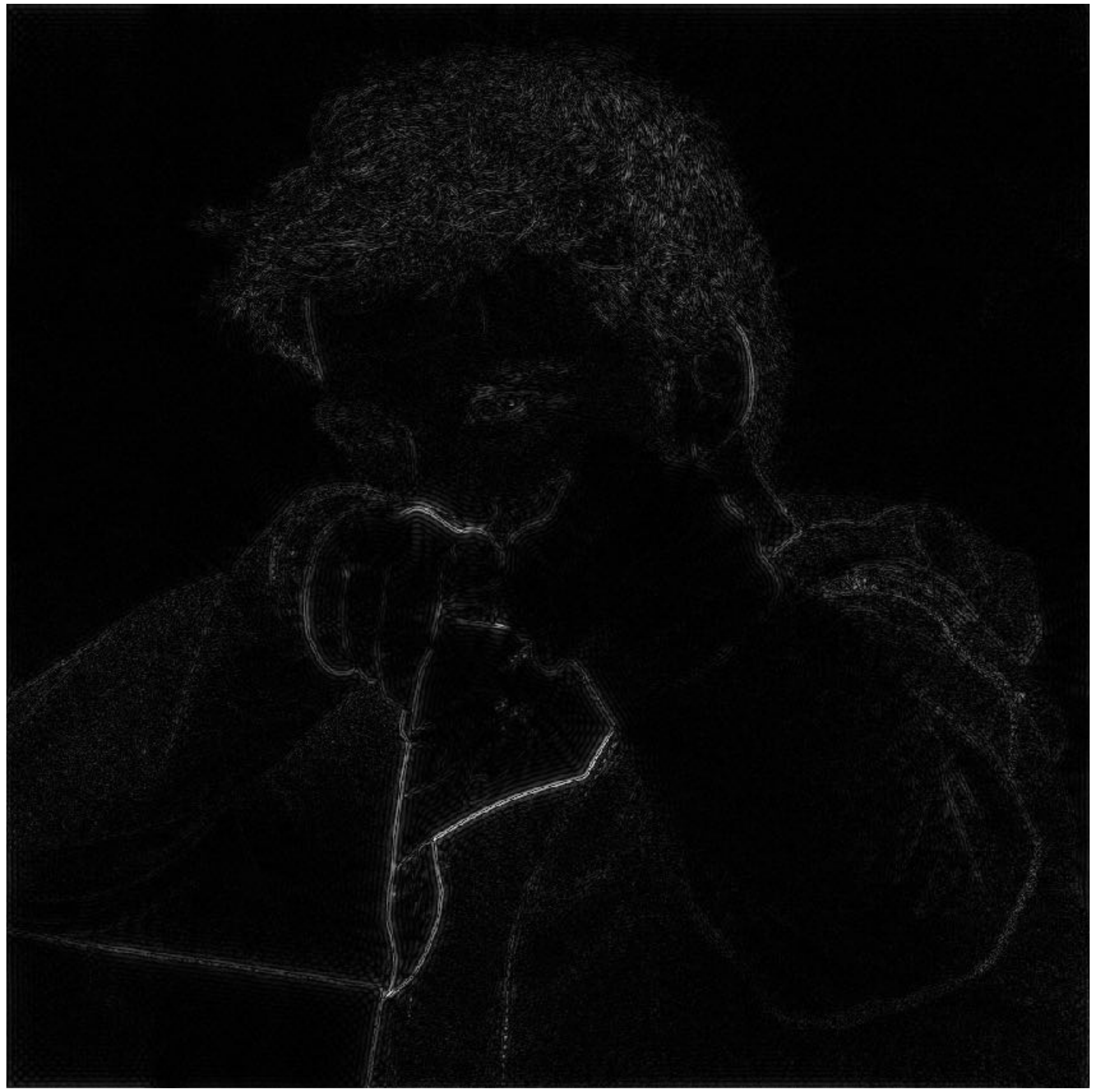


Frequency domain

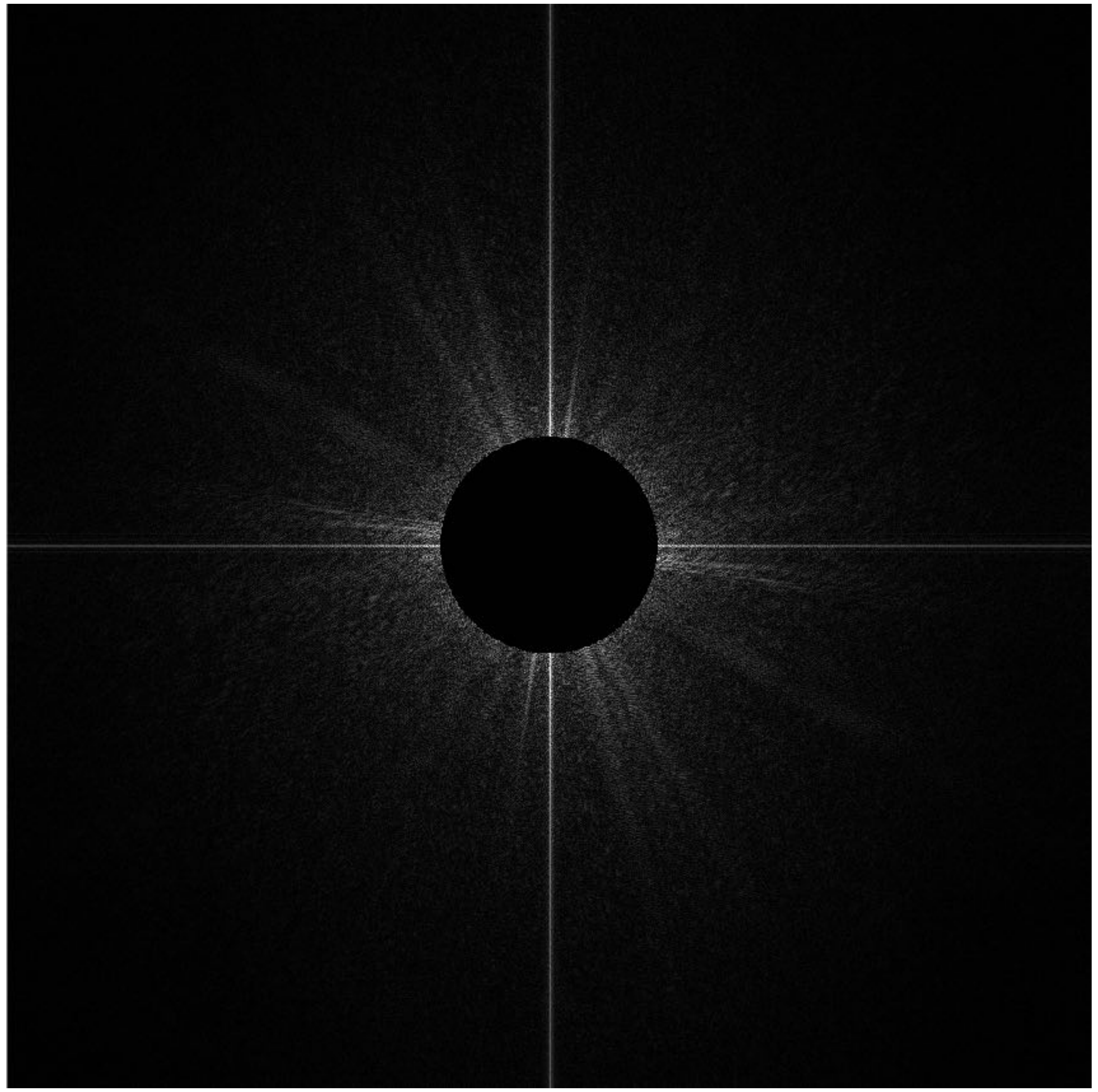
(after low-pass filter)

All frequencies above cutoff have 0 magnitude

Retain high frequencies only (edges)

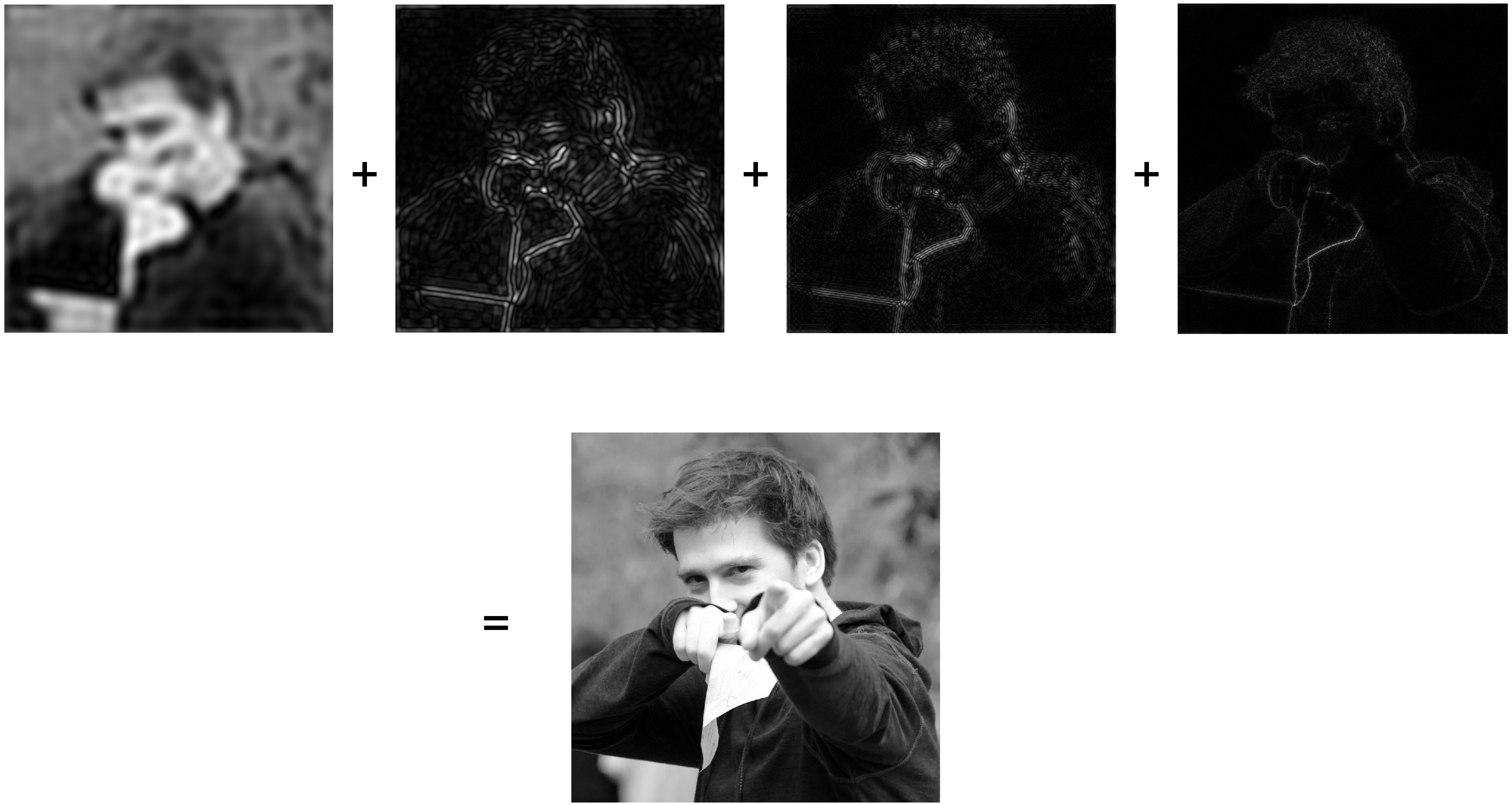


Spatial domain
(strongest edges)

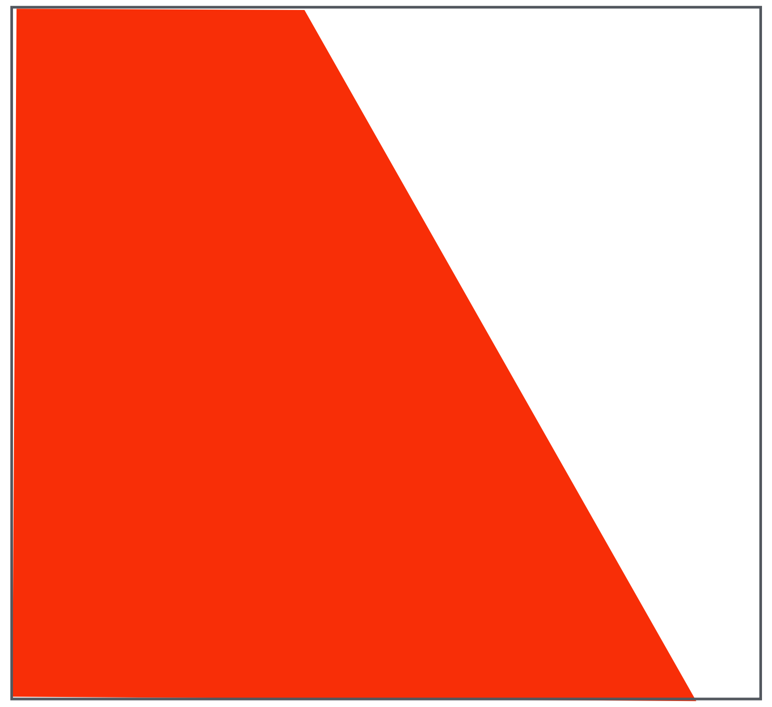


Frequency domain
(after high-pass filter)
All frequencies below threshold have 0 magnitude

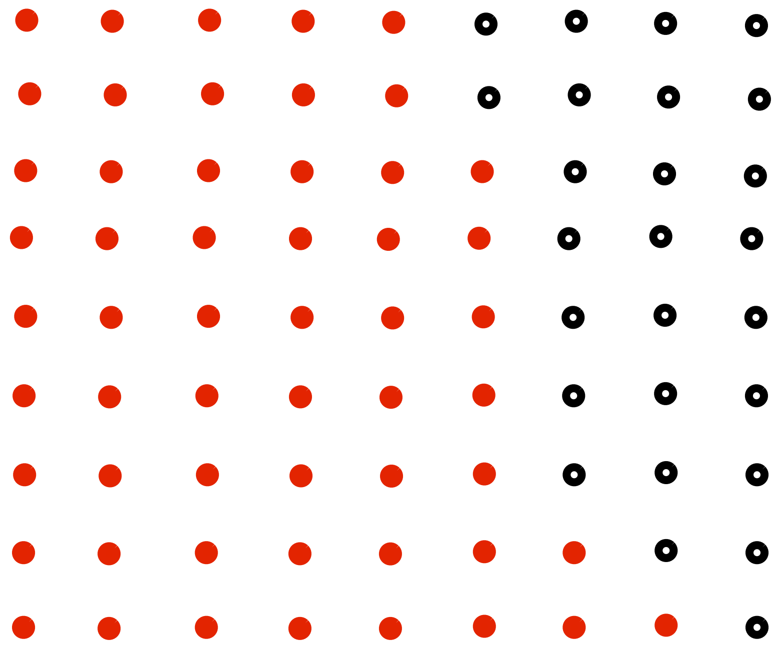
An image as a sum of its frequency components



Our anti-aliasing technique

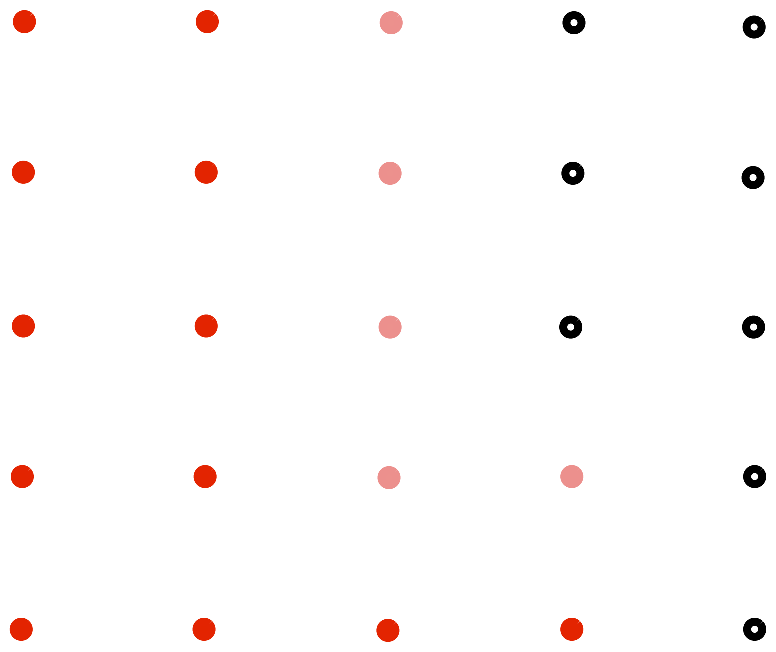


Original signal
(with high frequency edge)

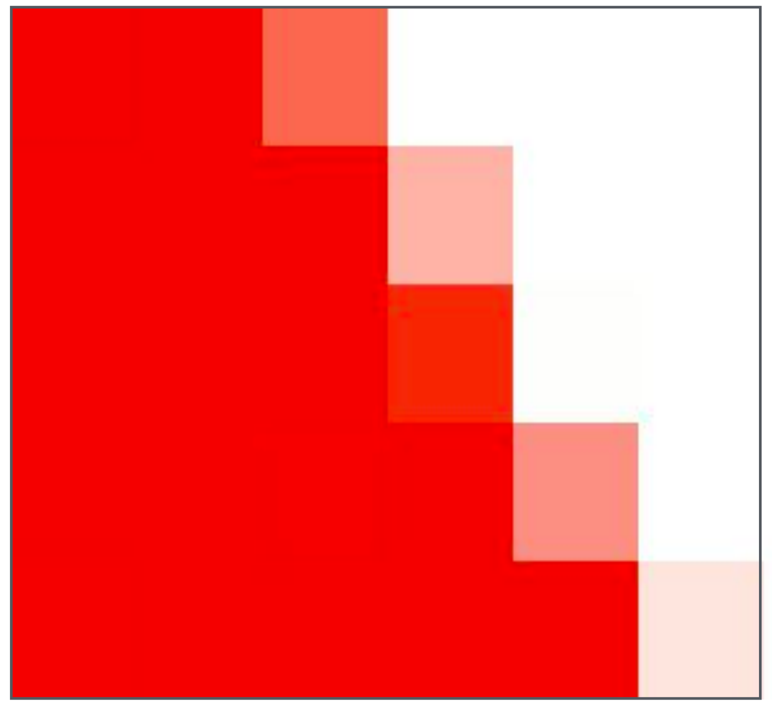


Dense sampling of signal
(supersampling)

Reconstructed signal with high frequencies reduced
(Blurring via averaging over pixel, etc)

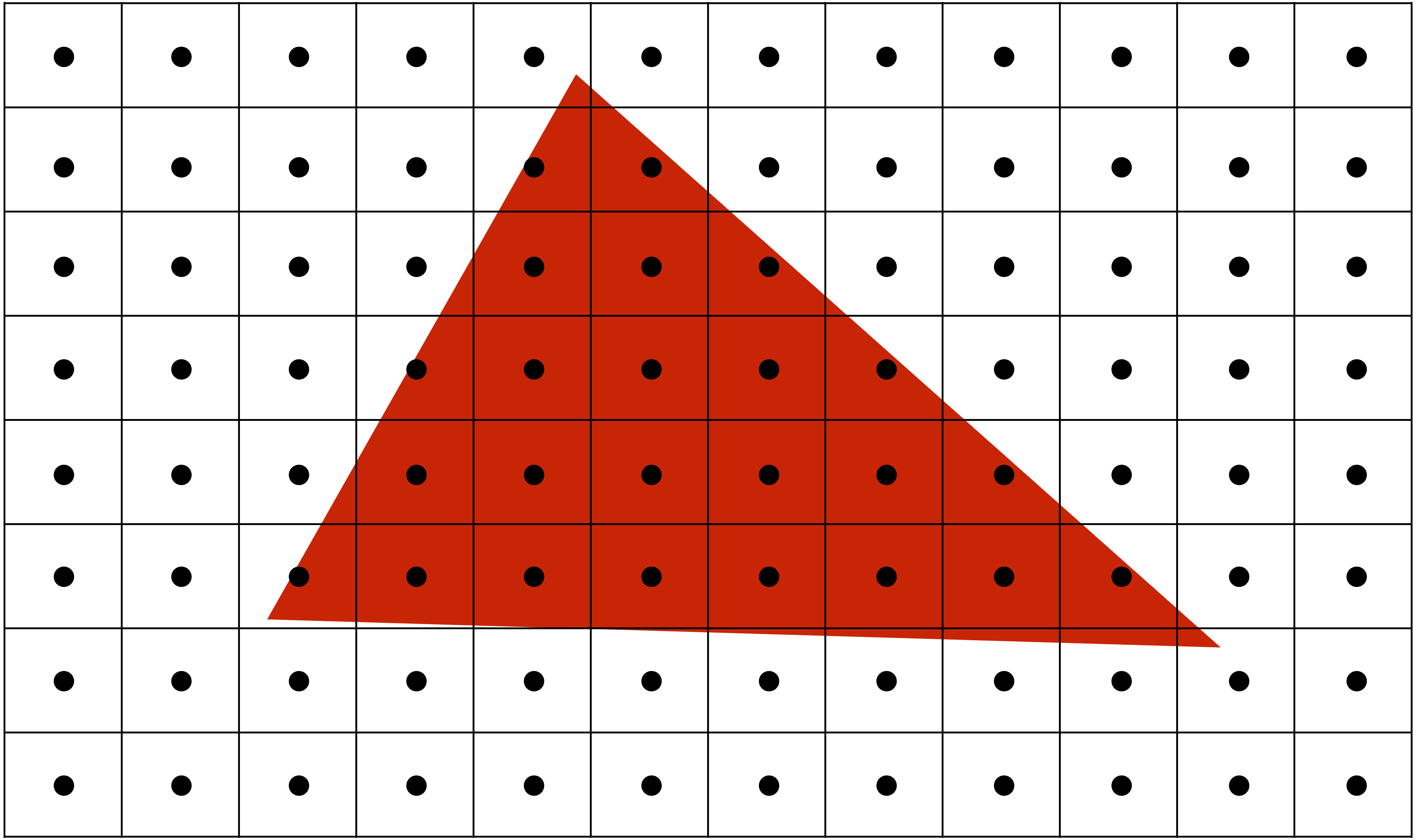


Coarsely sampled signal
(to store in image, or send to display)

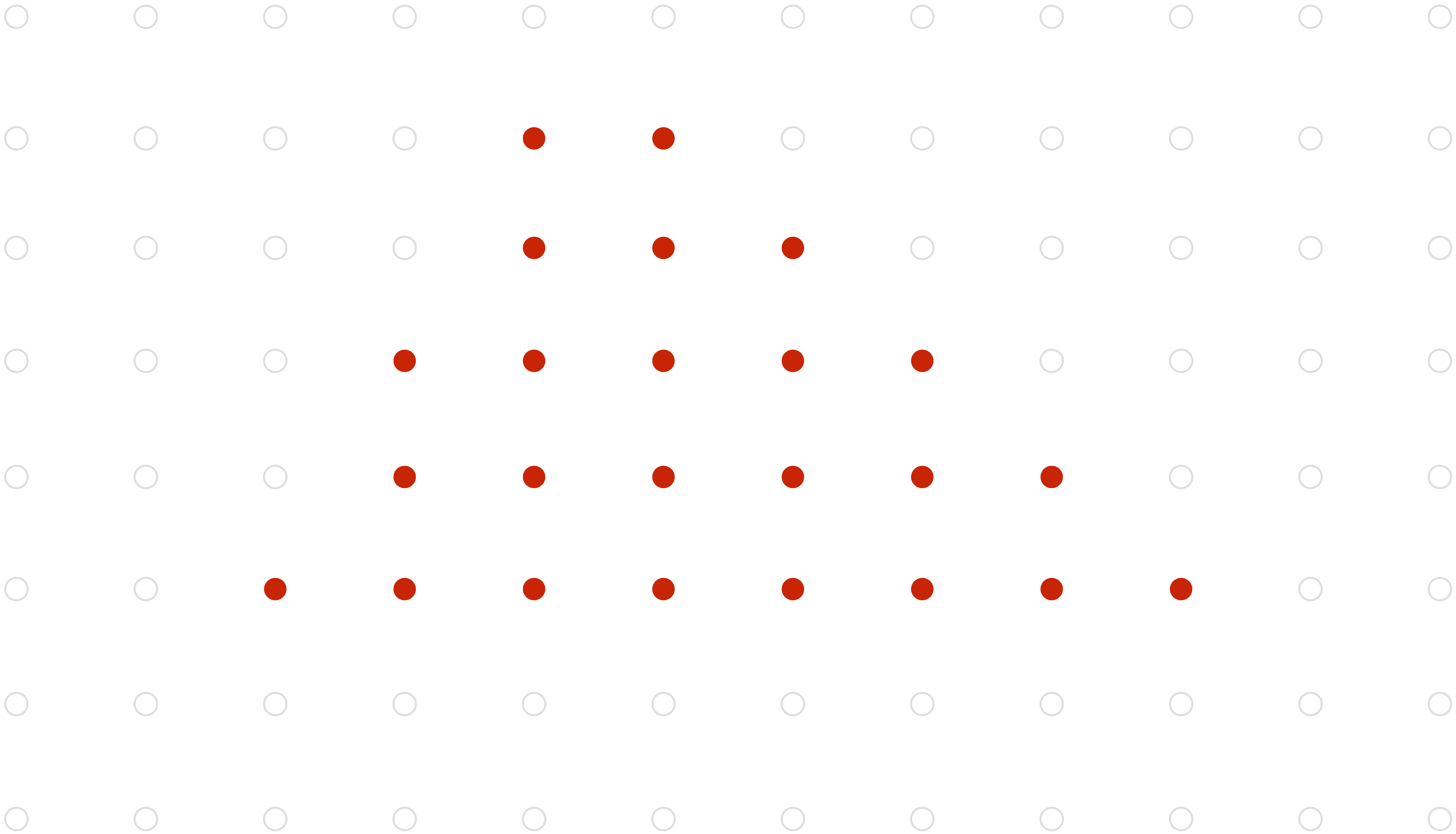


Reconstruction on display

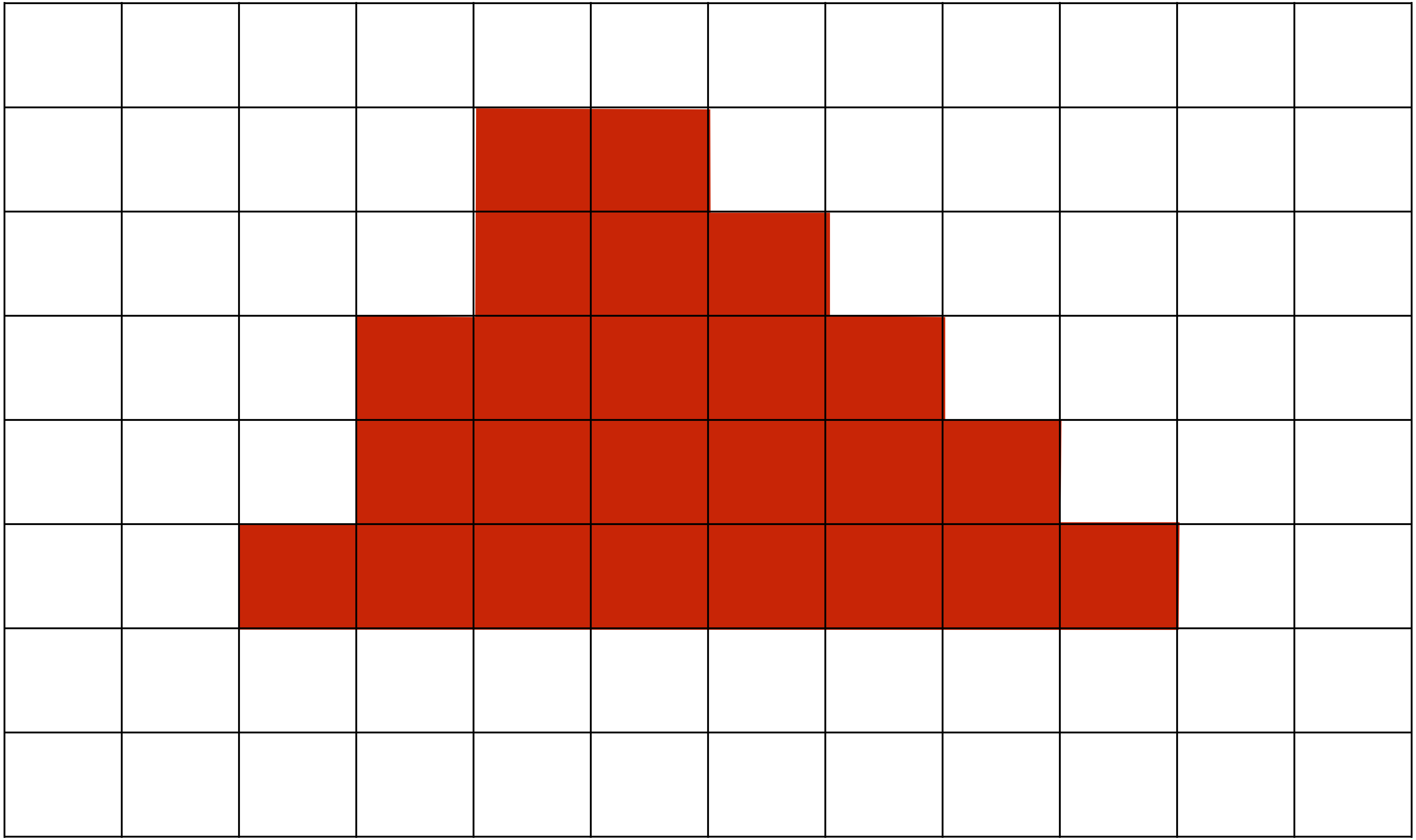
Sampling a triangle (1 sample/pixel)



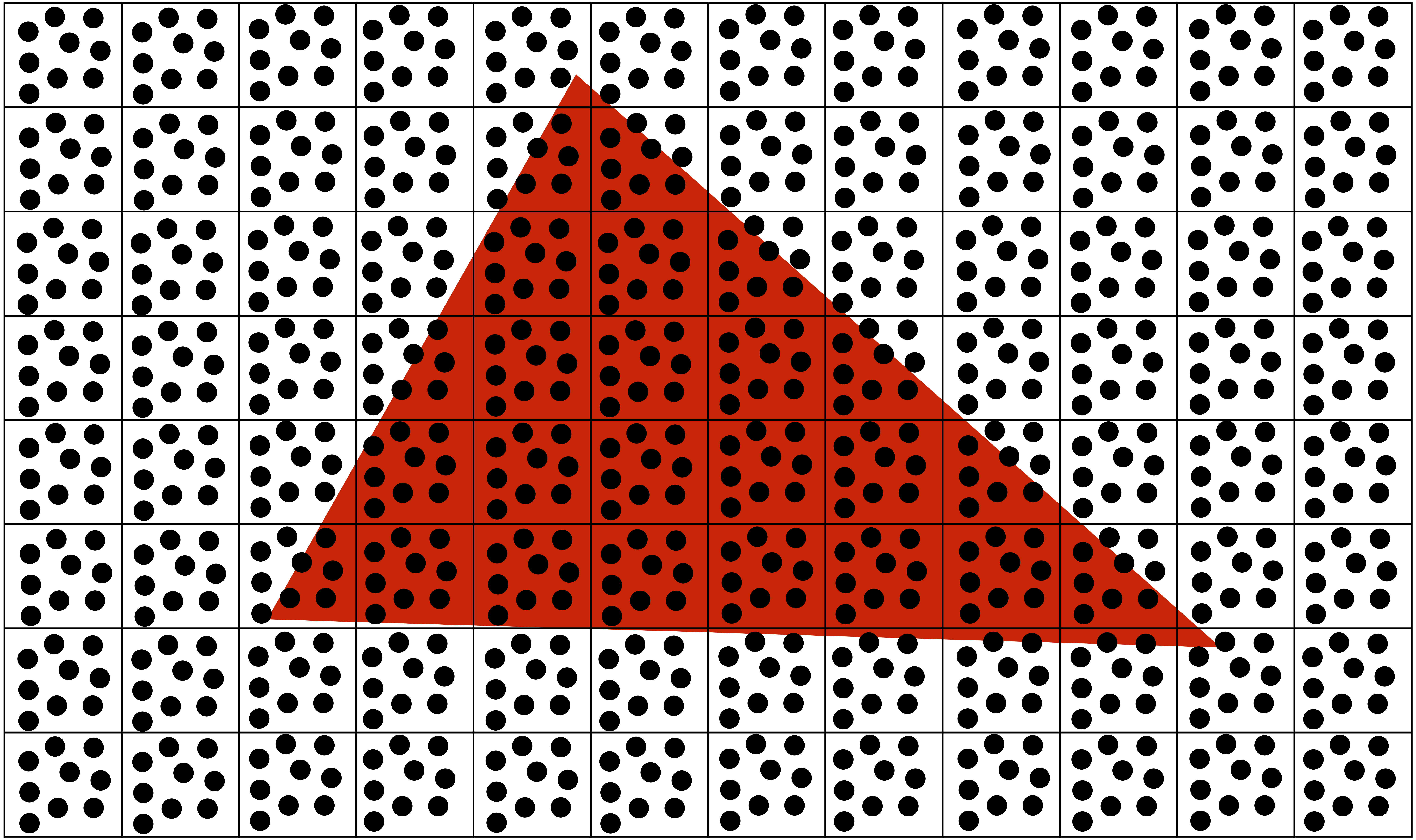
Results of sampling (1 sample/pixel)



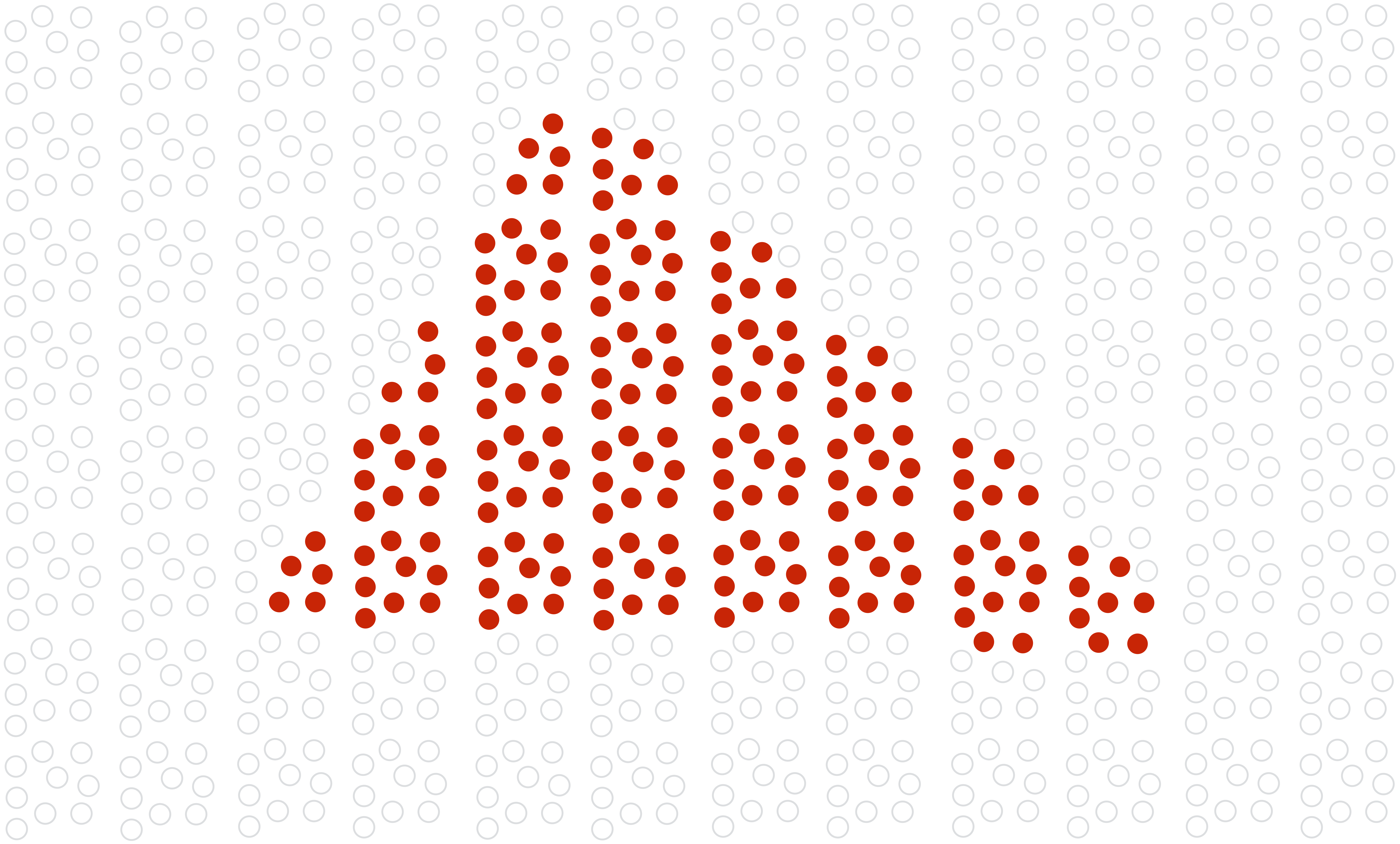
Displaying the results of sampling (square pixels on a display)



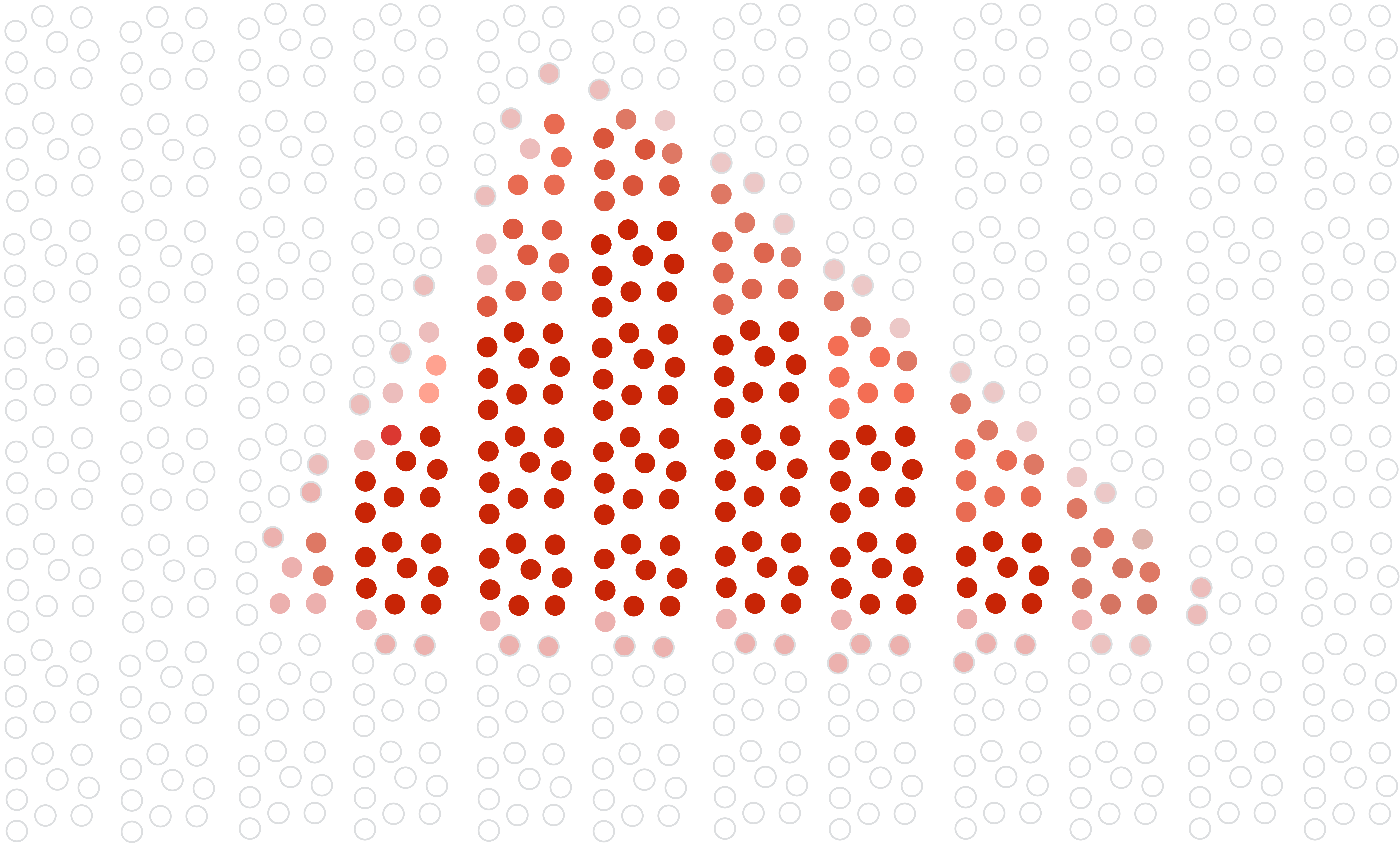
Sampling a triangle (9 samples/pixel)



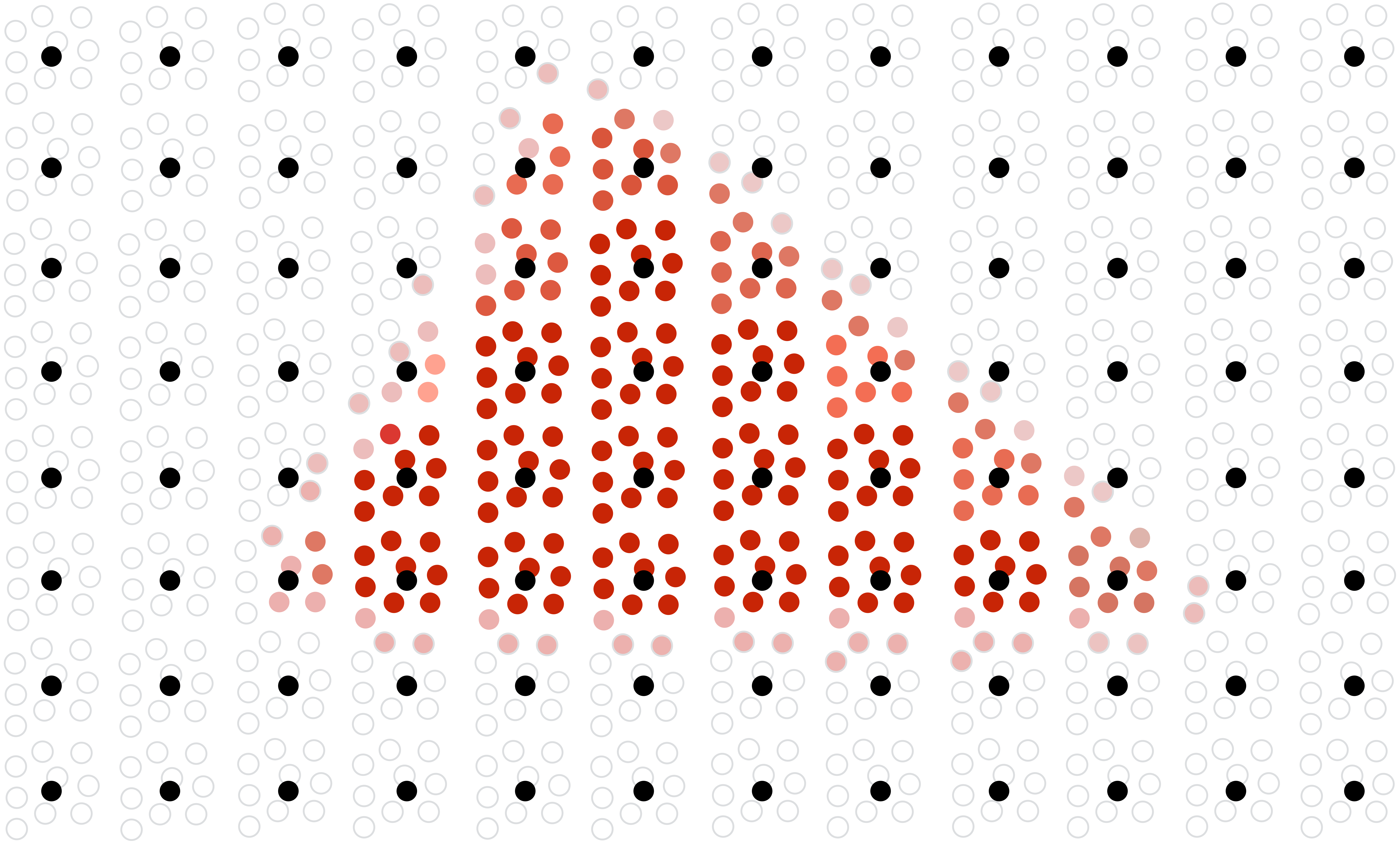
Results of sampling (9 samples/pixel)



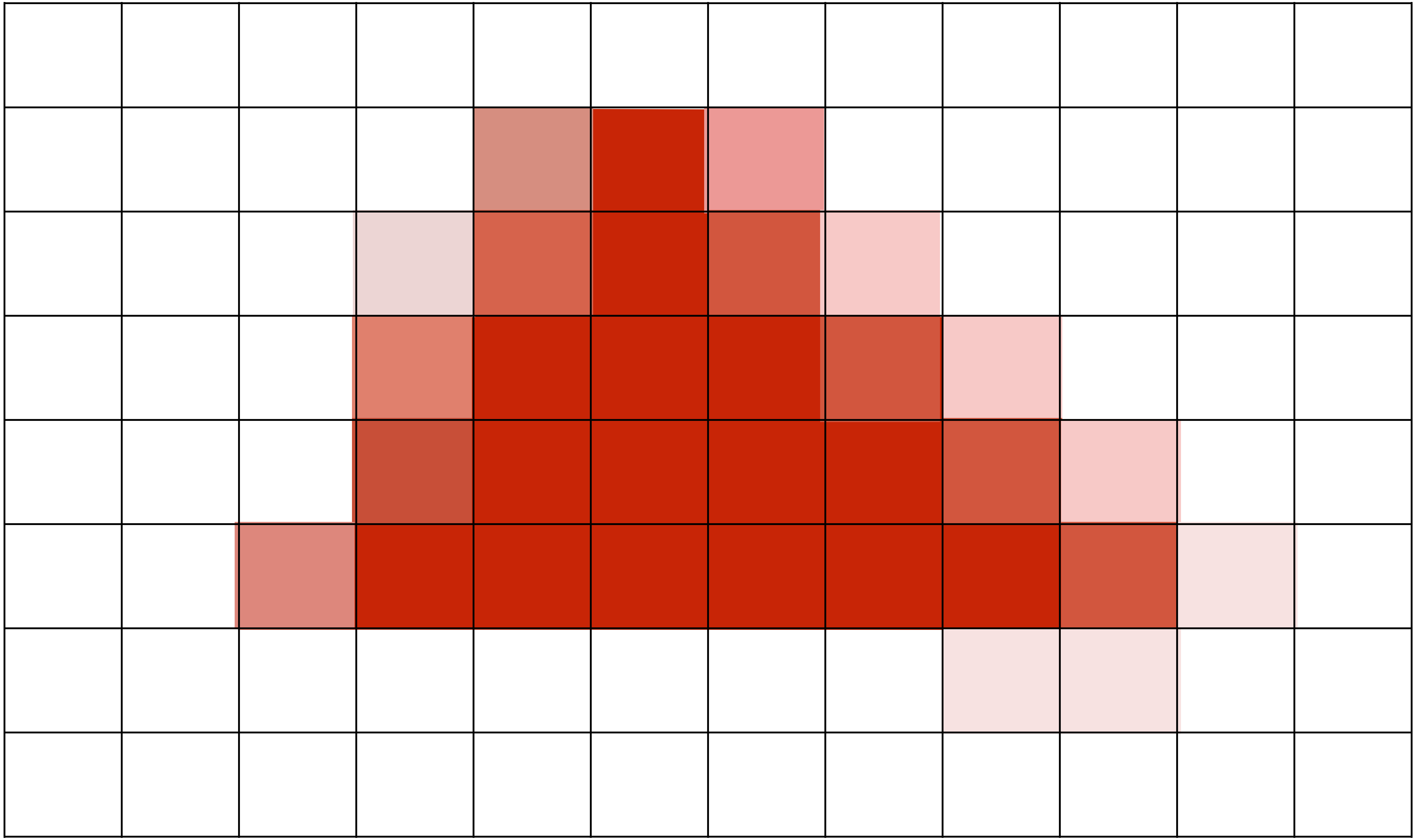
Blurring (“filtering) the densely sampled result (remove high frequencies)



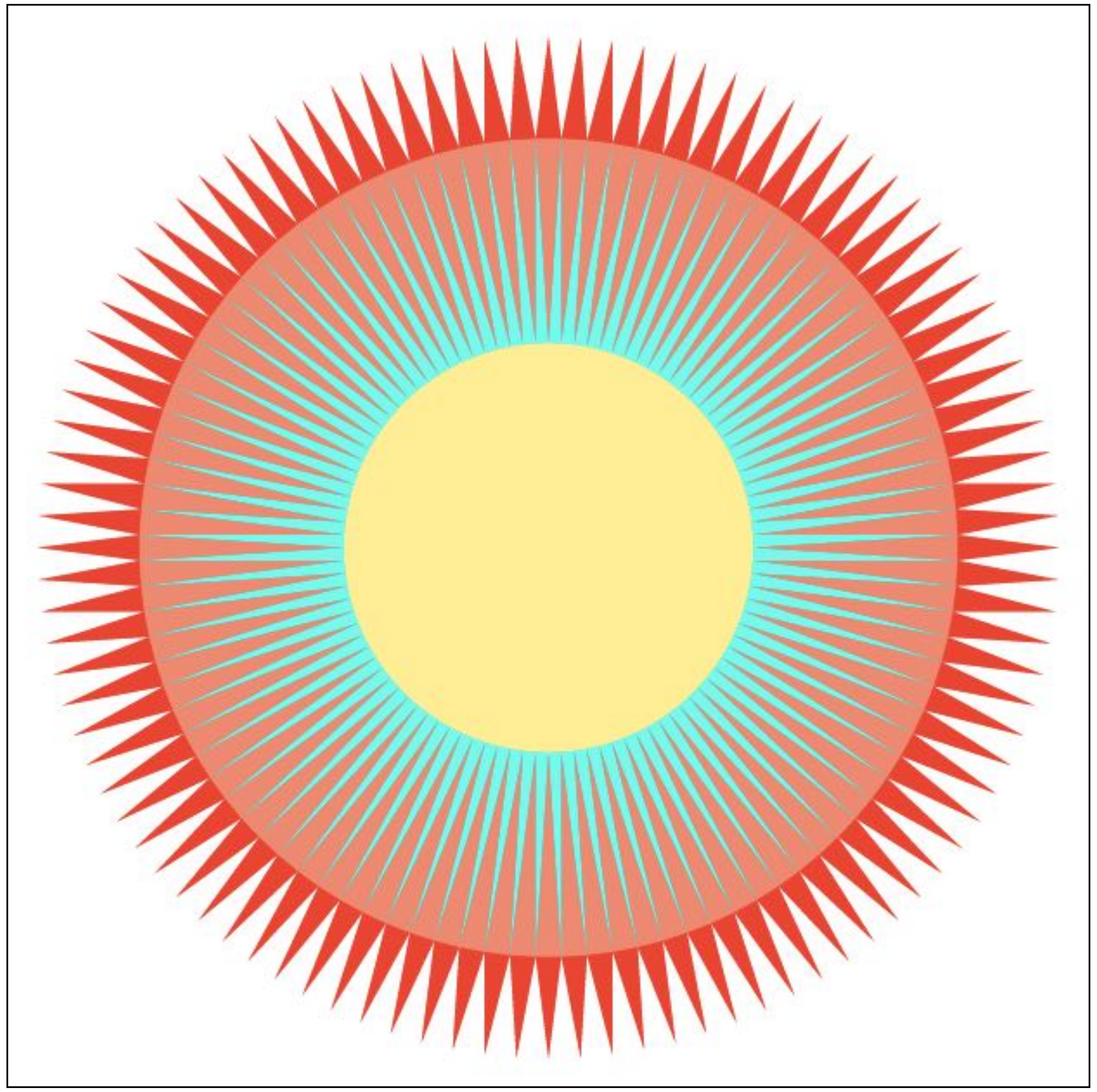
Resample blurred signal at pixel centers (one sample per pixel)

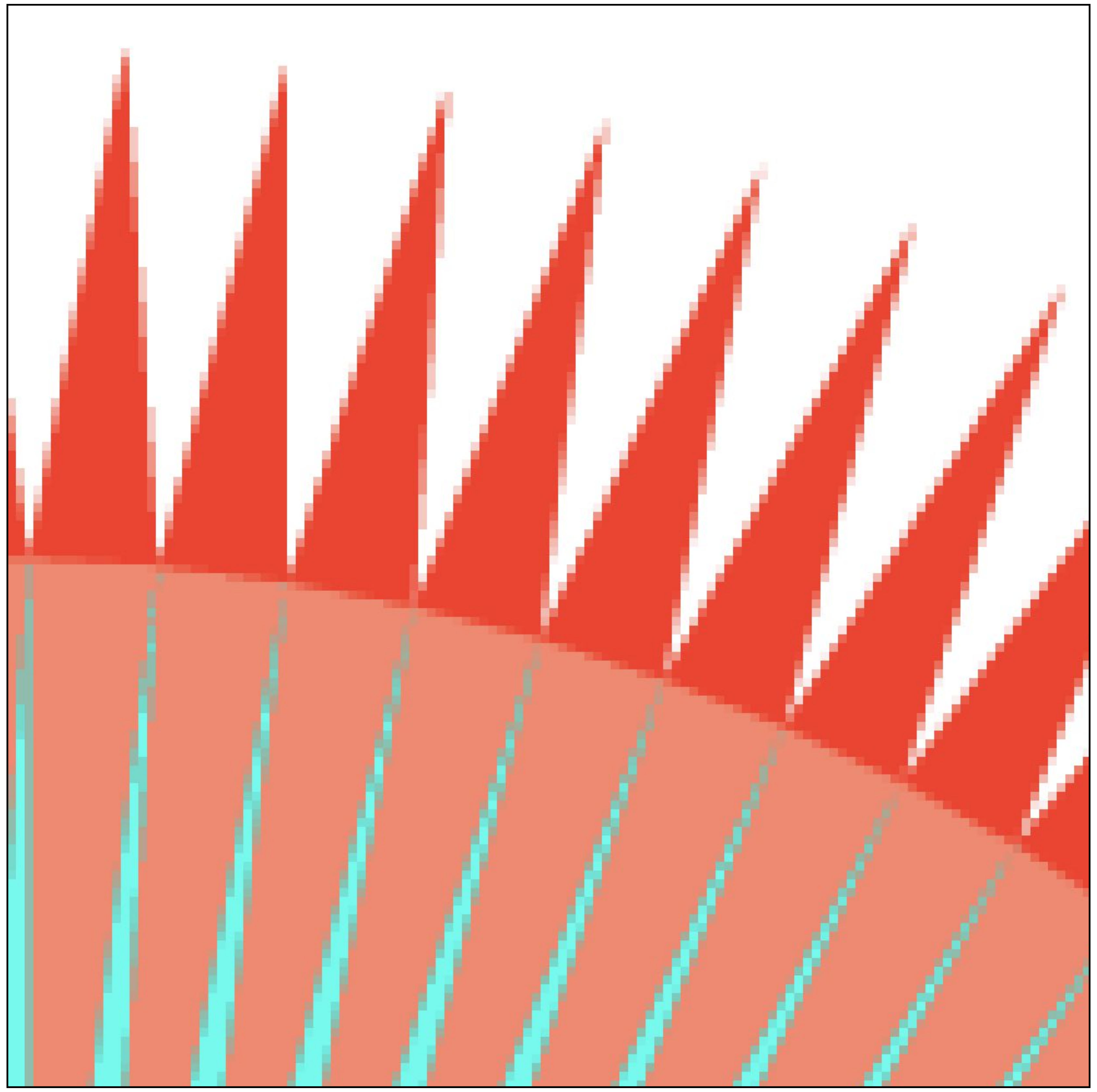


Displaying the results of sampling (square pixels on a display)



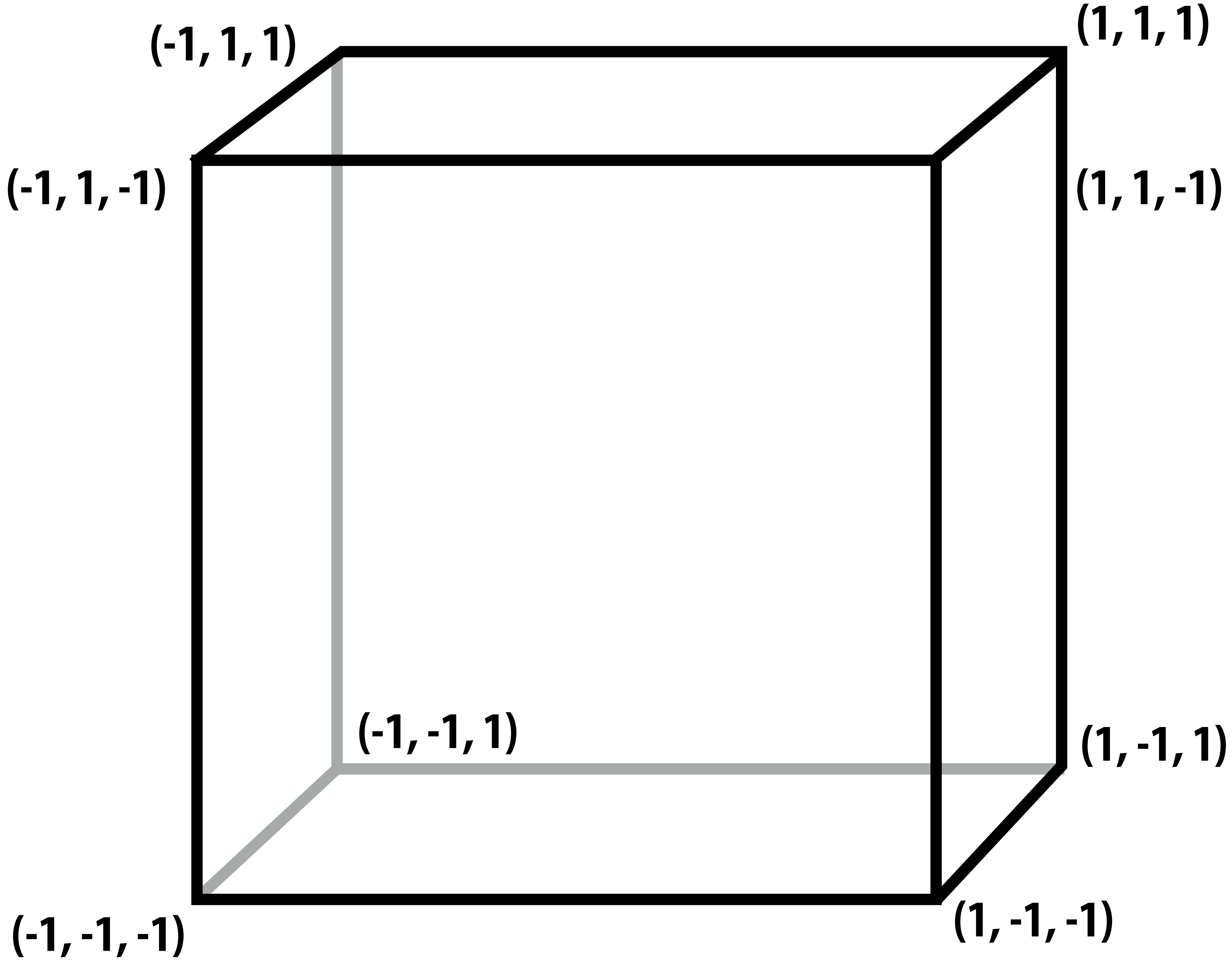
Example: anti-aliased results



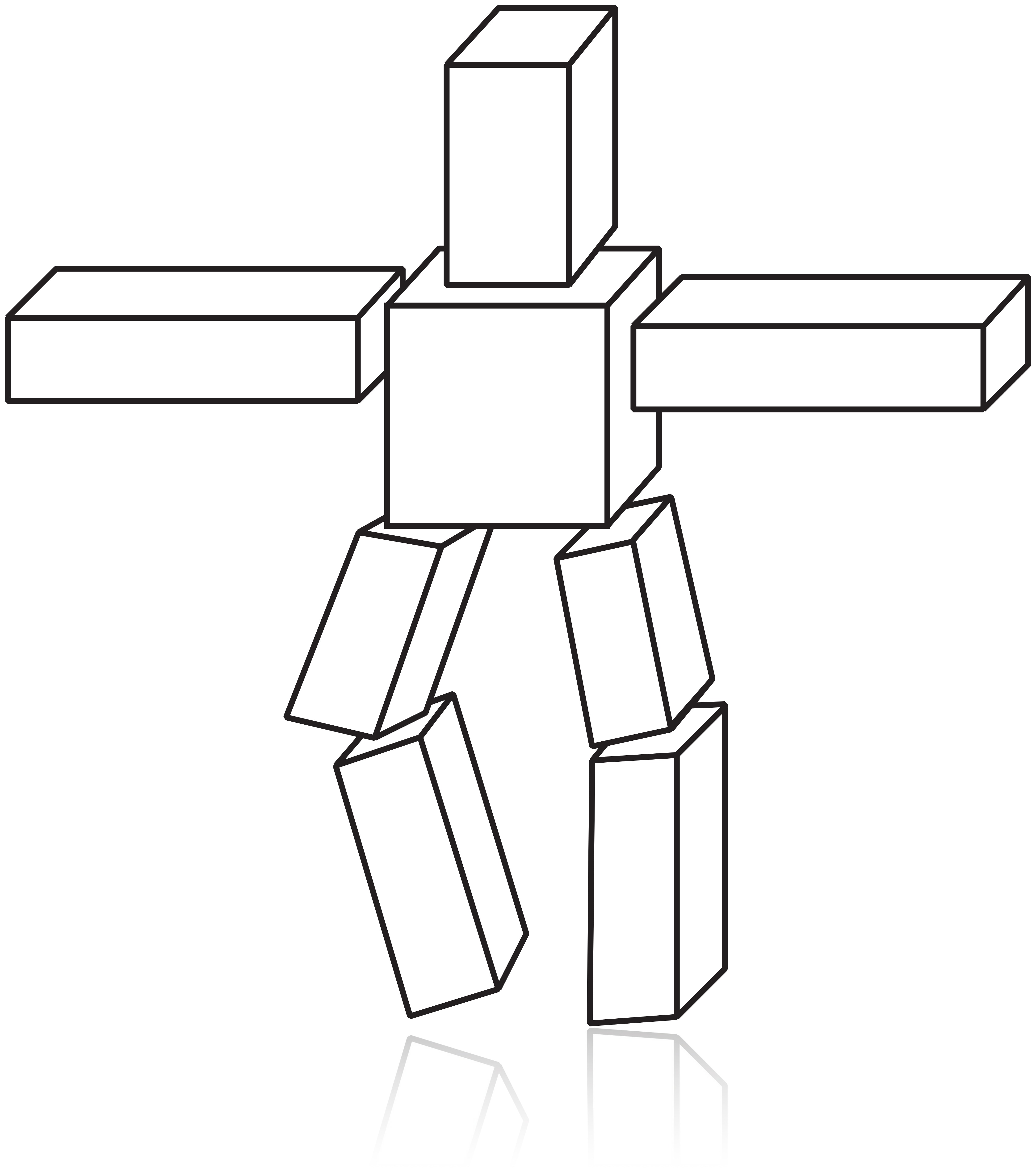


Transformations

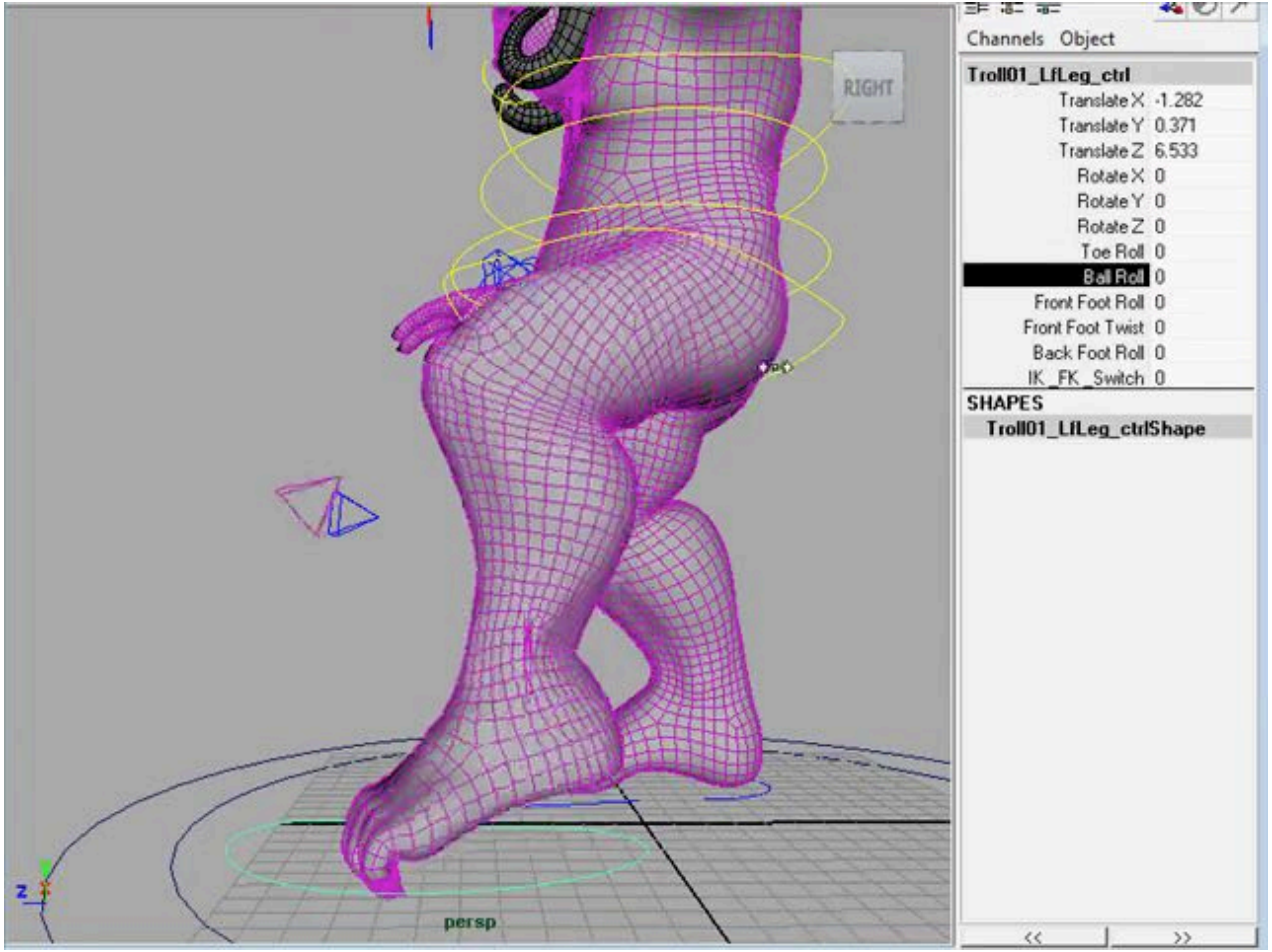
A cube, centered at the origin, with faces of size 2 x 2



Consider drawing a cube person



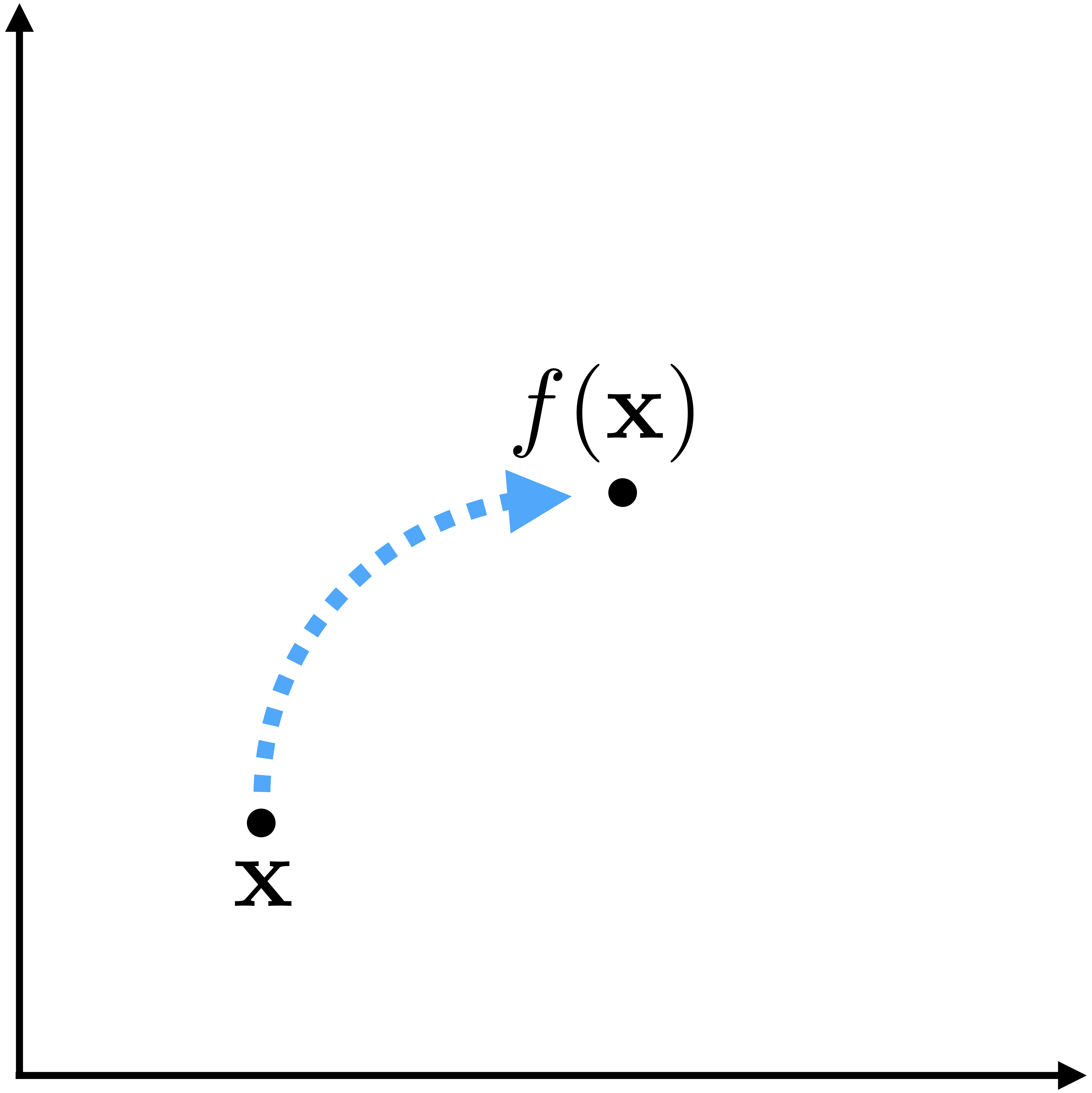
Transformations in character rigging



Transformations for geometry instancing

Credit: Jan-Walter Schliep, Burak Kahraman, and Timm Dapper

Basic idea: f transforms $\mathbf{x}$ to $f(\mathbf{x})$



What can we do with *linear* transformations?

- **What does *linear* mean?**

$$f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$$

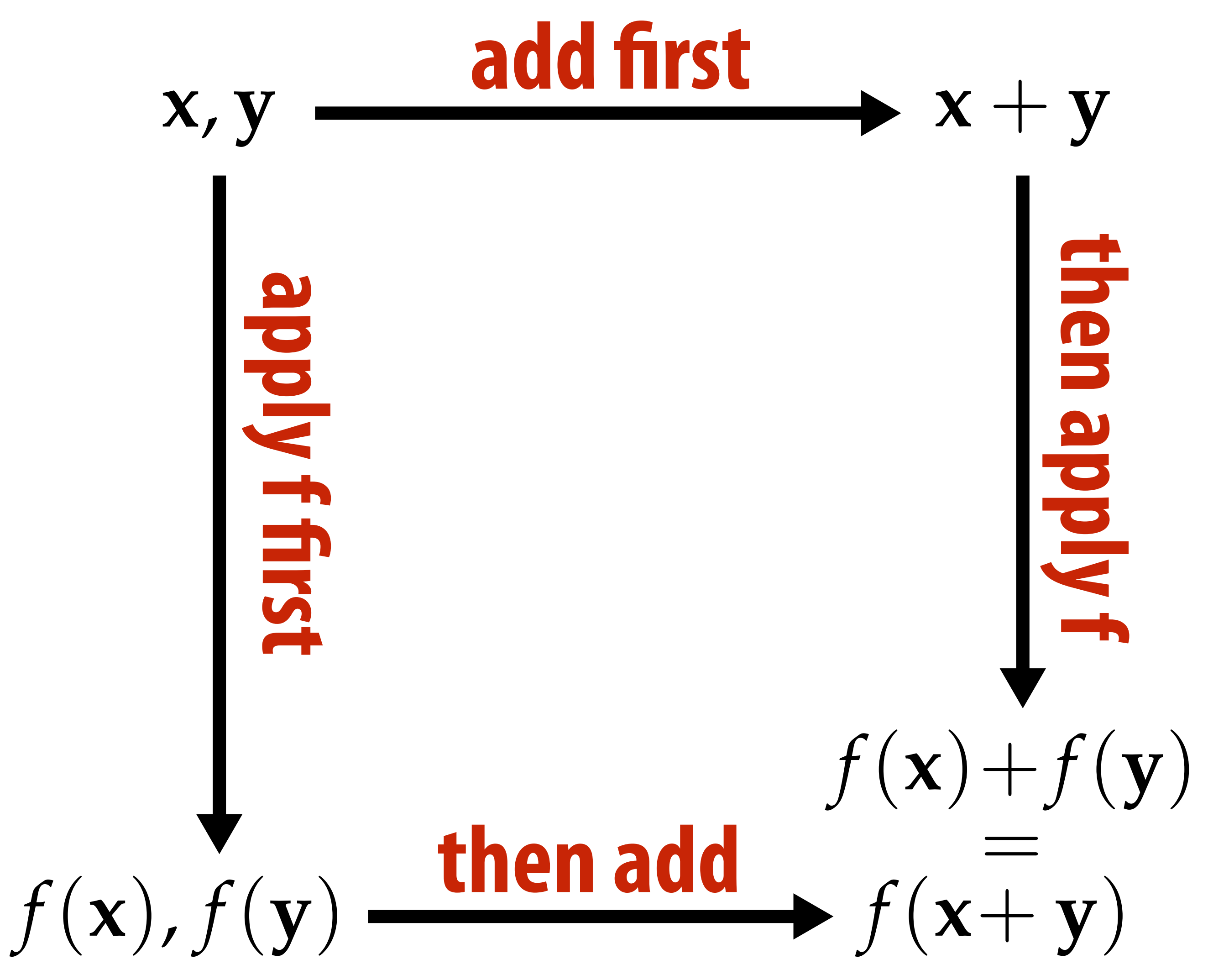
$$f(a\mathbf{x}) = af(\mathbf{x})$$

- **Cheap to compute**
- **Composition of linear transformations is linear**
  - **Leads to uniform representation of many types of transformations**

Linear transformation

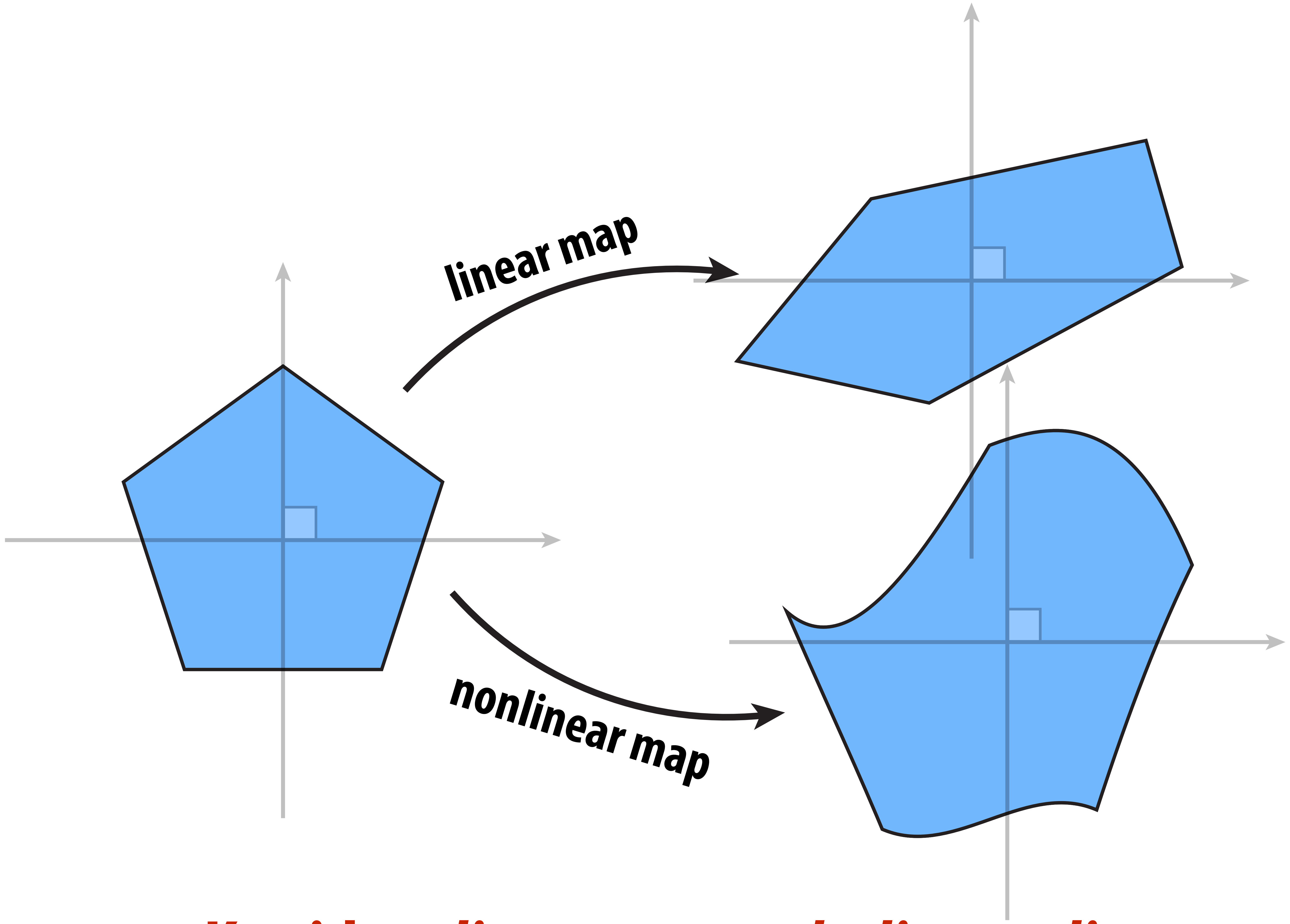
$$f(\mathbf{u}+\mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$$
$$f(a\mathbf{u}) = af(\mathbf{u})$$

- **In other words: if it doesn't matter whether we add the vectors and then apply the map, or apply the map and then add the vectors (and likewise for scaling):**



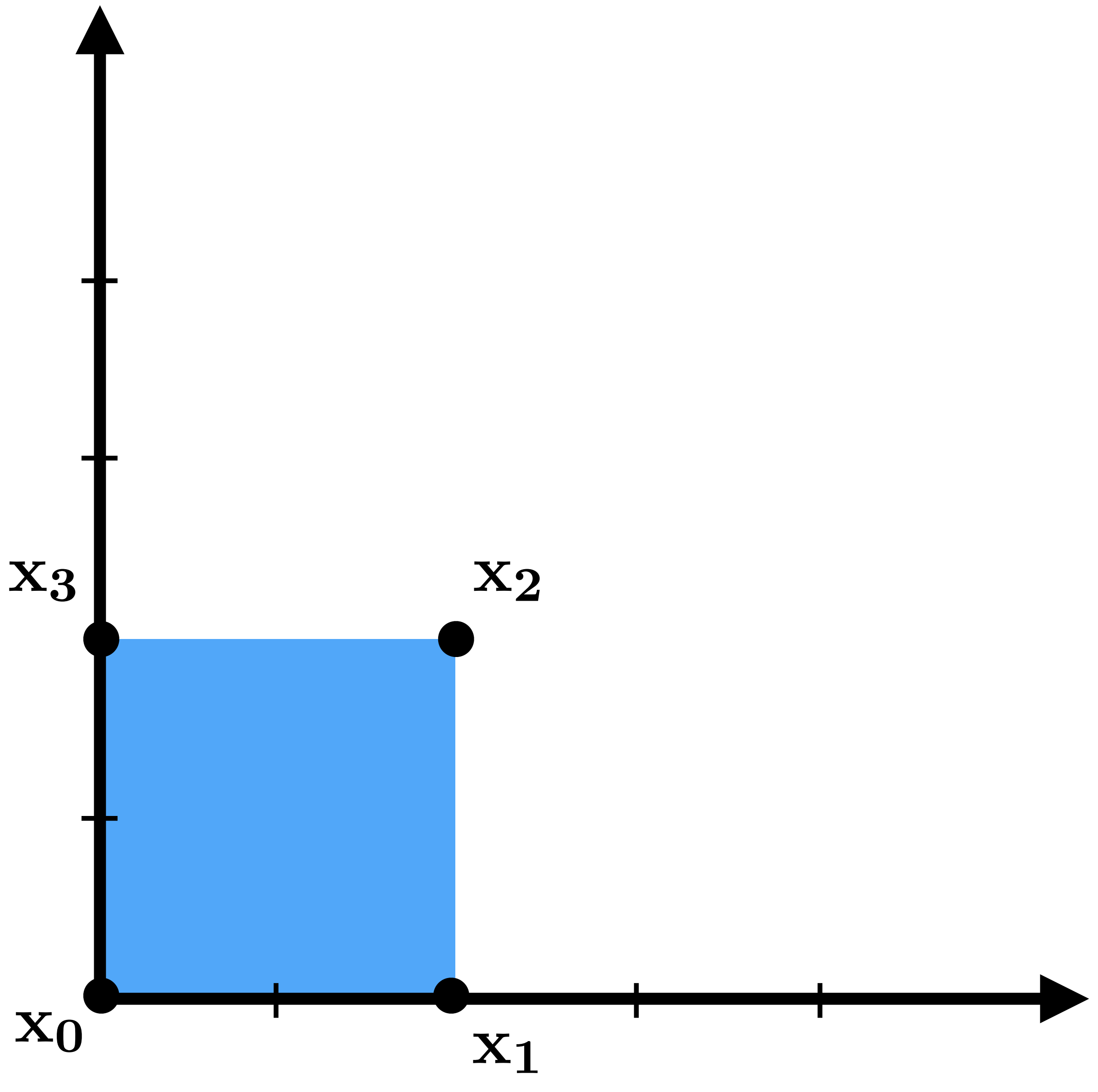
Linear transforms/maps—visualized

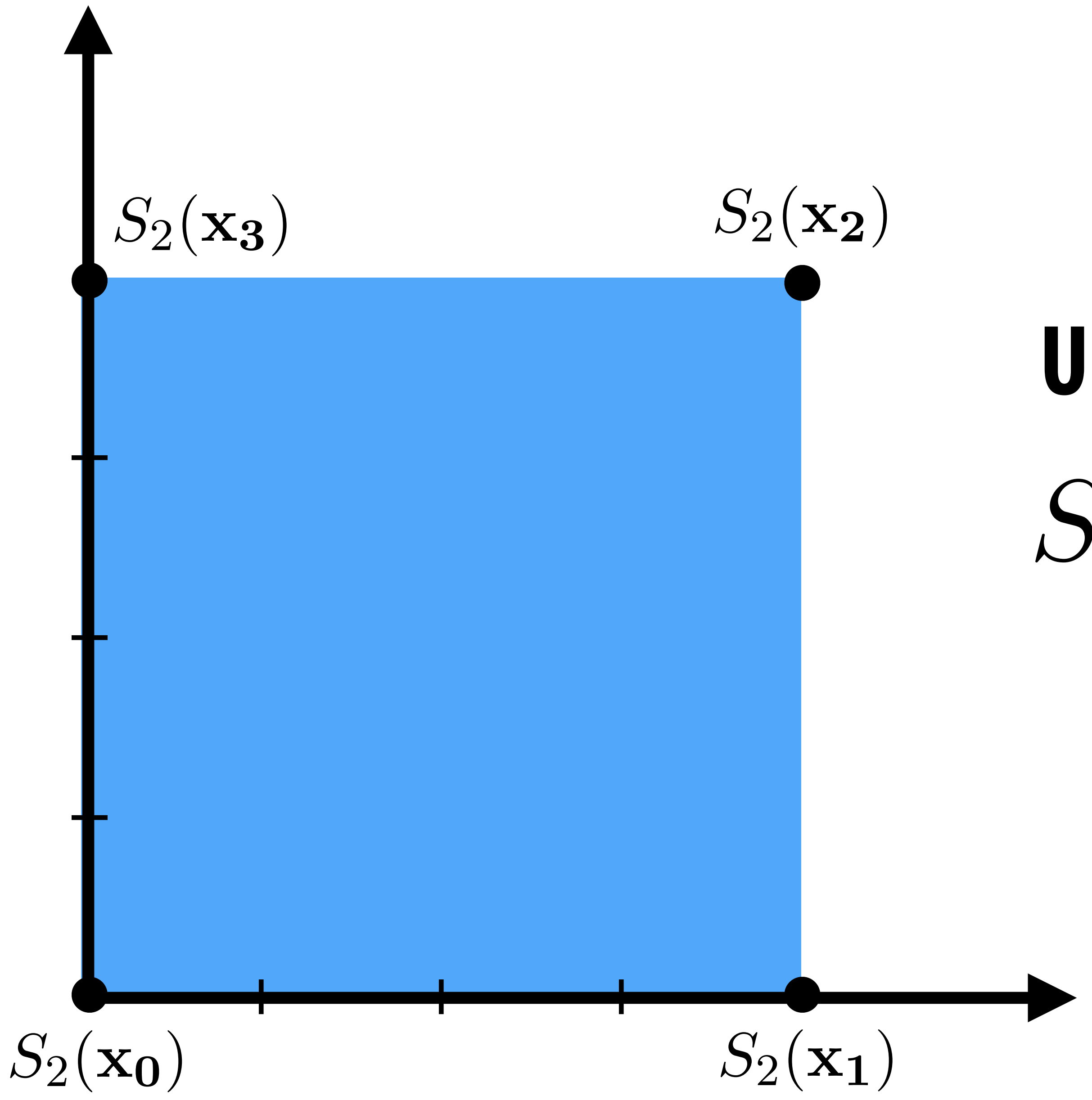
- **Example:**



Key idea: ***linear maps take lines to lines***

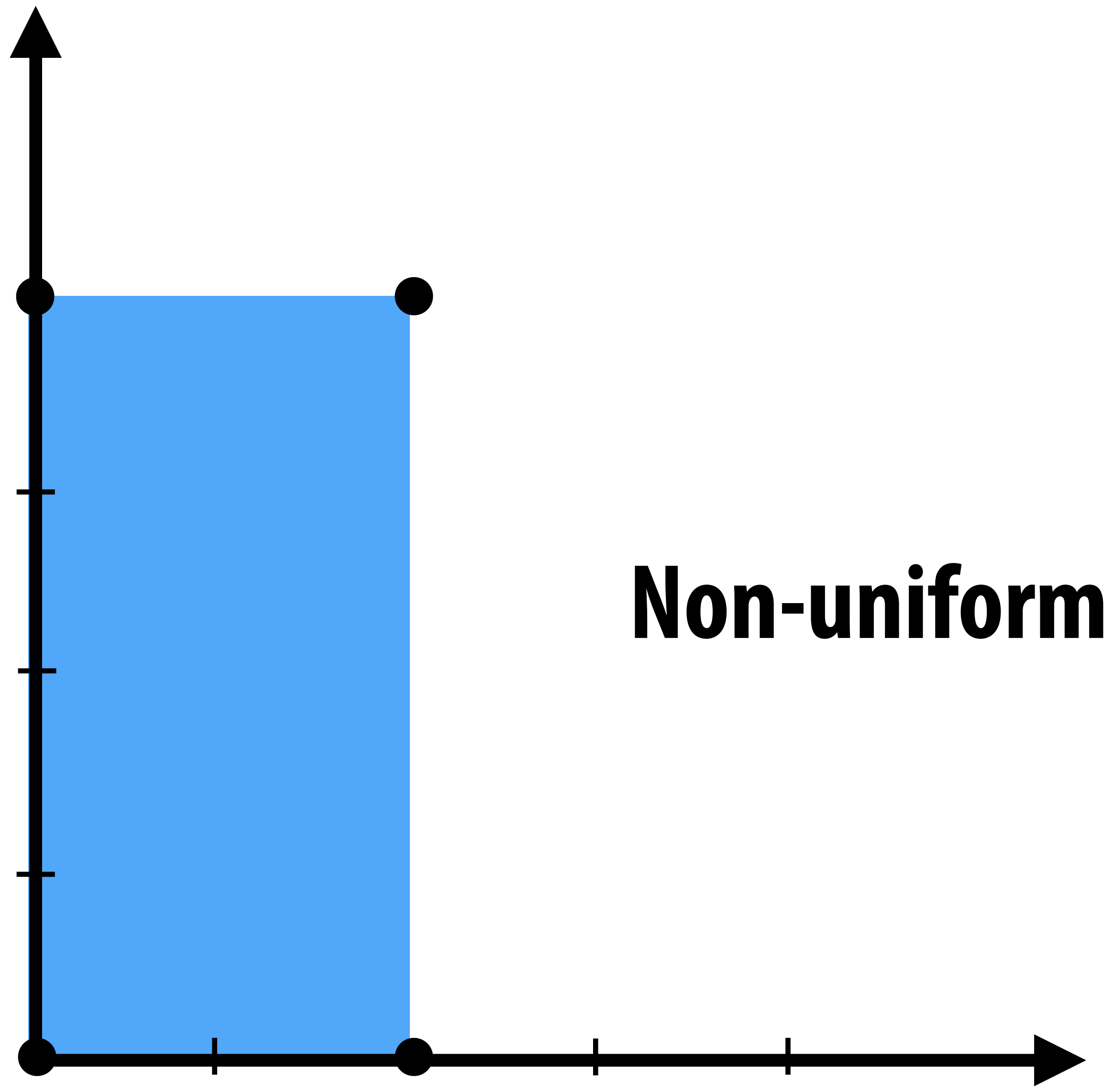
Scale





Uniform scale:

$$S_a(\mathbf{x}) = a\mathbf{x}$$



Non-uniform scale??

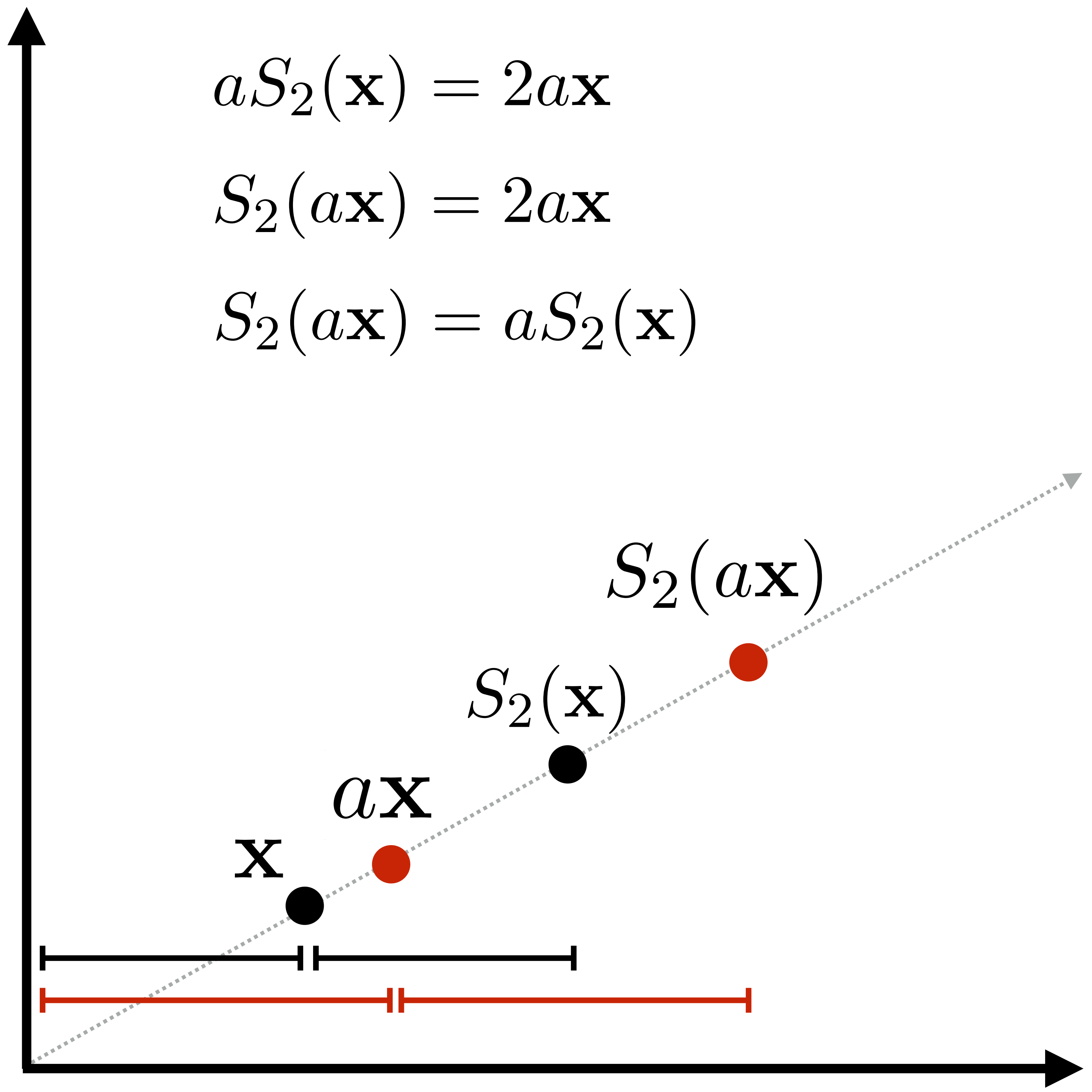
Is scale a linear transform?

$$S_2(\mathbf{x}) = 2\mathbf{x}$$

$$aS_2(\mathbf{x}) = 2a\mathbf{x}$$

$$S_2(a\mathbf{x}) = 2a\mathbf{x}$$

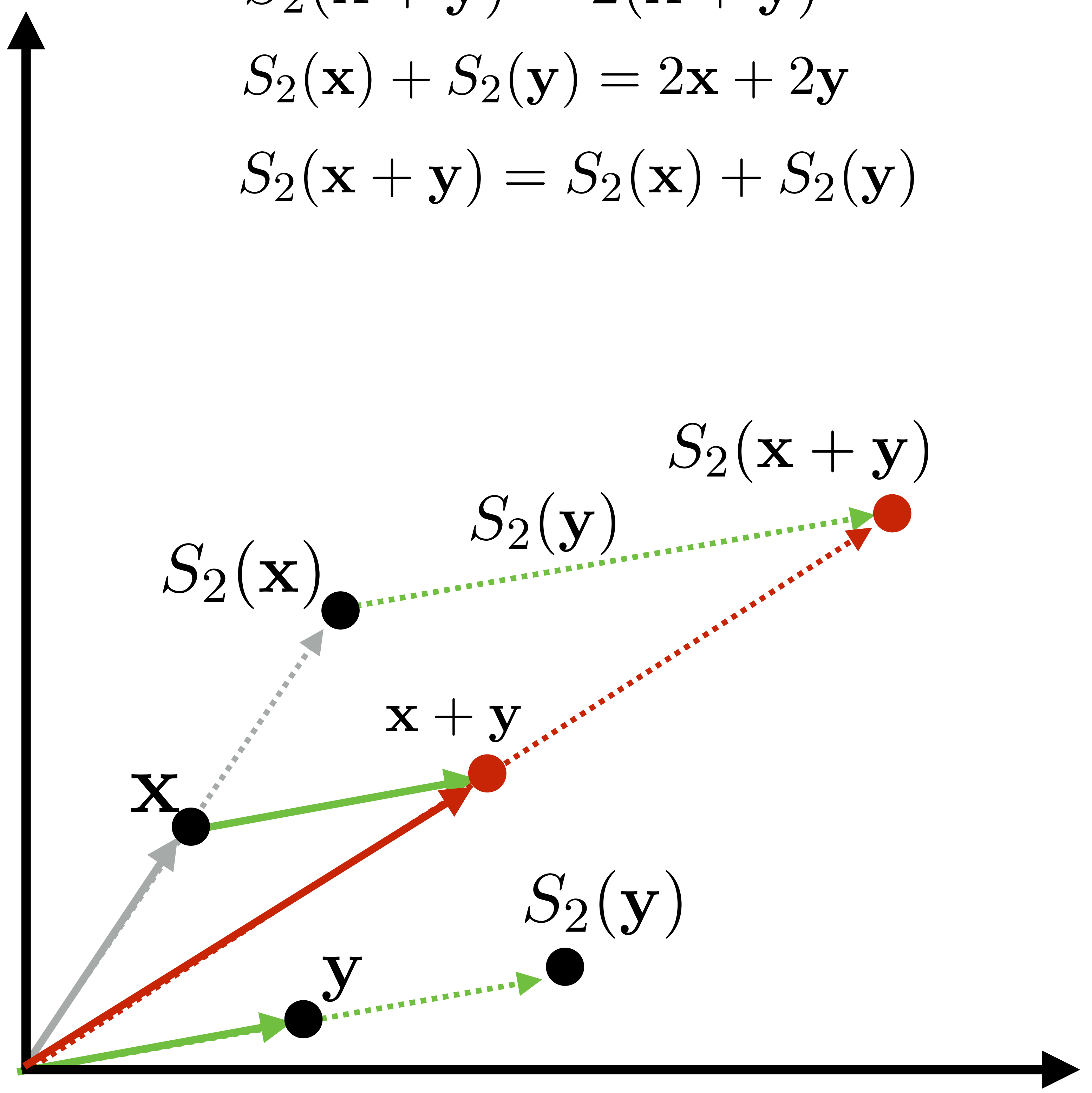
$$S_2(a\mathbf{x}) = aS_2(\mathbf{x})$$



$$S_2(\mathbf{x}+\mathbf{y}) = 2(\mathbf{x}+\mathbf{y})$$

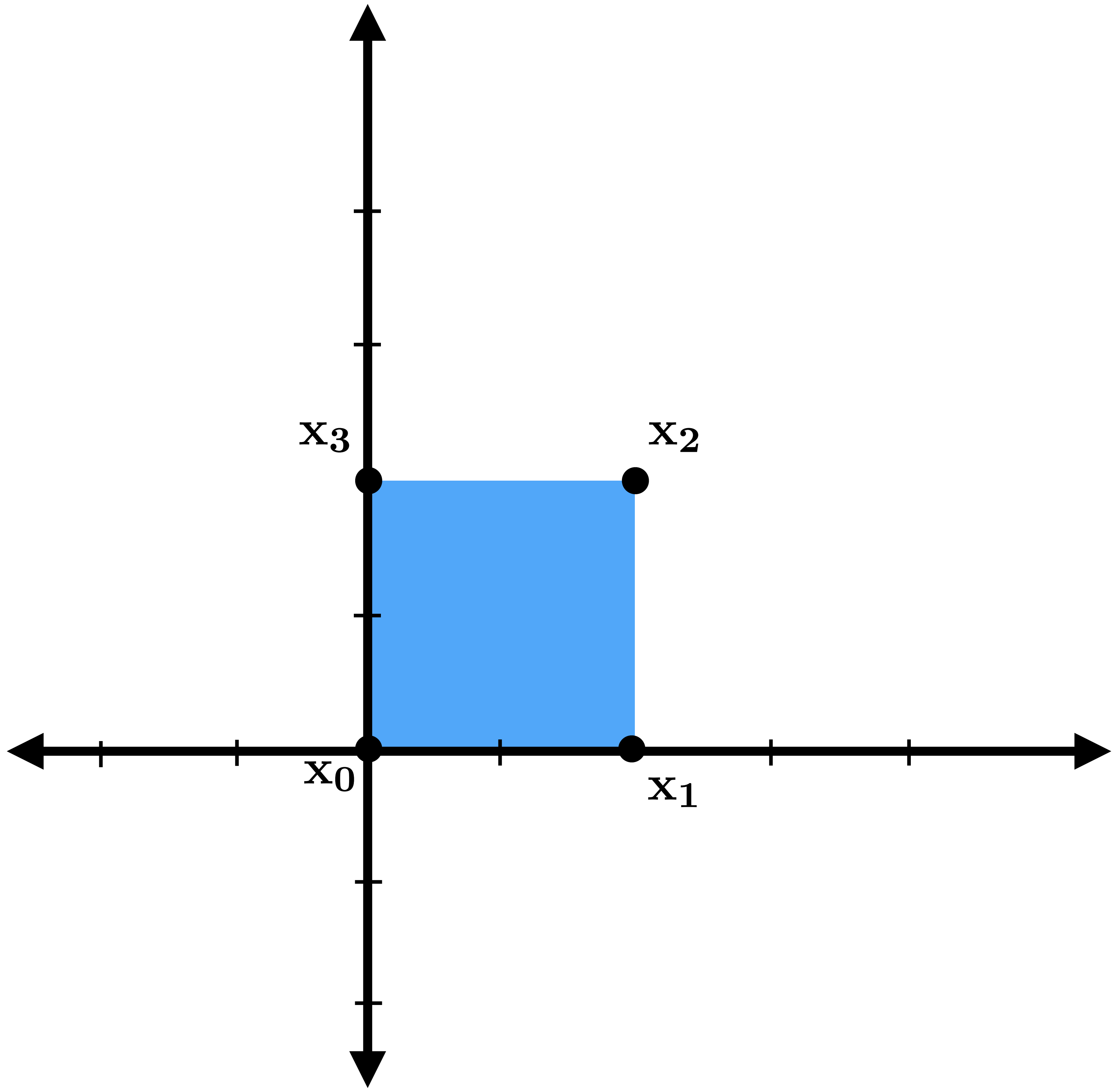
$$S_2(\mathbf{x}) + S_2(\mathbf{y}) = 2\mathbf{x}+2\mathbf{y}$$

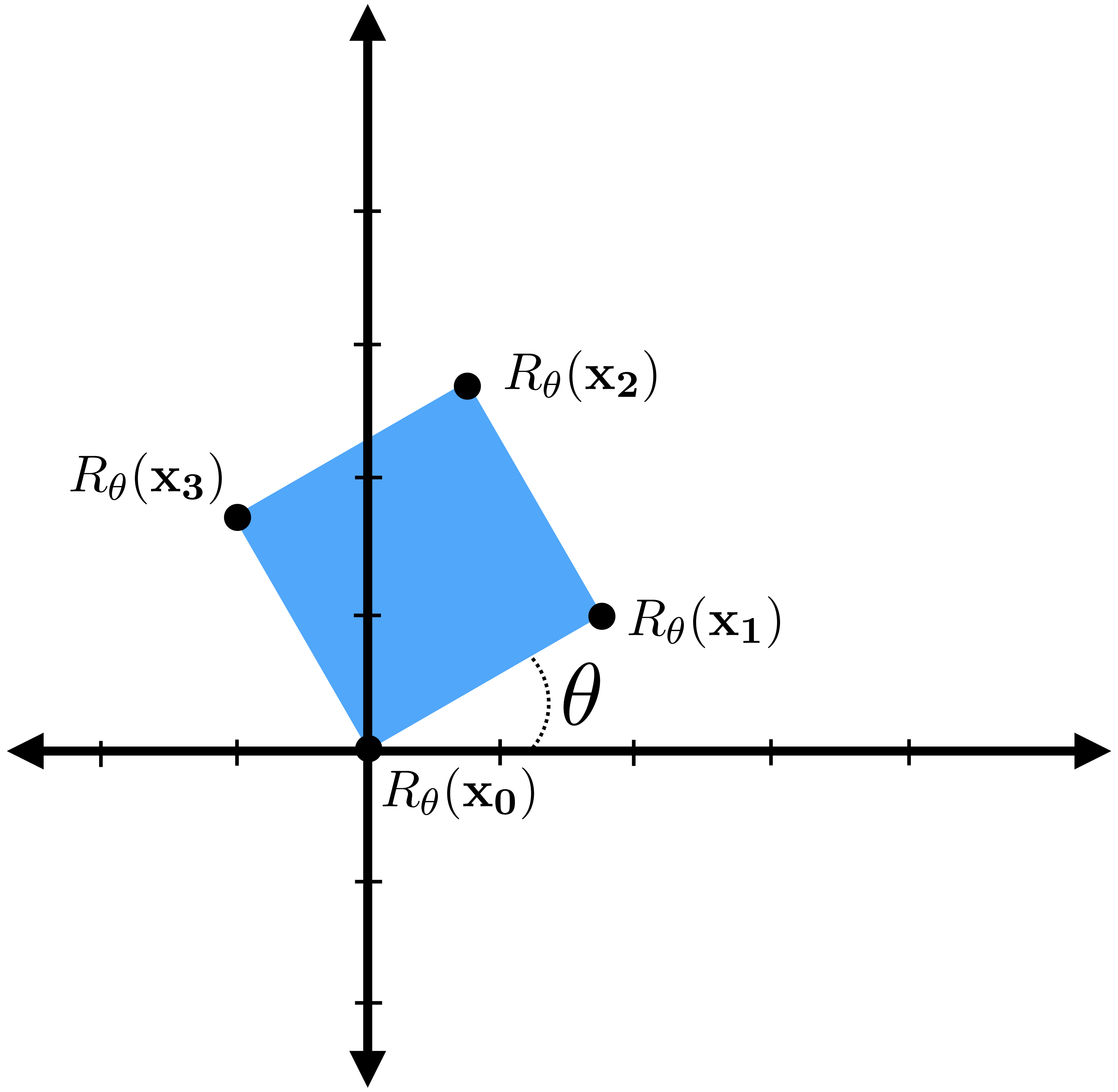
$$S_2(\mathbf{x}+\mathbf{y}) = S_2(\mathbf{x}) + S_2(\mathbf{y})$$



Yes!

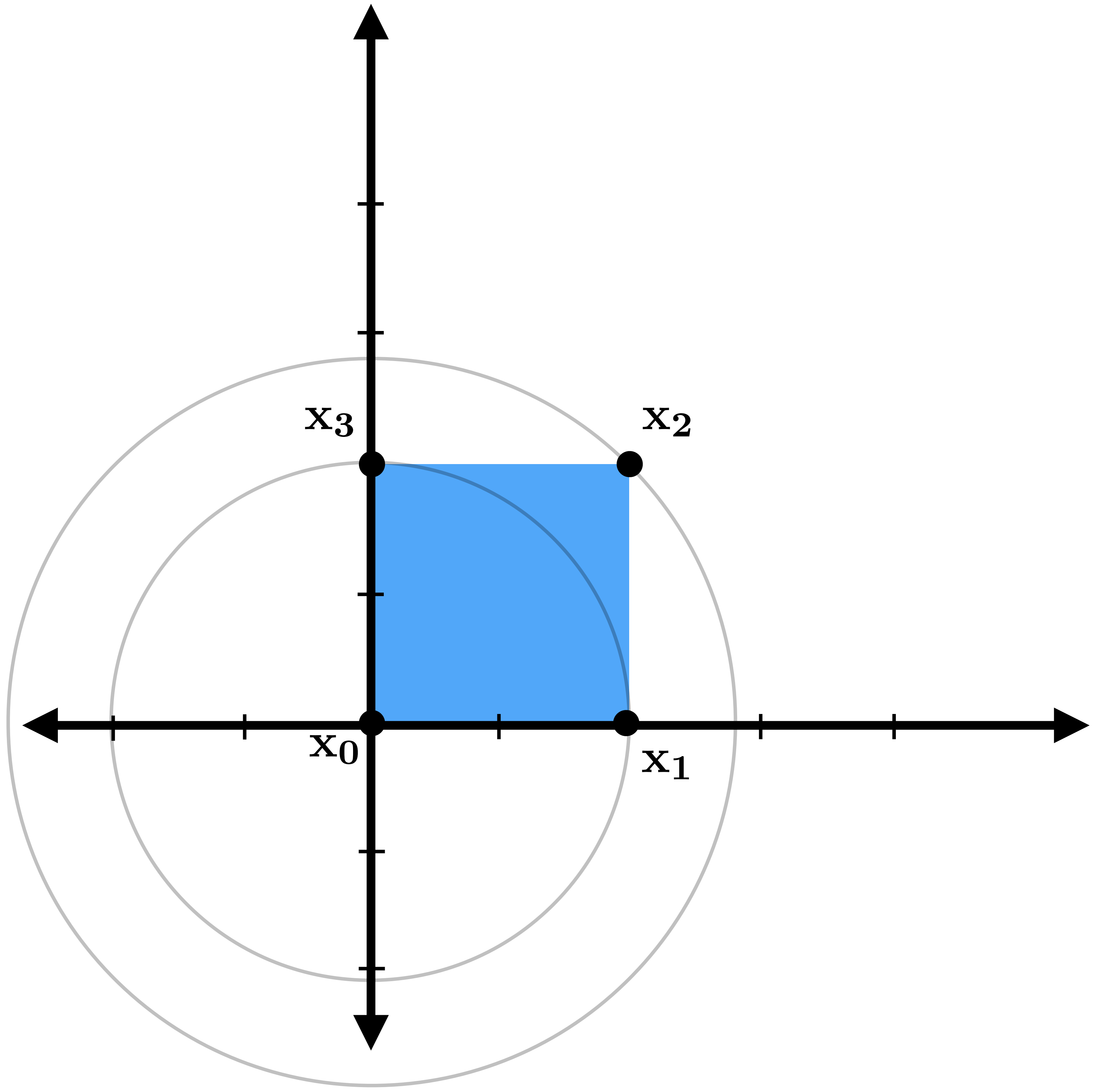
Rotation

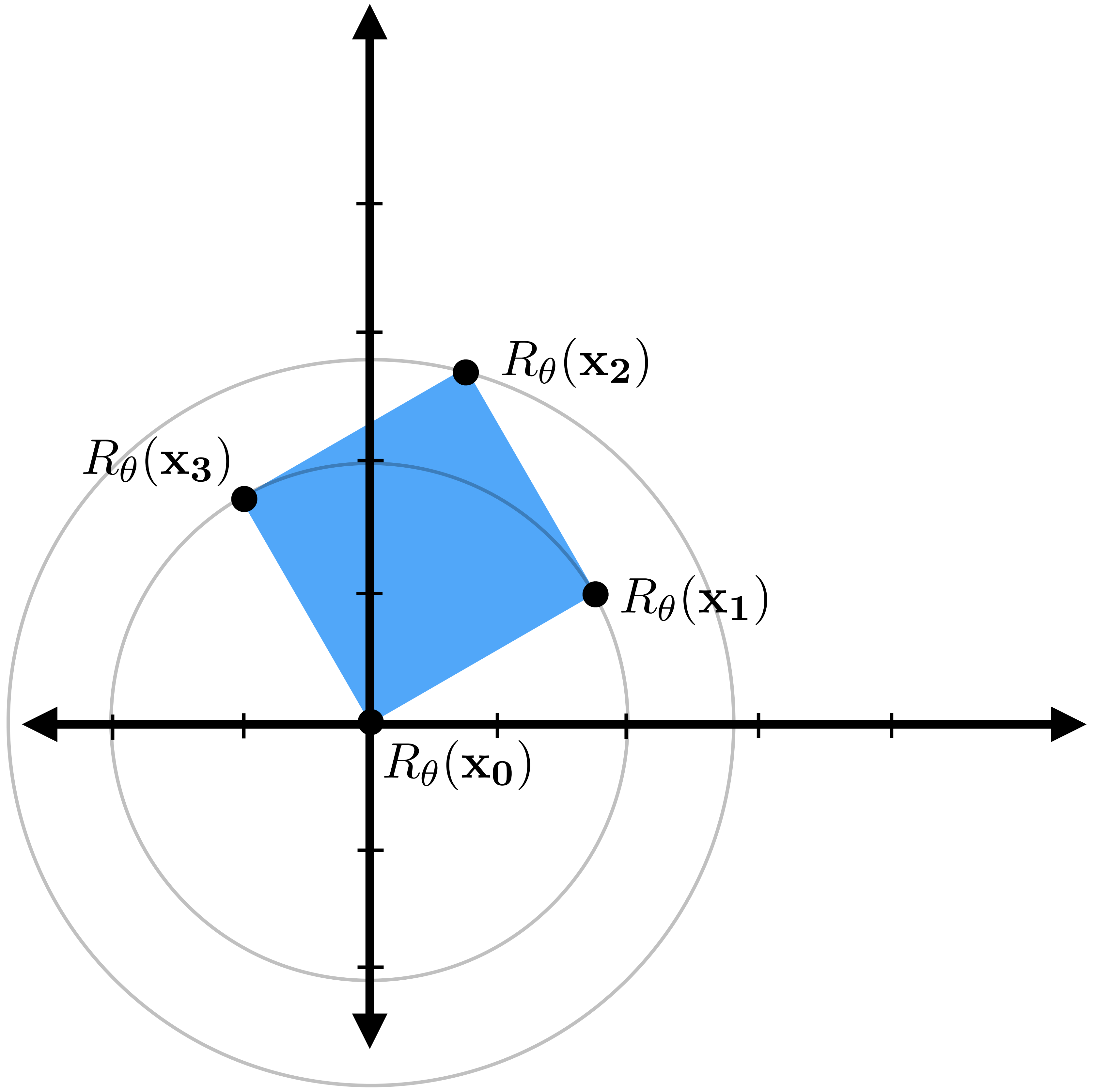




R_θ **= rotate counter-clockwise by** θ

Rotation as circular motion



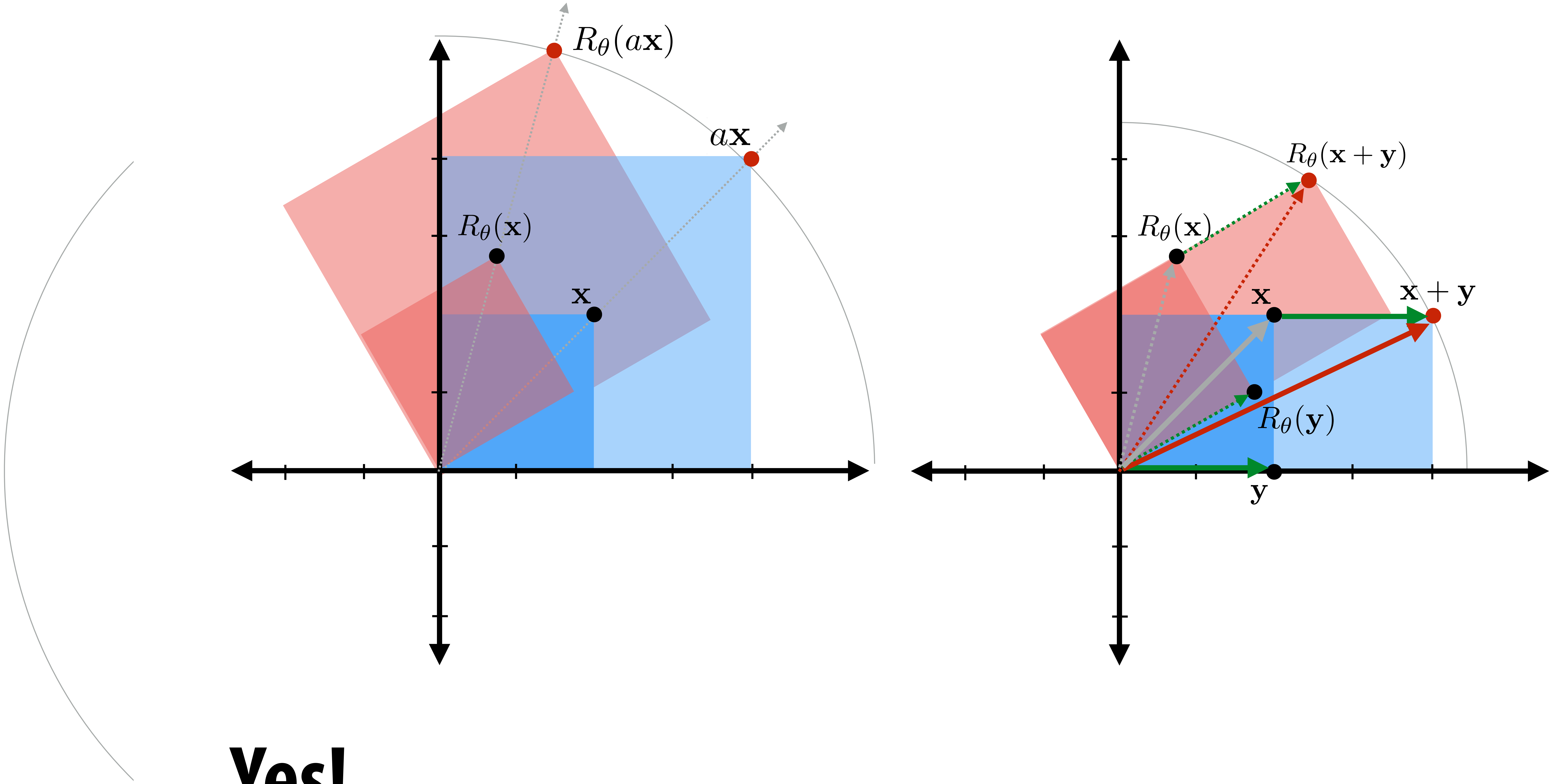


R_θ = **rotate counter-clockwise by** θ

As angle changes, points move along *circular* trajectories.

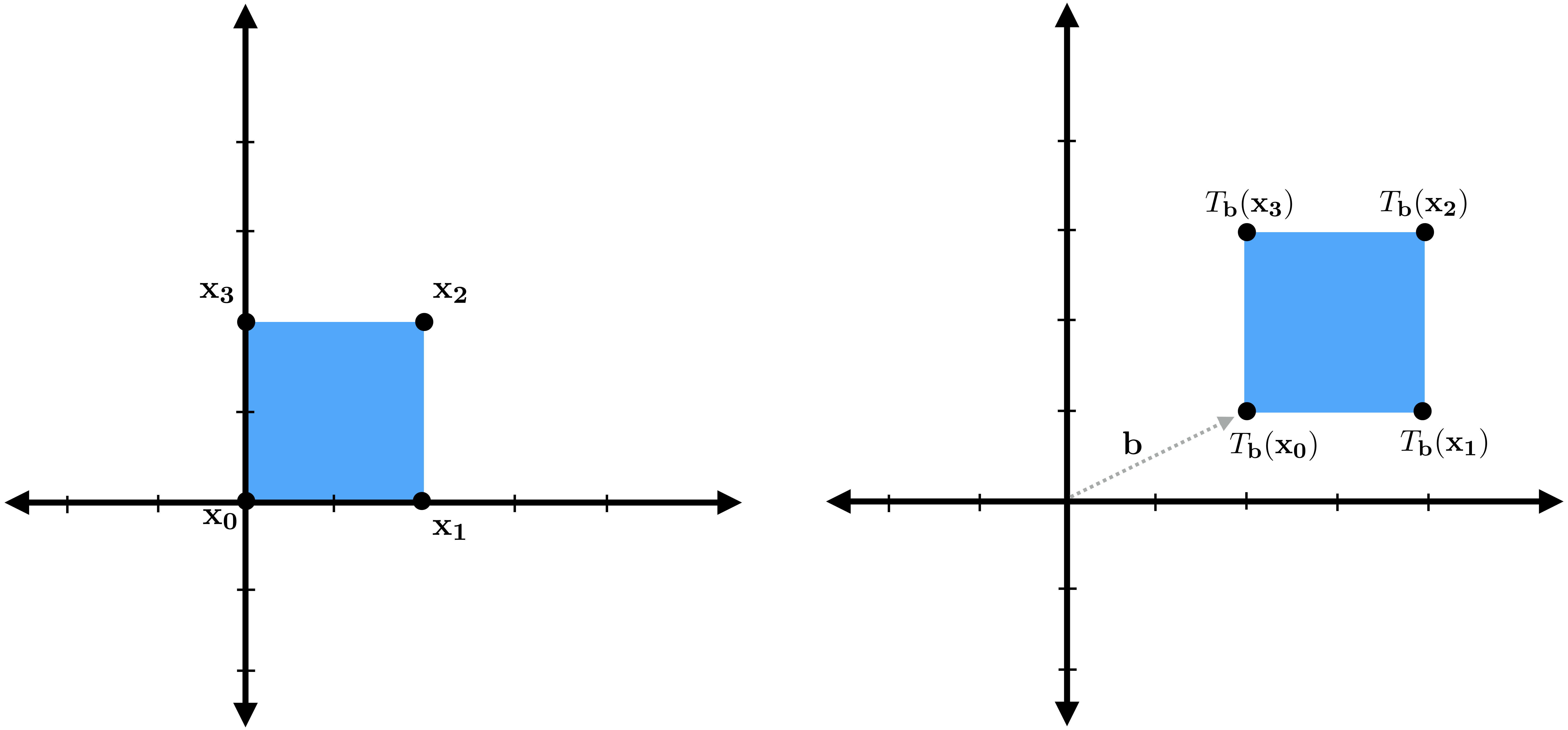
Hence, rotations preserve length of vectors: $|R_\theta(\mathbf{x})| = |\mathbf{x}|$

Is rotation linear?



Yes!

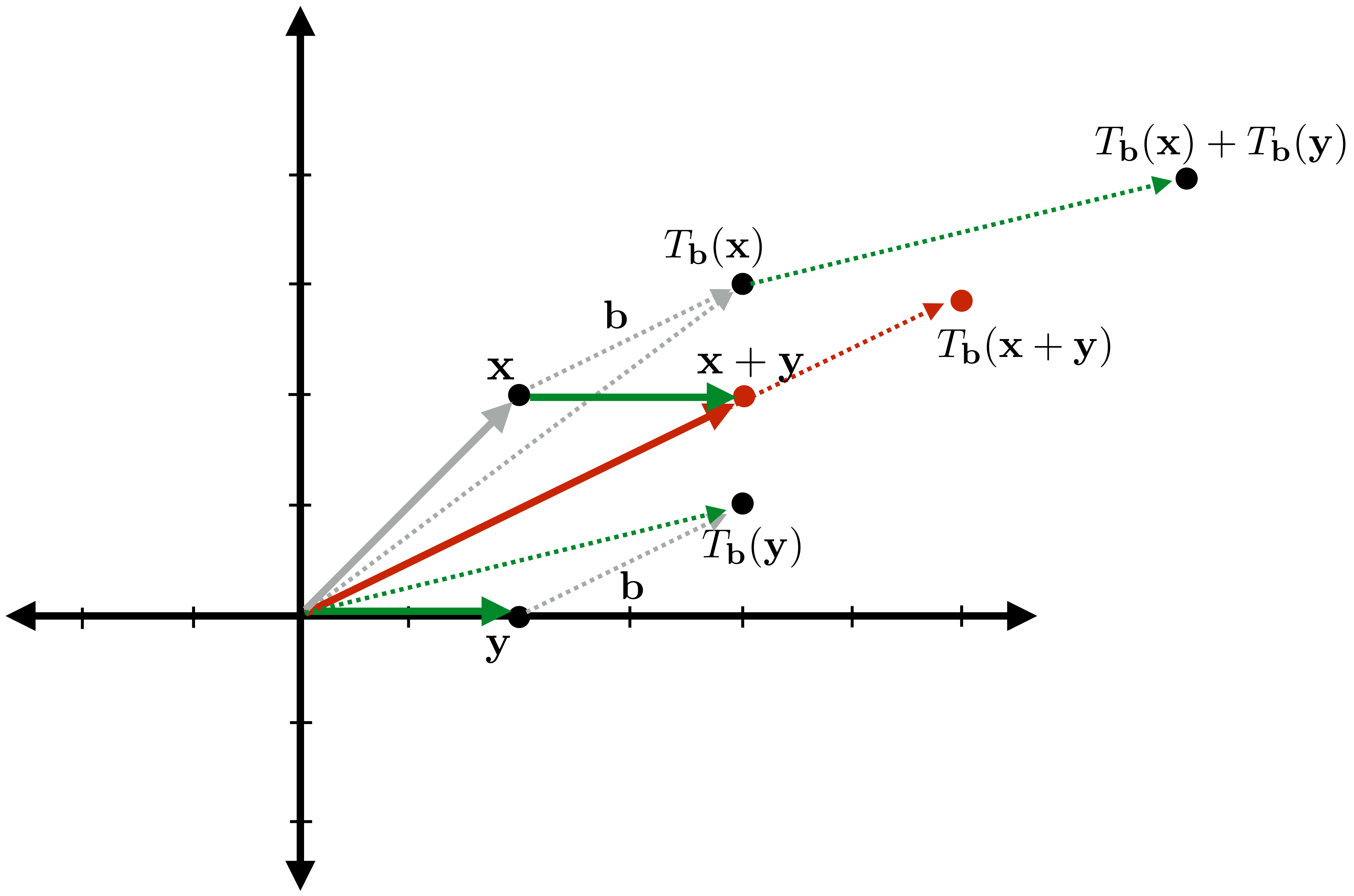
Translation



$T_{\mathbf{b}}$ — **"translate by b"**

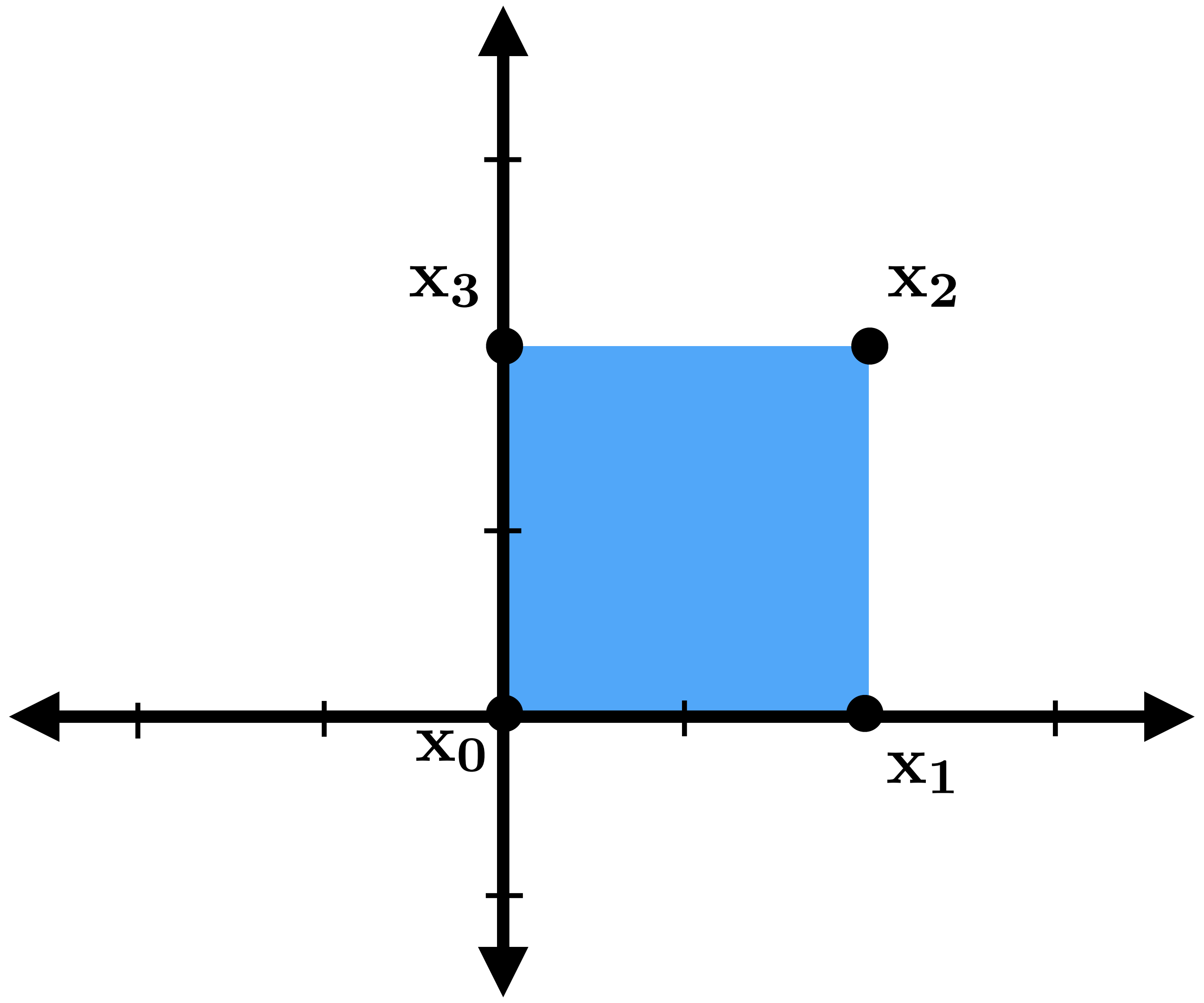
$$T_{\mathbf{b}}(\mathbf{x}) = \mathbf{x} + \mathbf{b}$$

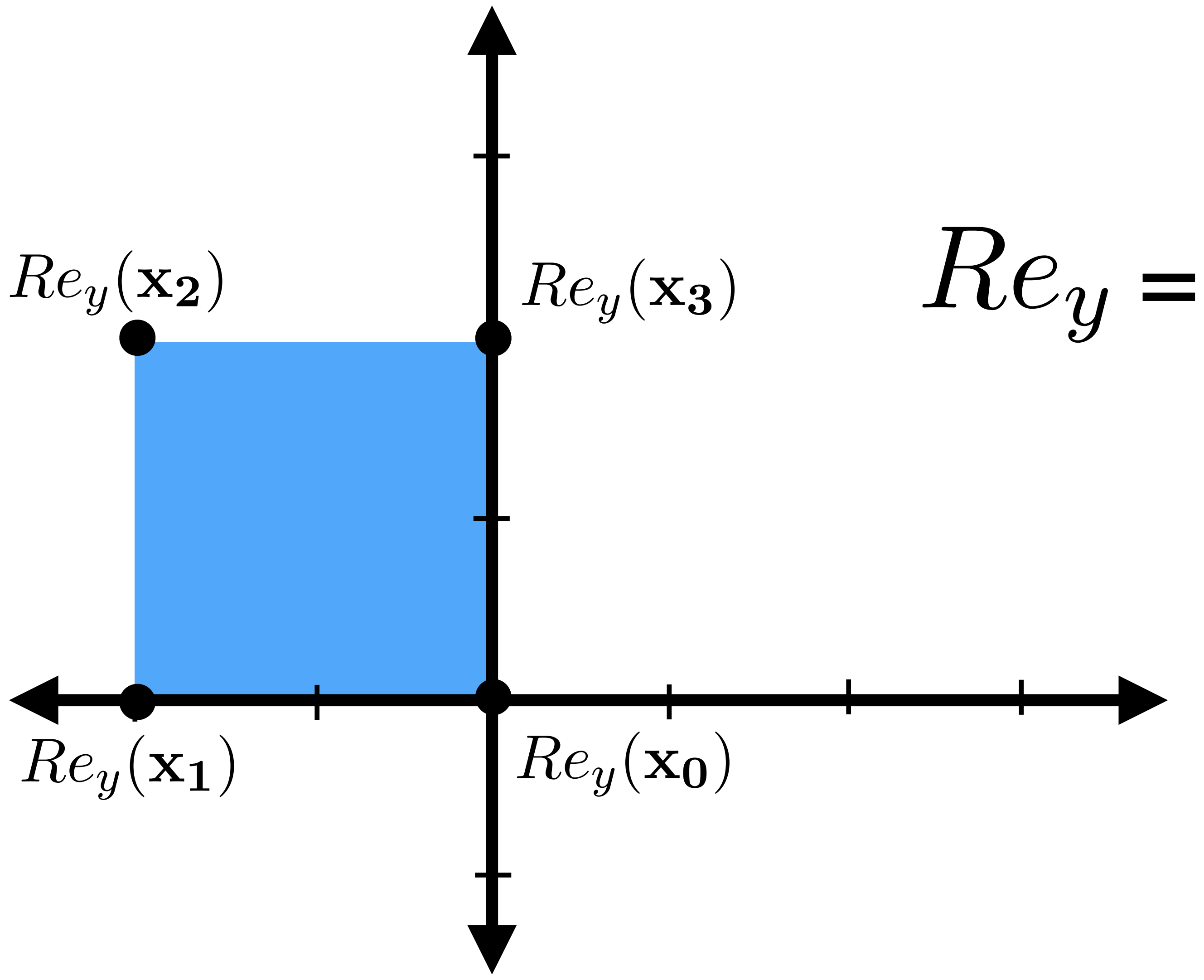
Is translation linear?



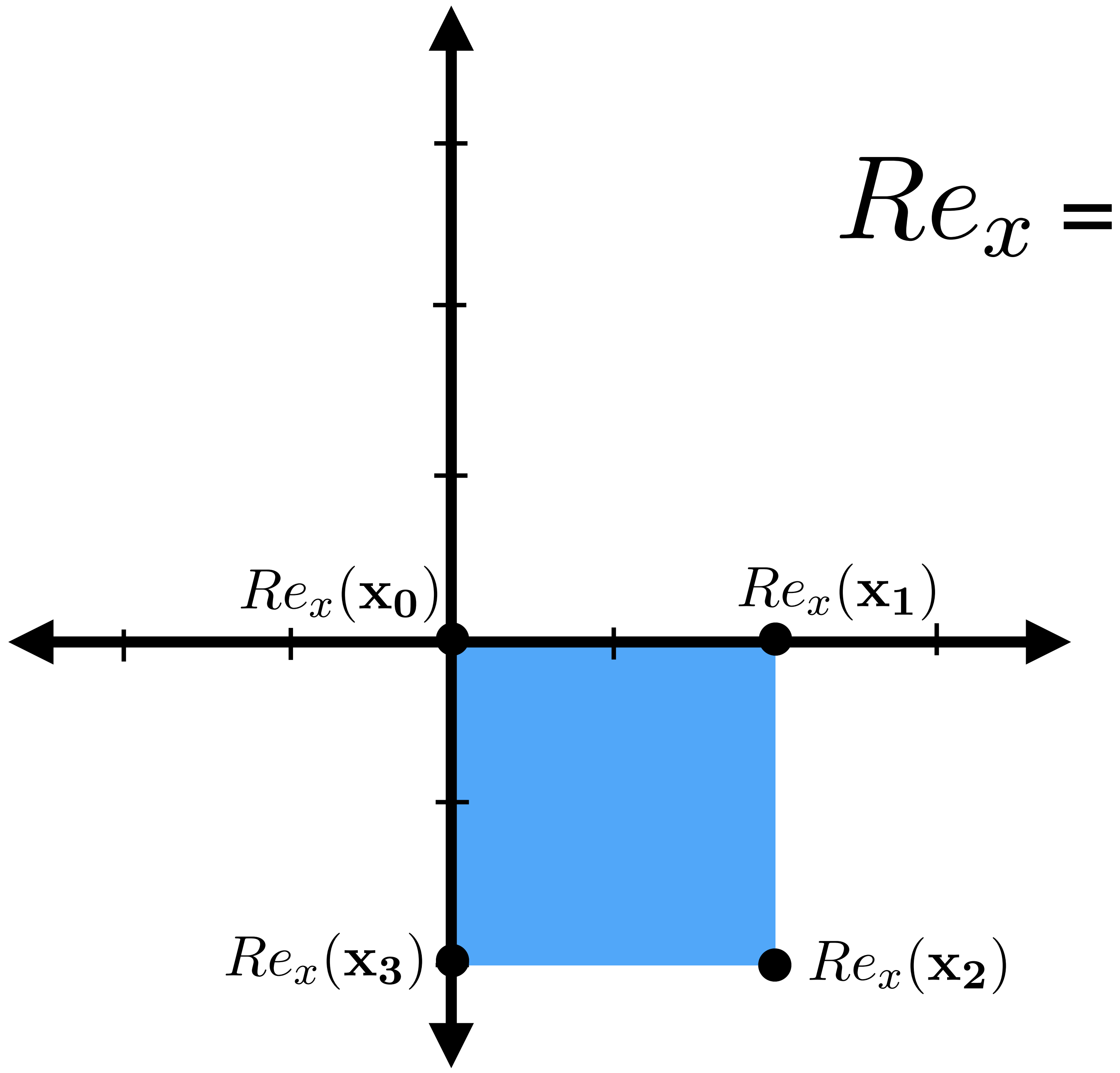
No. Translation is affine.

Reflection



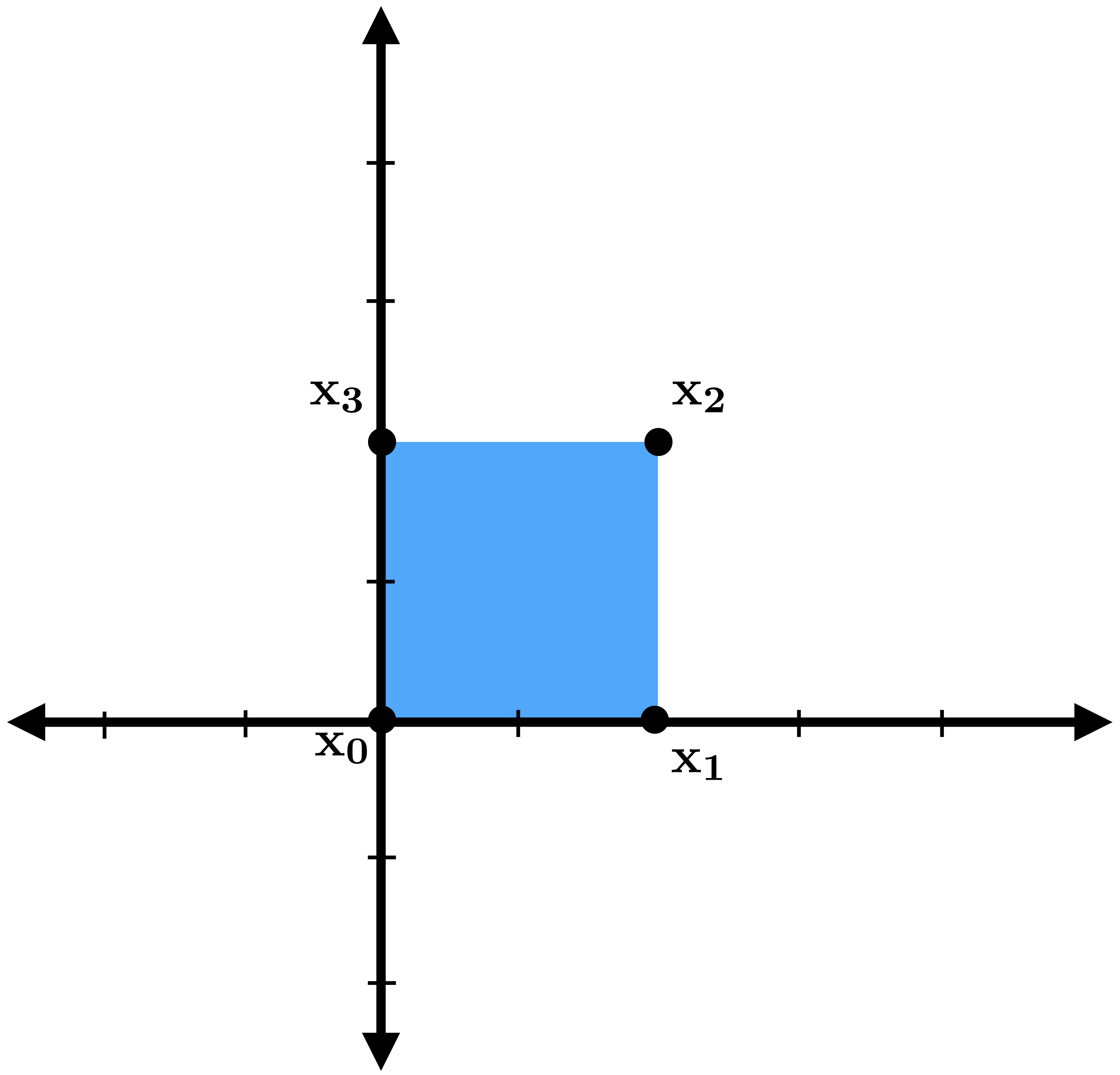


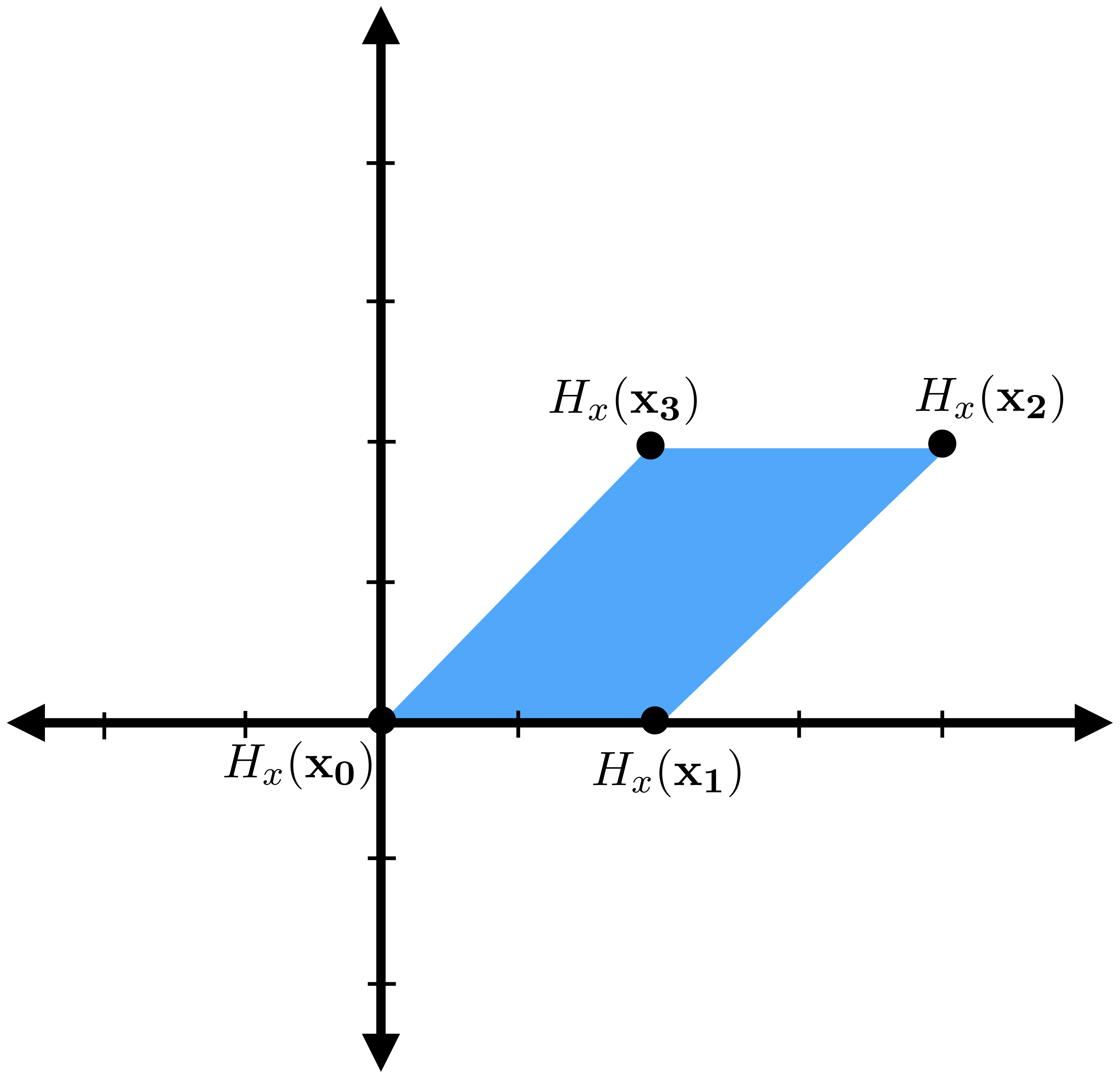
Re_y = **reflection about *y***



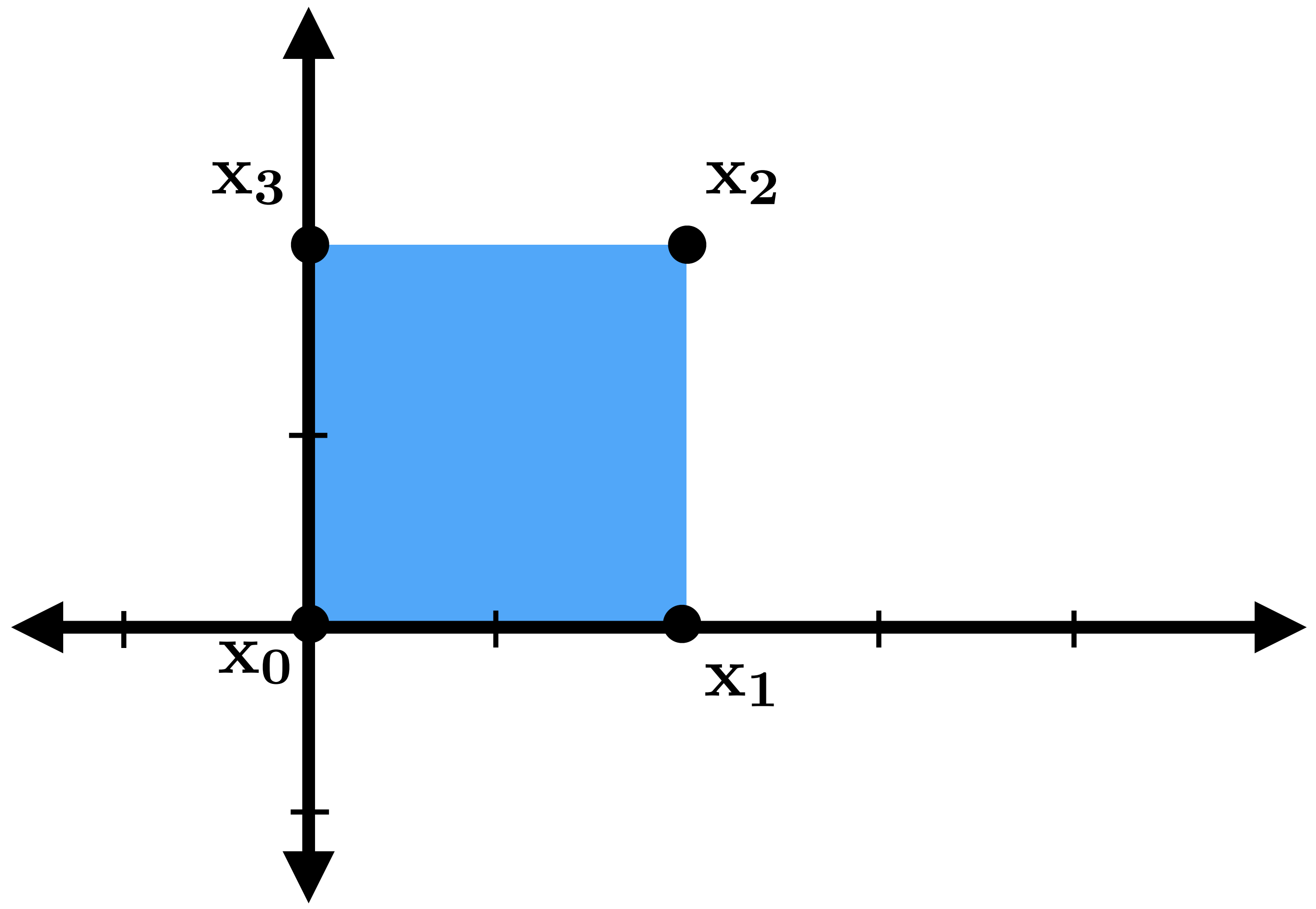
Re_x = **reflection about *x***

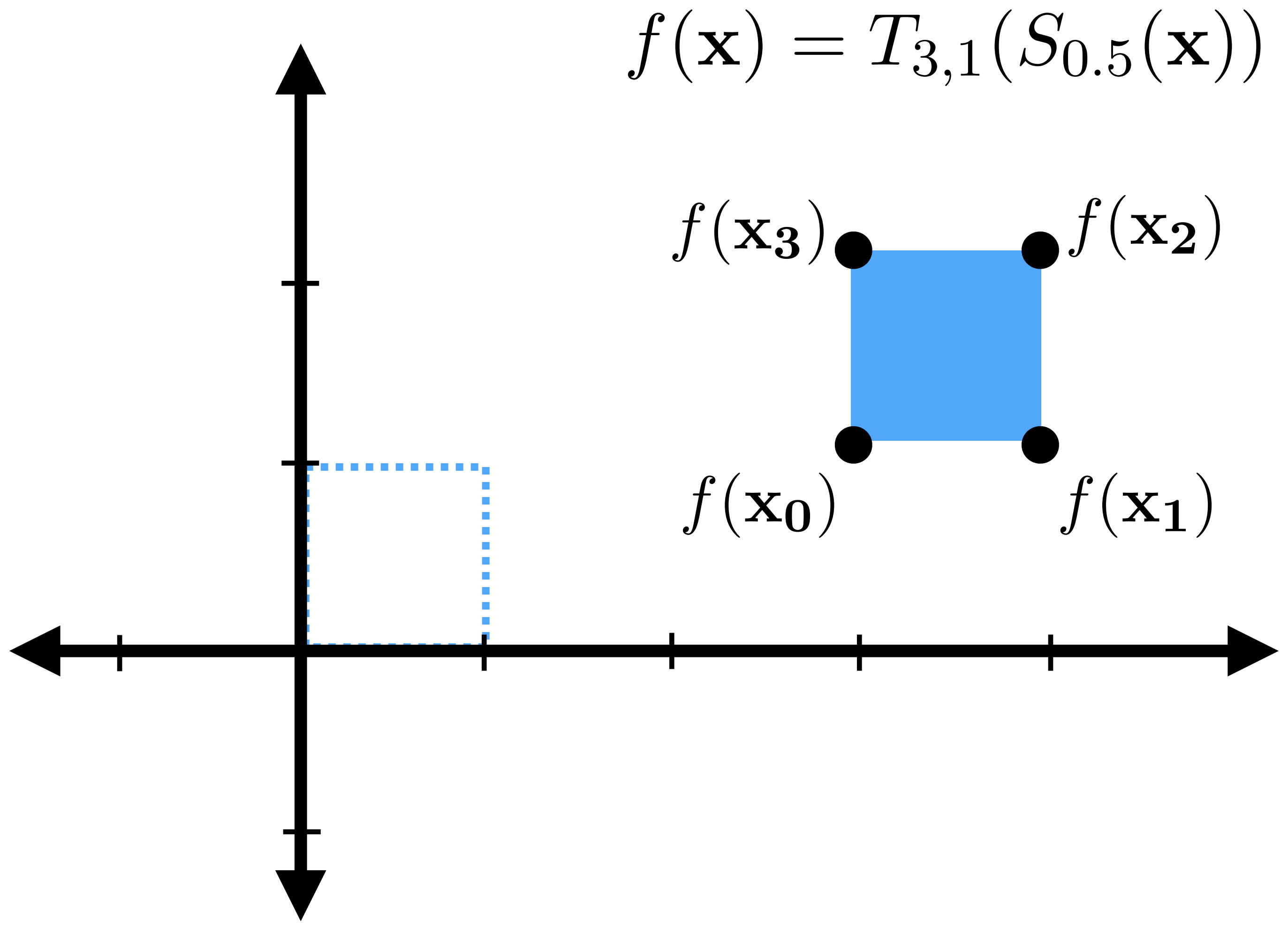
Shear (in x direction)

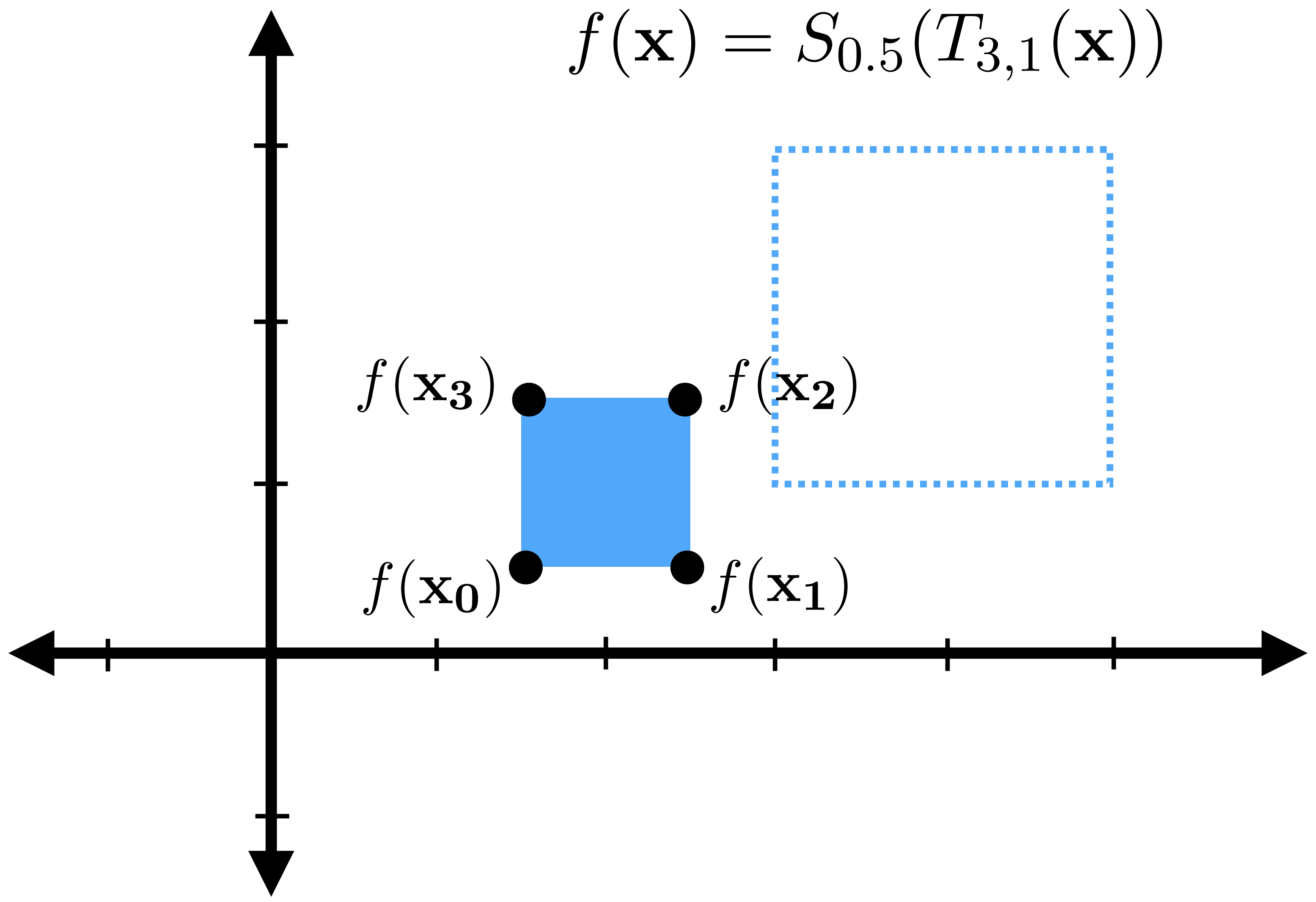




Compose basic transformations to construct more complicated ones



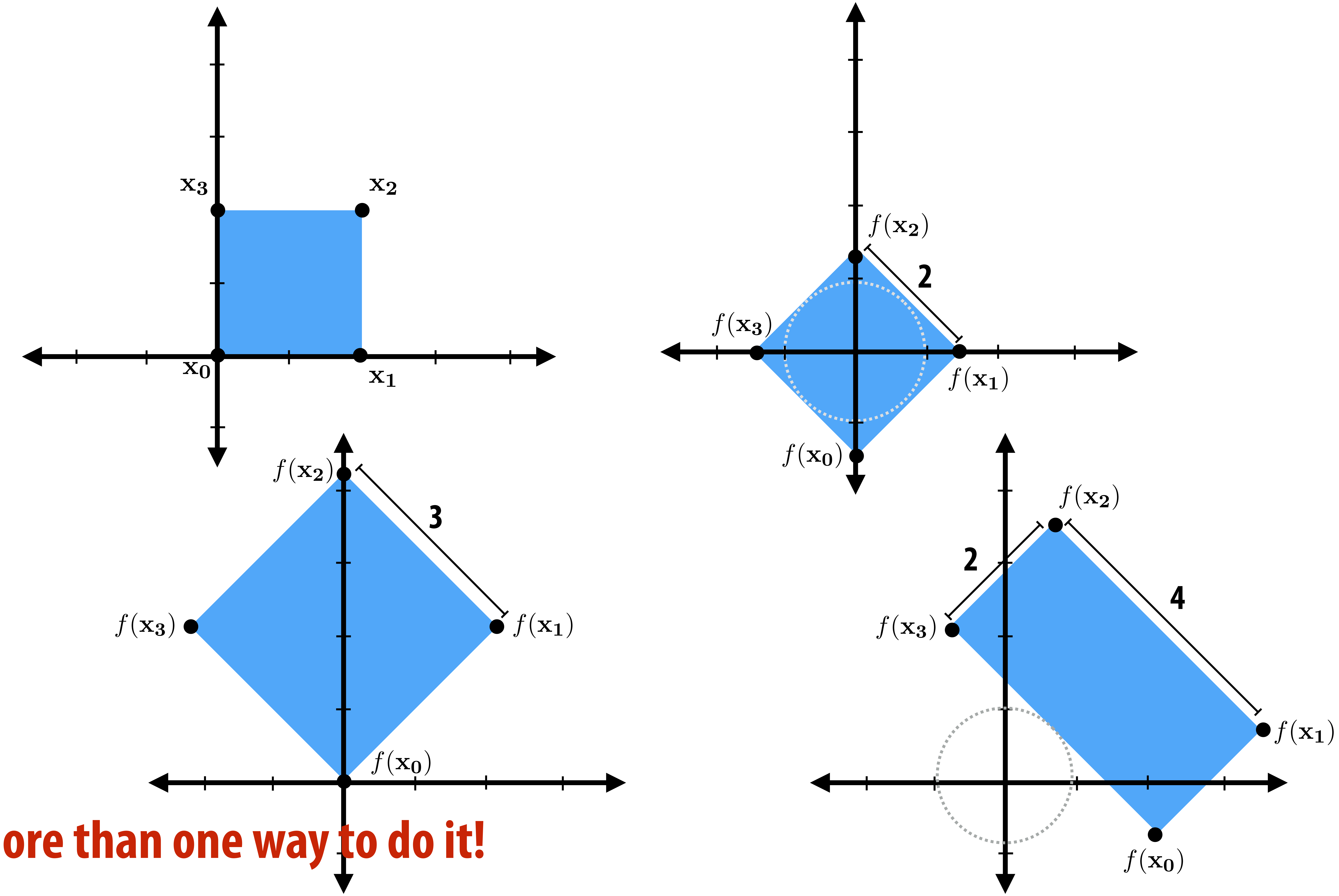




Note: order of composition matters

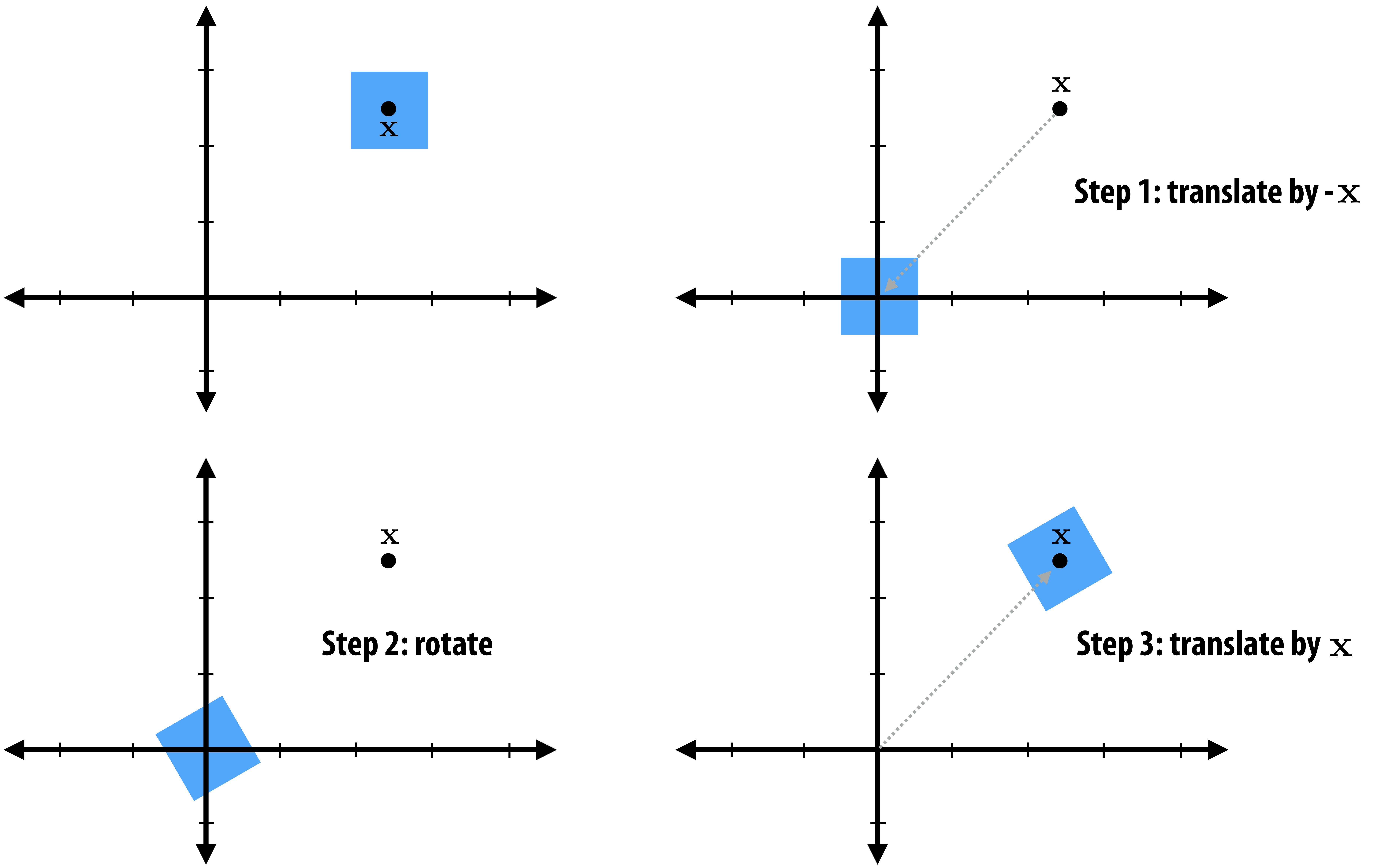
Top-right: scale, then translate
Bottom-right: translate, then scale

How would you perform these transformations?



Usually more than one way to do it!

Common task: rotate about a point x



Summary of basic transformations

Linear:

$$f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$$
$$f(a\mathbf{x}) = af(\mathbf{x})$$

Scale
Rotation
Reflection
Shear

Not linear:

Translation

Affine:

Composition of linear transform + translation
(all examples on previous two slides)

$$f(\mathbf{x}) = g(\mathbf{x}) + \mathbf{b}$$

Not affine: perspective projection (will discuss later)

Euclidean: (Isometries)

Preserve distance between points (preserves length)

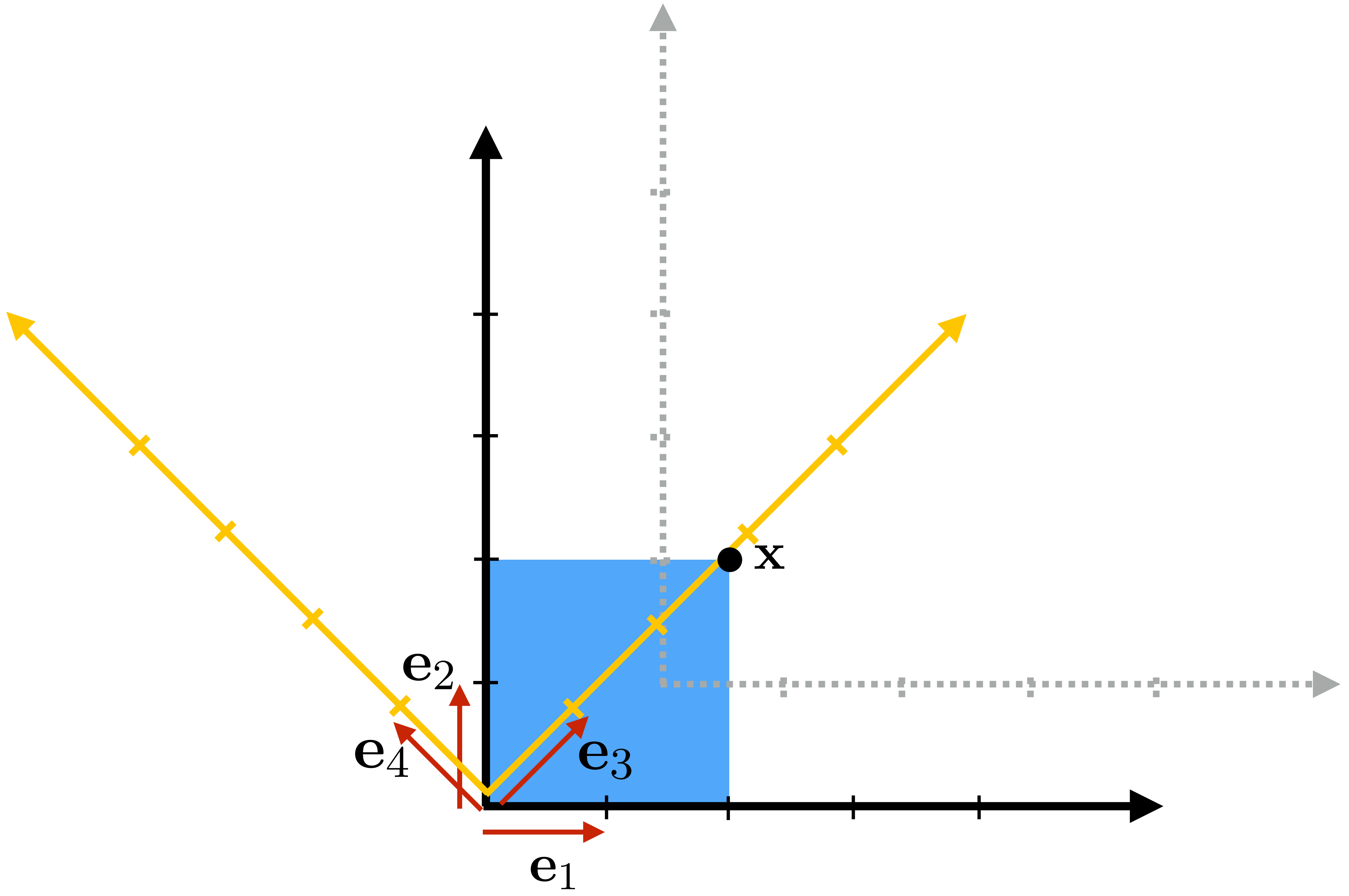
$$|f(\mathbf{x}) - f(\mathbf{y})| = |\mathbf{x} - \mathbf{y}|$$

Translation
Rotation
Reflection

"Rigid body" transformations are distance-preserving motions that also preserve *orientation* (i.e., does not include reflection)

Representing Transformations in Coordinates

Review: representing points in a coordinate space



It's the same point: *x*
But *x* is represented via different coordinates in difference coordinate spaces!

Consider coordinate space defined by orthogonal vectors $\mathbf{e}_1$ and $\mathbf{e}_2$

$$\mathbf{x} = 2\mathbf{e}_1 + 2\mathbf{e}_2$$

$$\mathbf{x} = \begin{bmatrix} 2 & 2 \end{bmatrix}$$

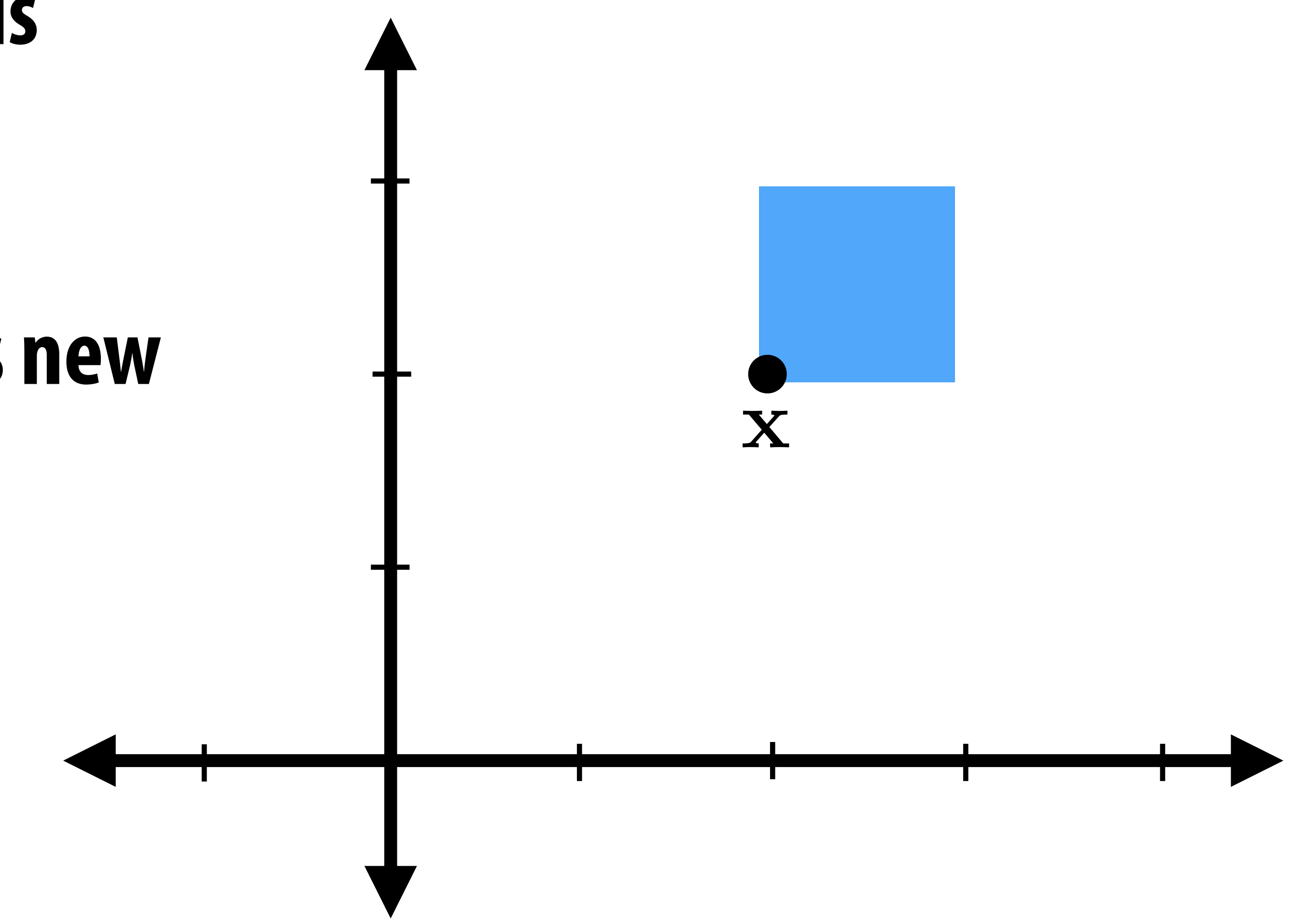
$\mathbf{x} = \begin{bmatrix} 0.5 & 1 \end{bmatrix}$ **in coordinate space defined by $\mathbf{e}_1$ and $\mathbf{e}_2$, with origin at (1.5, 1)**

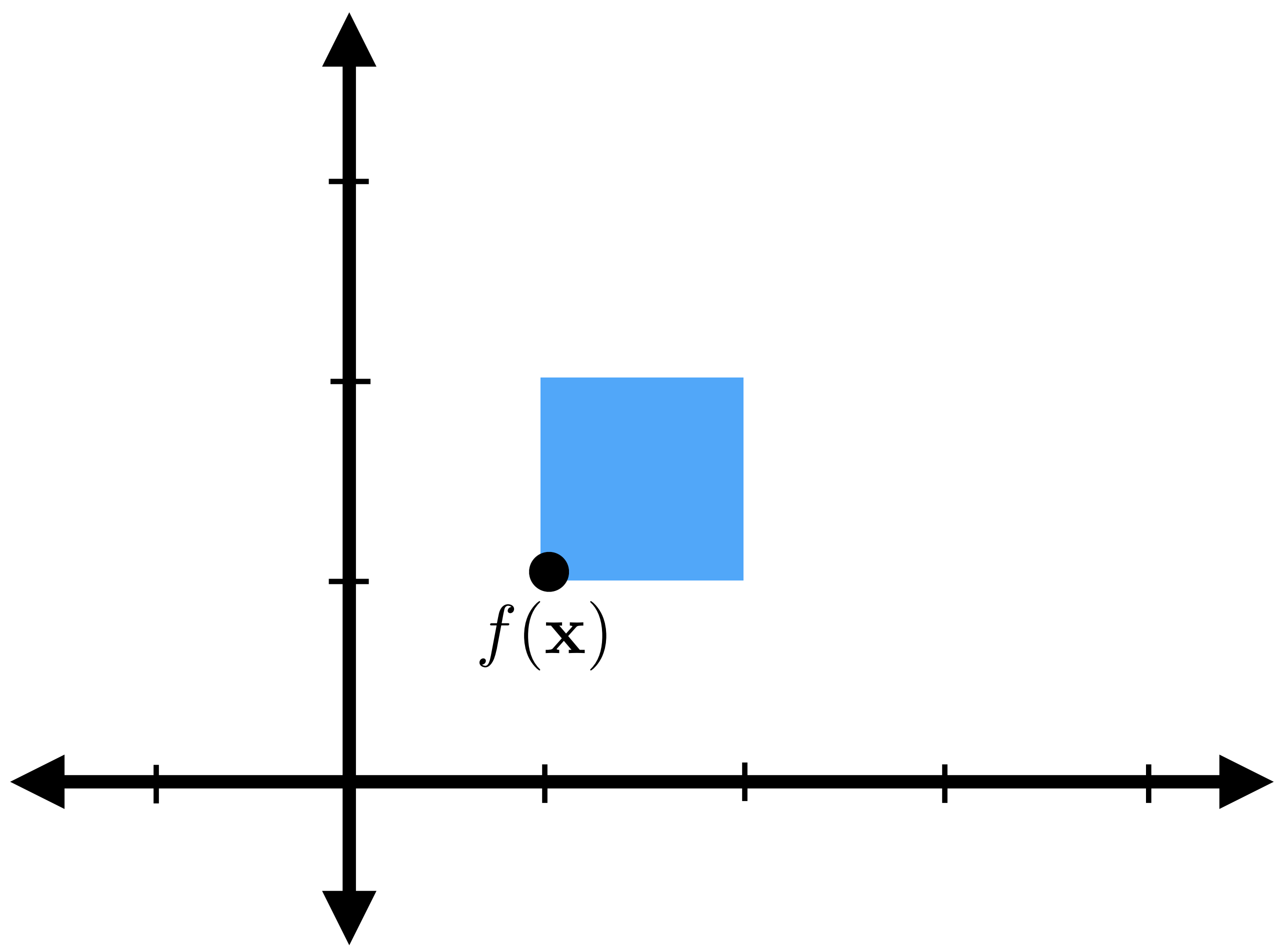
$\mathbf{x} = \begin{bmatrix} \sqrt{8} & 0 \end{bmatrix}$ **in coordinate space defined by $\mathbf{e}_3$ and $\mathbf{e}_4$, with origin at (0, 0)**

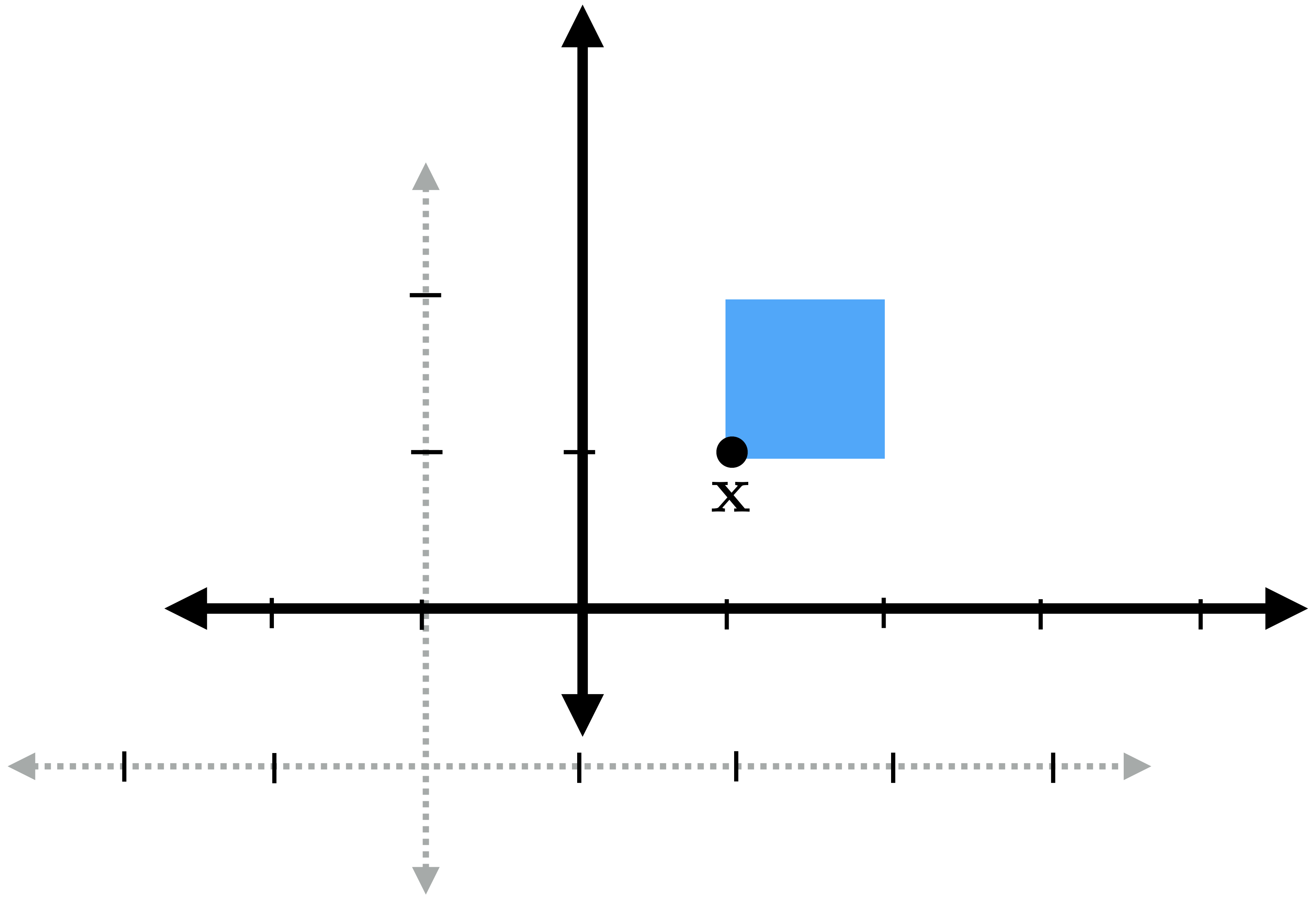
Another way to think about transformations: change of coordinates

Interpretation of transformations so far in this lecture: ***transformations modify (move) points***

Point $\mathbf{x}$ moved to new position $f(\mathbf{x})$ so it has new coordinates in same coordinate space.







Alternative interpretation:

Transformations induce of change of coordinate frame: Representation of $\mathbf{x}$ changes since point is now expressed in new coordinates

Review: 2D matrix multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} =$$

$$x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix} =$$

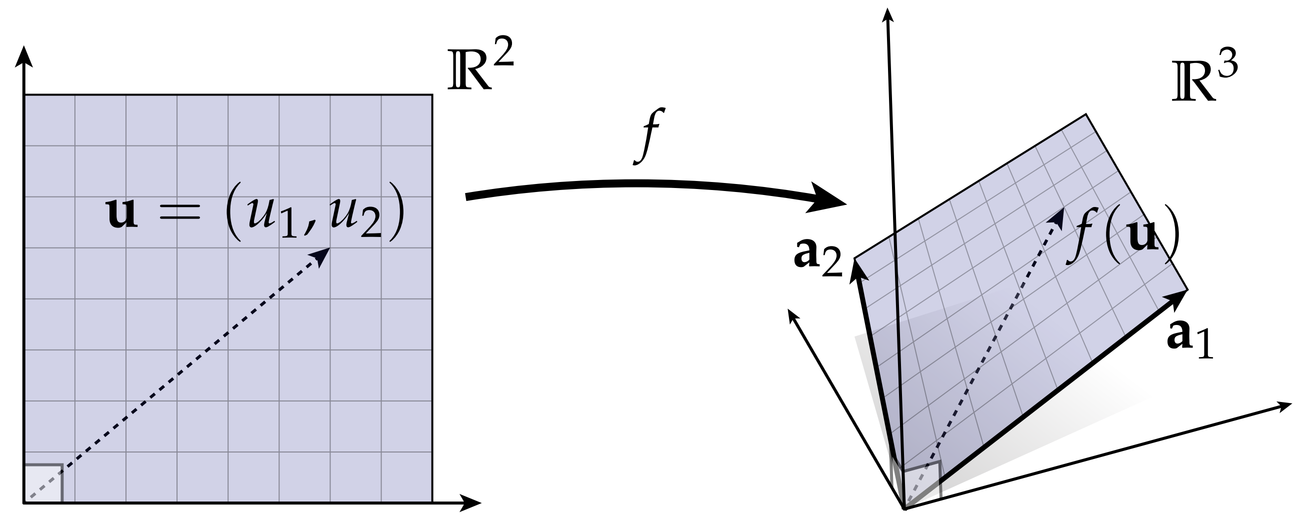
$$\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

- **Matrix multiplication is linear combination of columns**
- **Encodes a linear map!**

Linear maps via matrices

- **Example: suppose I have a linear map**

$$f(\mathbf{u}) = u_1\mathbf{a}_1 + u_2\mathbf{a}_2$$



- **Encoding as a matrix: "a" vectors become matrix columns:**

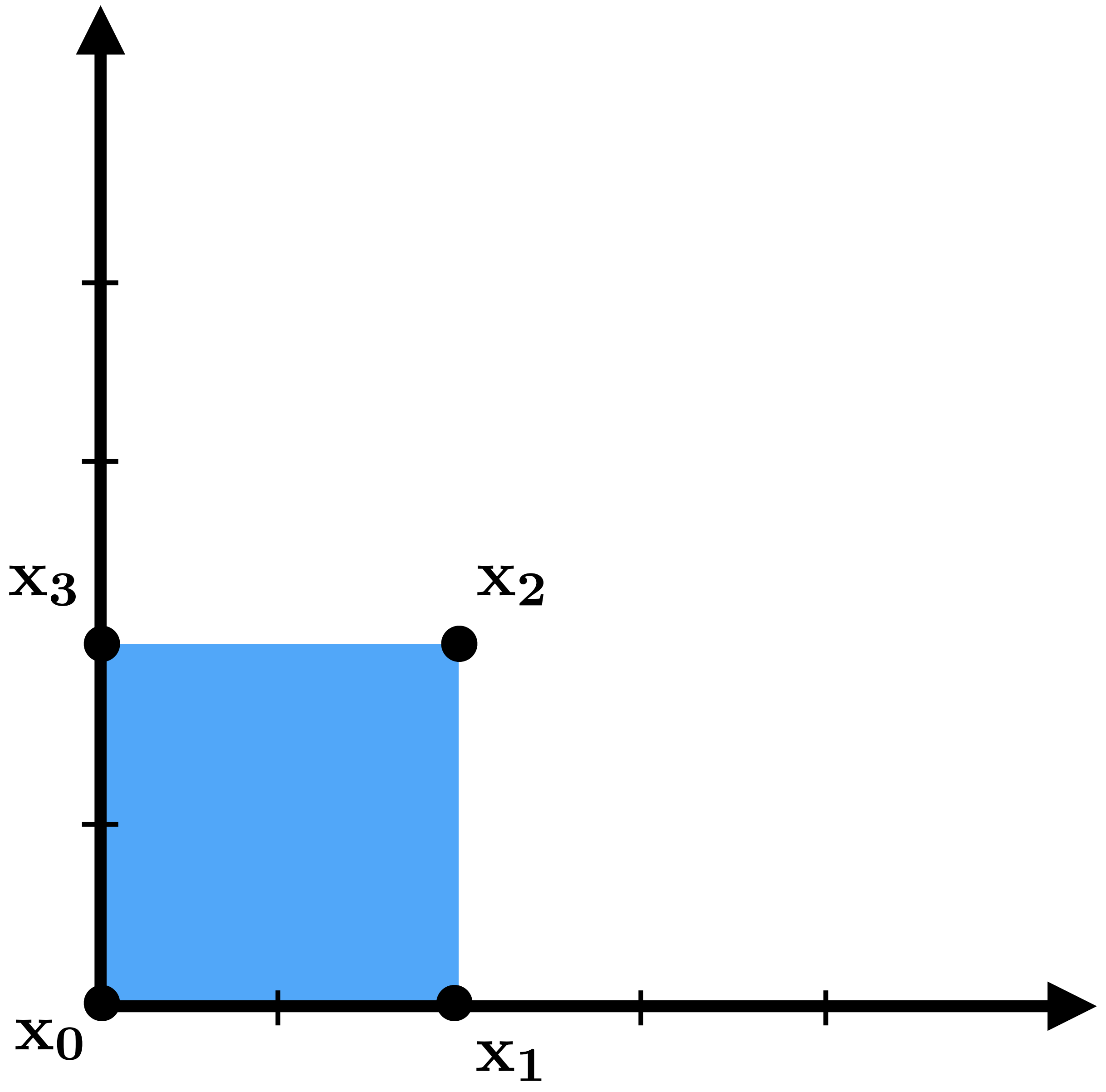
$$A := \begin{bmatrix} a_{1,x} & a_{2,x} \\ a_{1,y} & a_{2,y} \\ a_{1,z} & a_{2,z} \end{bmatrix}$$

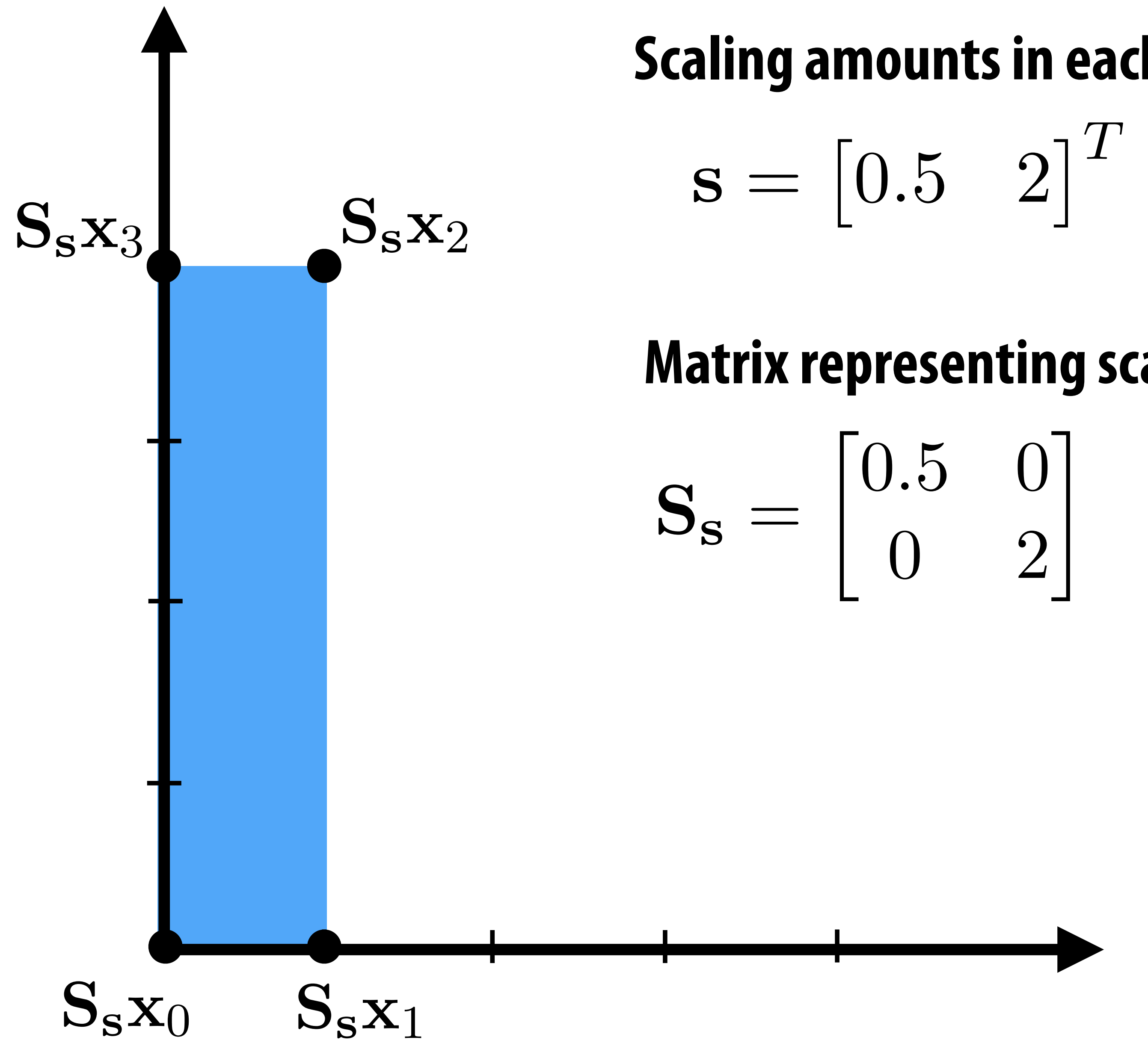
- **Matrix-vector multiply computes same output as original map:**

$$\begin{bmatrix} a_{1,x} & a_{2,x} \\ a_{1,y} & a_{2,y} \\ a_{1,z} & a_{2,z} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} a_{1,x}u_1 + a_{2,x}u_2 \\ a_{1,y}u_1 + a_{2,y}u_2 \\ a_{1,z}u_1 + a_{2,x}u_2 \end{bmatrix} = u_1\mathbf{a}_1 + u_2\mathbf{a}_2$$

Linear transformations in 2D can be represented as 2x2 matrices

Consider non-uniform scale: $\mathbf{S_s} = \begin{bmatrix} \mathbf{s}_x & 0 \\ 0 & \mathbf{s}_y \end{bmatrix}$





Scaling amounts in each direction:

$$\mathbf{s} = \begin{bmatrix} 0.5 & 2 \end{bmatrix}^T$$

Matrix representing scale transform:

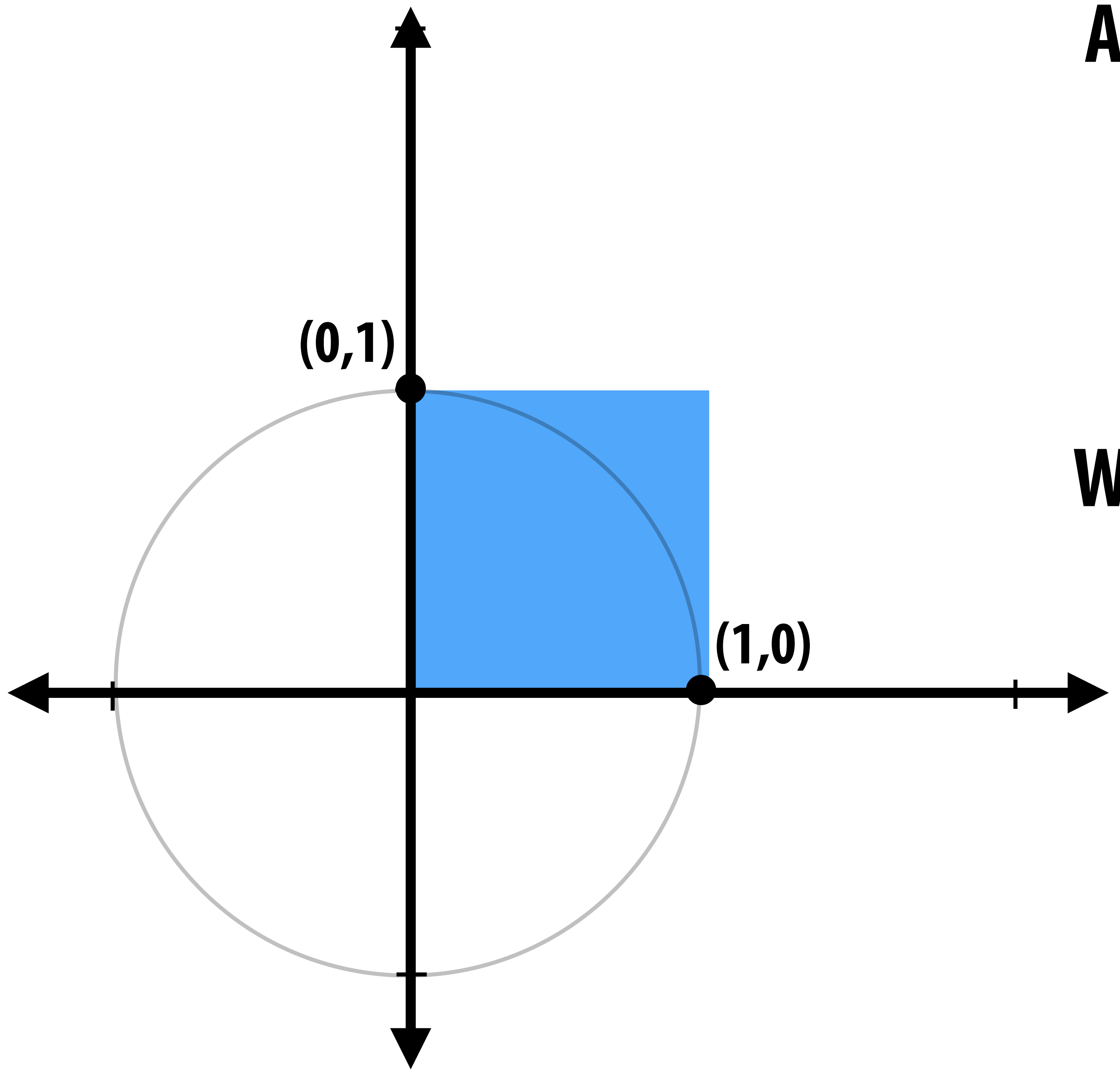
$$\mathbf{S_s} = \begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix}$$

Rotation matrix (2D)

Question: what happens to (1, 0) and (0,1) after rotation by θ?

Reminder: rotation moves points along circular trajectories.

(Recall that $\cos\theta$ and $\sin\theta$ are the coordinates of a point on the unit circle.)



Answer:

$$R_\theta(1,0) = (\cos(\theta), \sin(\theta))$$

$$R_\theta(0,1) = (\cos(\theta + \pi/2), \sin(\theta + \pi/2))$$

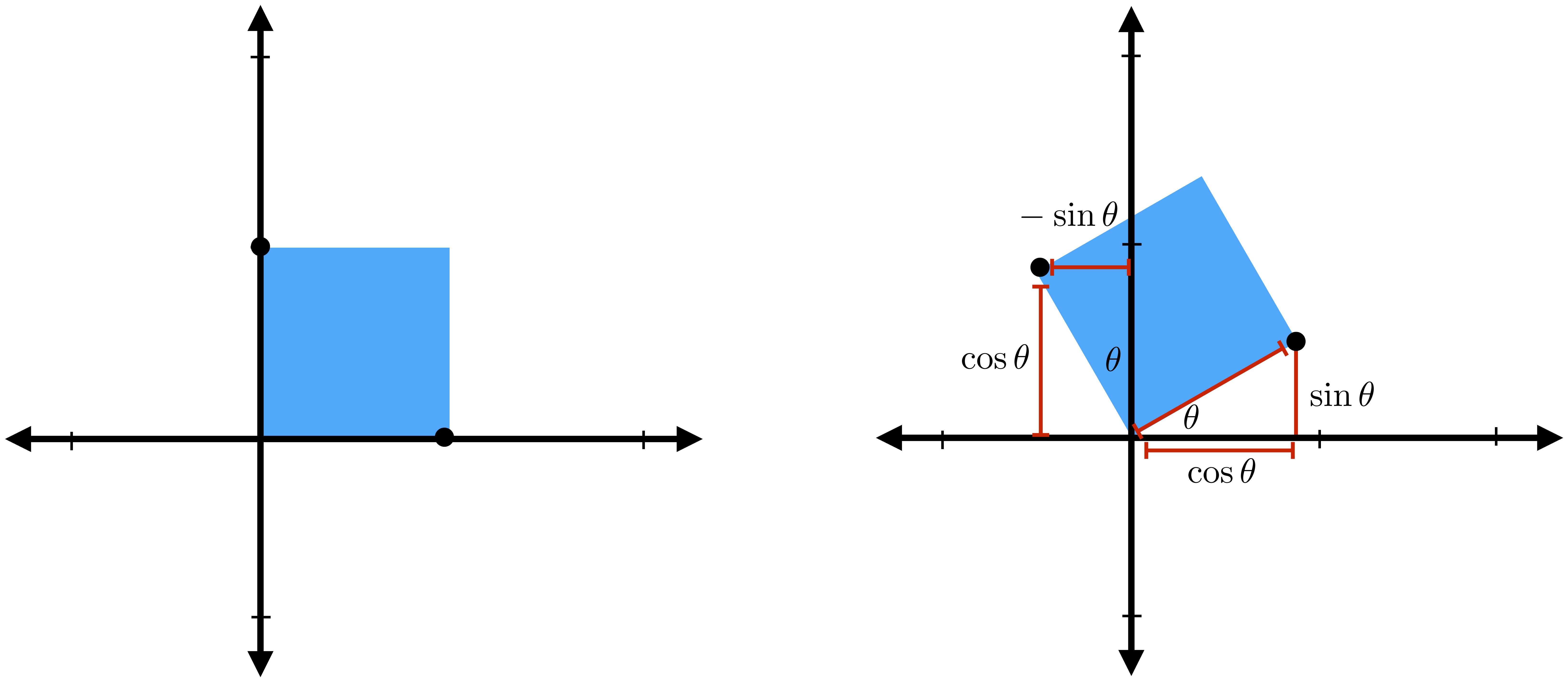
Which means the matrix must look like:

$$R_\theta = \begin{bmatrix} \cos(\theta) & \cos(\theta + \pi/2) \\ \sin(\theta) & \sin(\theta + \pi/2) \end{bmatrix}$$

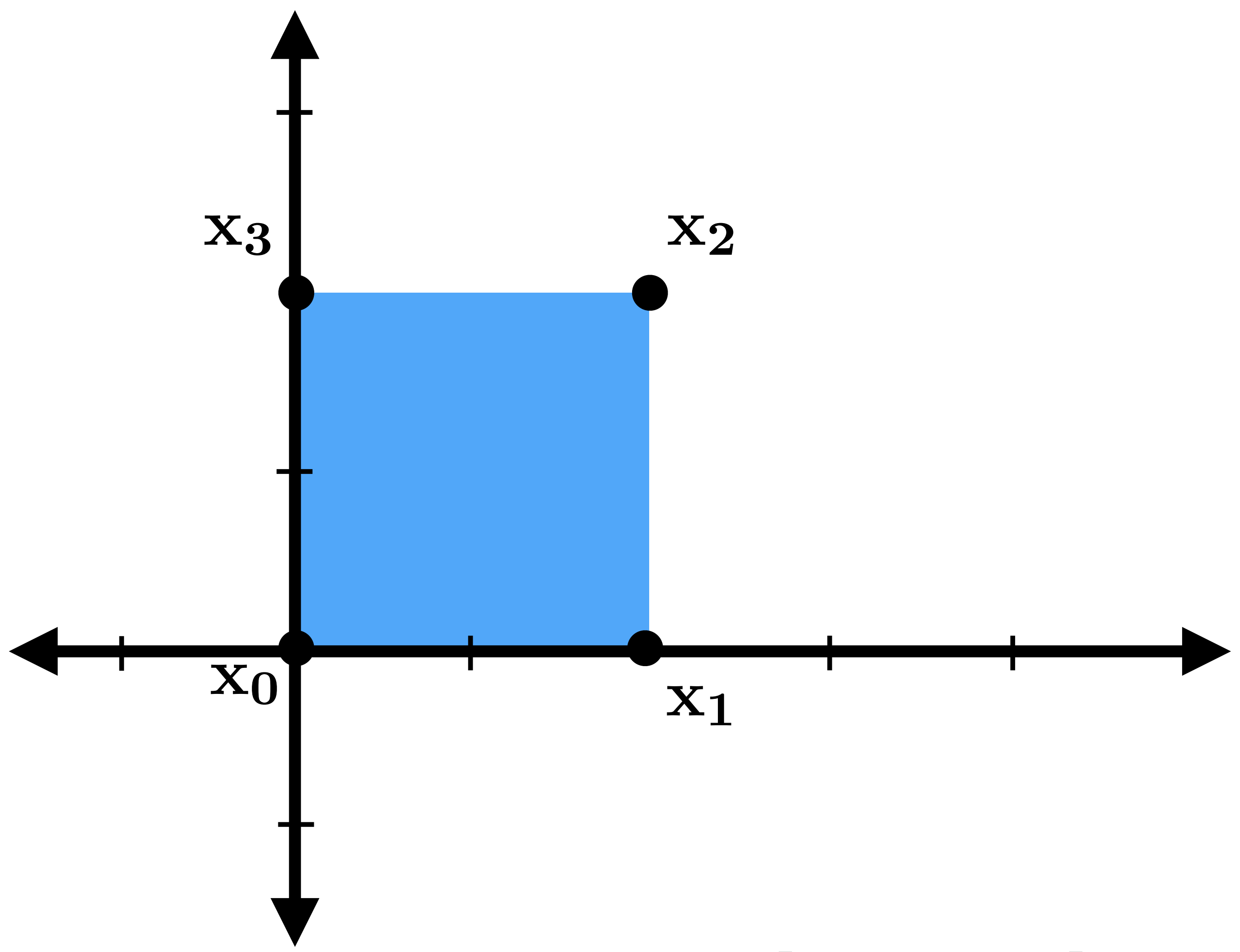
$$= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

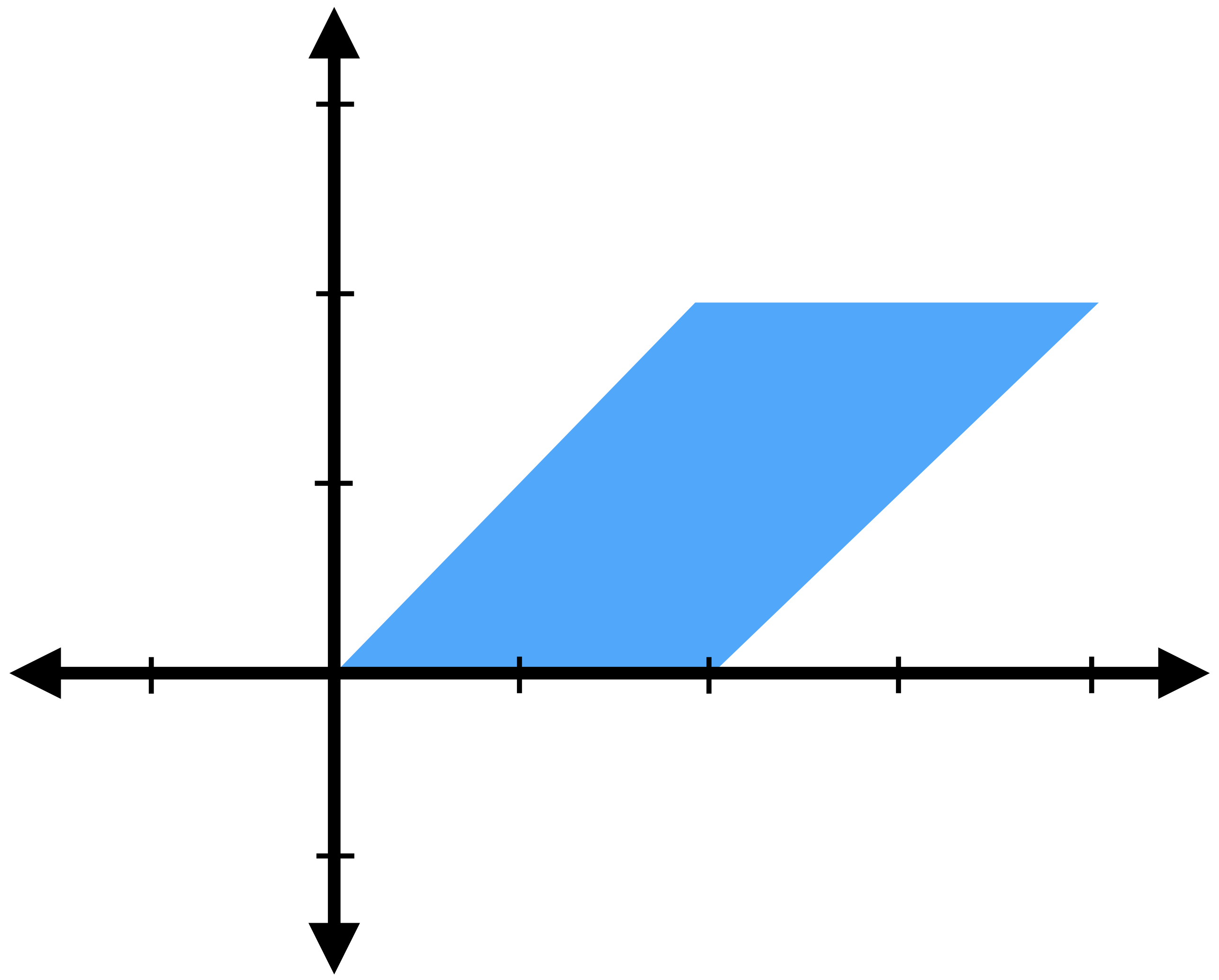
Rotation matrix (2D): another way…

$$\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



Shear



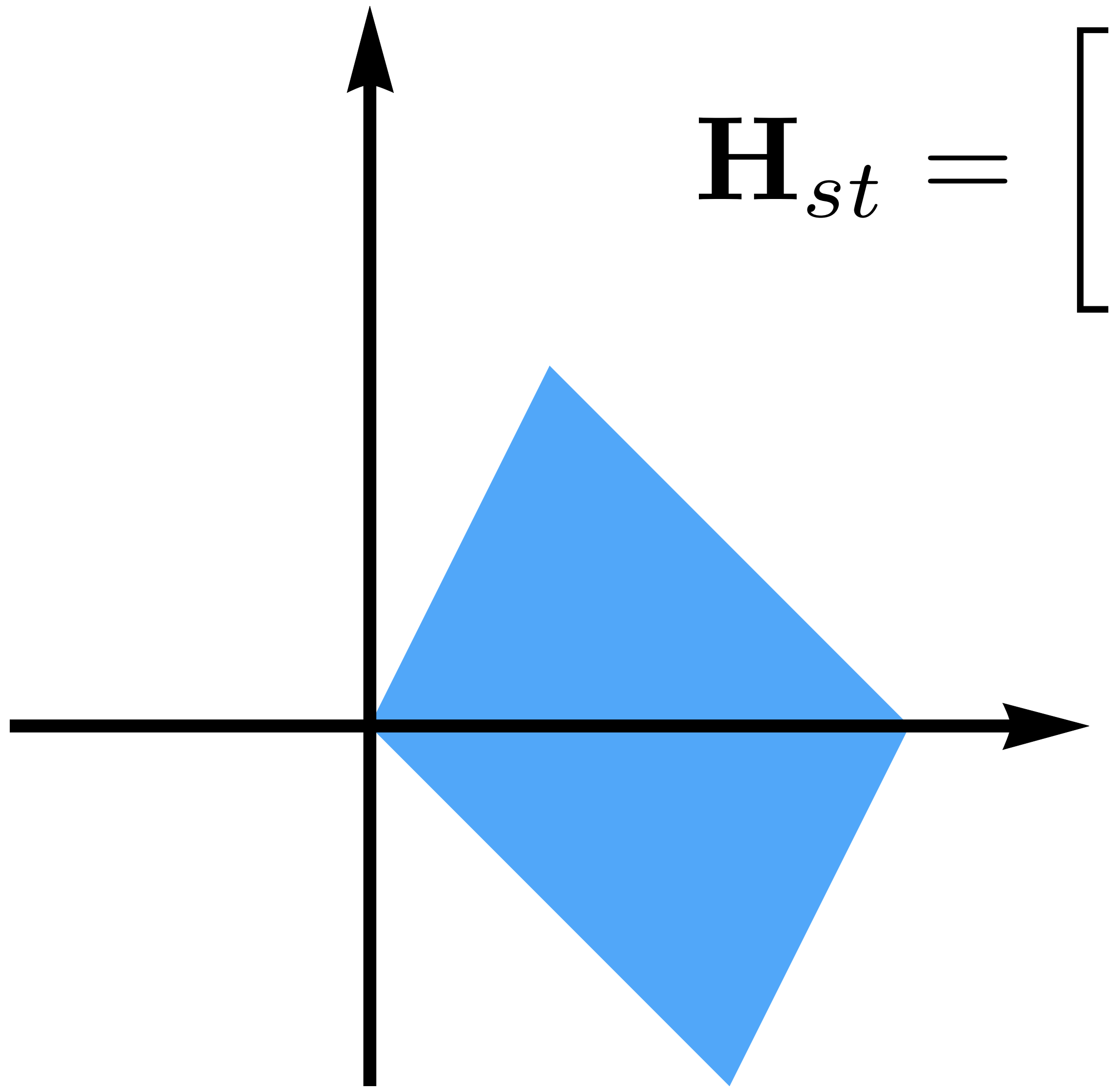


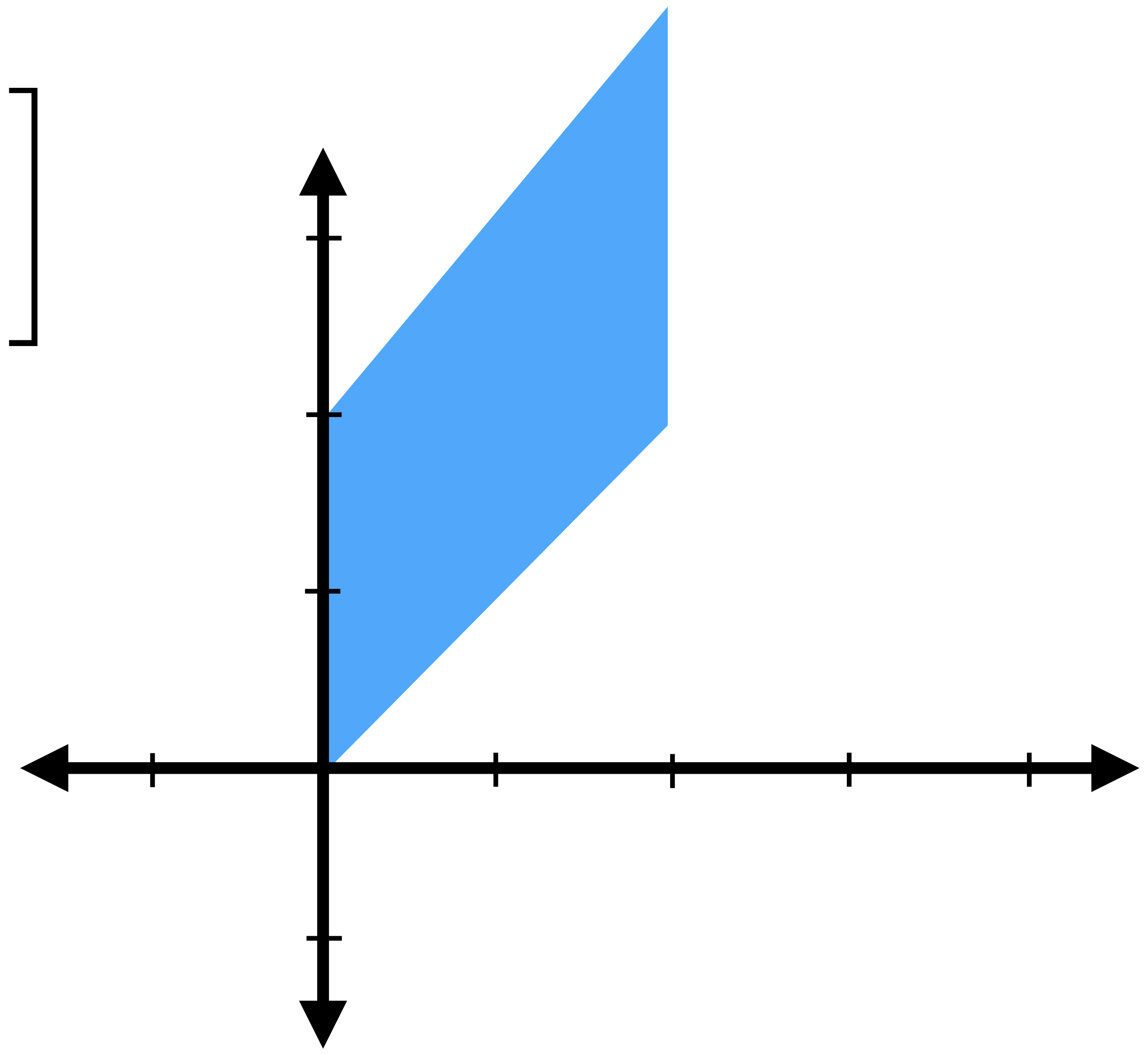
Shear in x:

$$\mathbf{H}_{xs} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

Arbitrary shear:

$$\mathbf{H}_{st} = \begin{bmatrix} 1 & s \\ t & 1 \end{bmatrix}$$

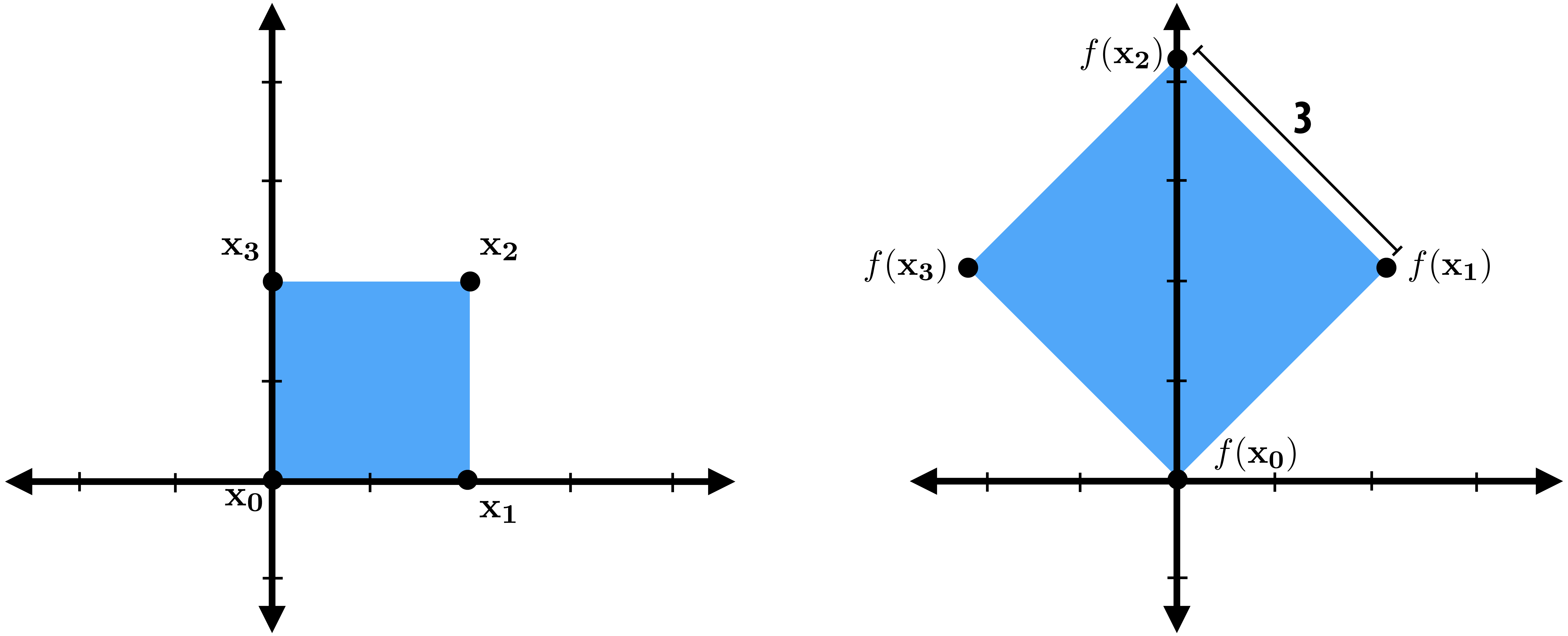




Shear in y:

$$\mathbf{H}_{ys} = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

How do we compose linear transformations?



Compose linear transformations via matrix multiplication.

This example: uniform scale, followed by rotation

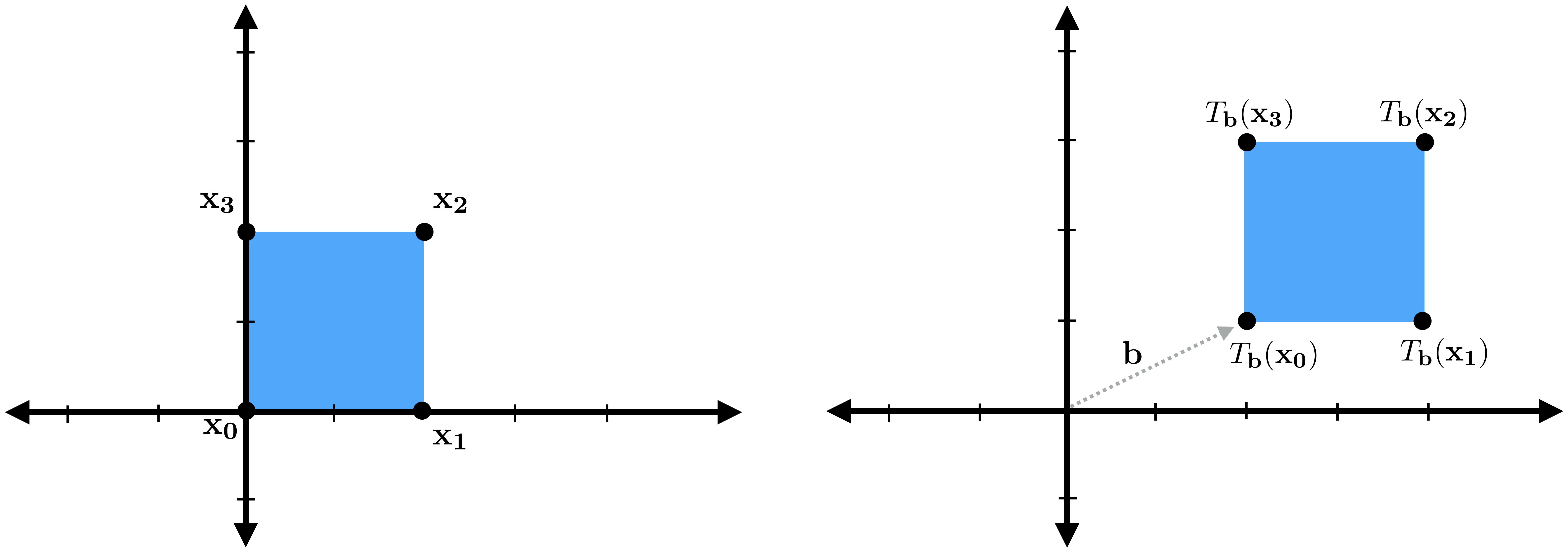
$$f(\mathbf{x}) = R_{\pi/4}\mathbf{S}_{[1.5,1.5]}\mathbf{x} = \mathbf{M}\mathbf{x}$$

Where: $\mathbf{M} = R_{\pi/4}\mathbf{S}_{[1.5,1.5]}$

Enables simple, efficient implementation: reduce complex chain of transformations to a single matrix multiplication!

How do we deal with translation? (Not linear)

$$T_{\mathbf{b}}(\mathbf{x}) = \mathbf{x} + \mathbf{b}$$



Recall: translation is not a linear transform

→ Output coefficients are not a linear combination of input coefficients

→ Translation operation cannot be represented by a 2x2 matrix

$$\mathbf{x}_{\mathbf{out}_x} = \mathbf{x}_x + \mathbf{b}_x$$

$$\mathbf{x}_{\mathbf{out}_y} = \mathbf{x}_y + \mathbf{b}_y$$

Translation math

2D homogeneous coordinates (2D-H)

Idea: represent 2D points with THREE values ("homogeneous coordinates")

So the point (x, y) **is represented as the 3-vector:** $\begin{bmatrix} x & y & 1 \end{bmatrix}^T$

And transformations are represented a 3x3 matrices that transform these vectors.

Recover final 2D coordinates by dividing by "extra" (third) coordinate

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow \begin{bmatrix} x/w \\ y/w \end{bmatrix}$$

(More on this later…)

Example: scale and rotation in 2D-H coords

- **For transformations that are already linear, not much changes:**

$$\mathbf{S_s} = \begin{bmatrix} \mathbf{S}_x & 0 & 0 \\ 0 & \mathbf{S}_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that the last row/column doesn't do anything interesting. E.g., for scaling:

$$\mathbf{S_s} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{S}_x x \\ \mathbf{S}_y y \\ 1 \end{bmatrix}$$

Now we divide by the 3rd coordinate to get our final 2D coordinates (not too exciting!)

$$\begin{bmatrix} \mathbf{S}_x x \\ \mathbf{S}_y y \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{S}_x x/1 \\ \mathbf{S}_y y/1 \end{bmatrix} = \begin{bmatrix} \mathbf{S}_x x \\ \mathbf{S}_y y \end{bmatrix}$$

(Will get more interesting when we talk about *perspective*…)

Translation in 2D homogeneous coordinates

Translation expressed as 3x3 matrix multiplication:

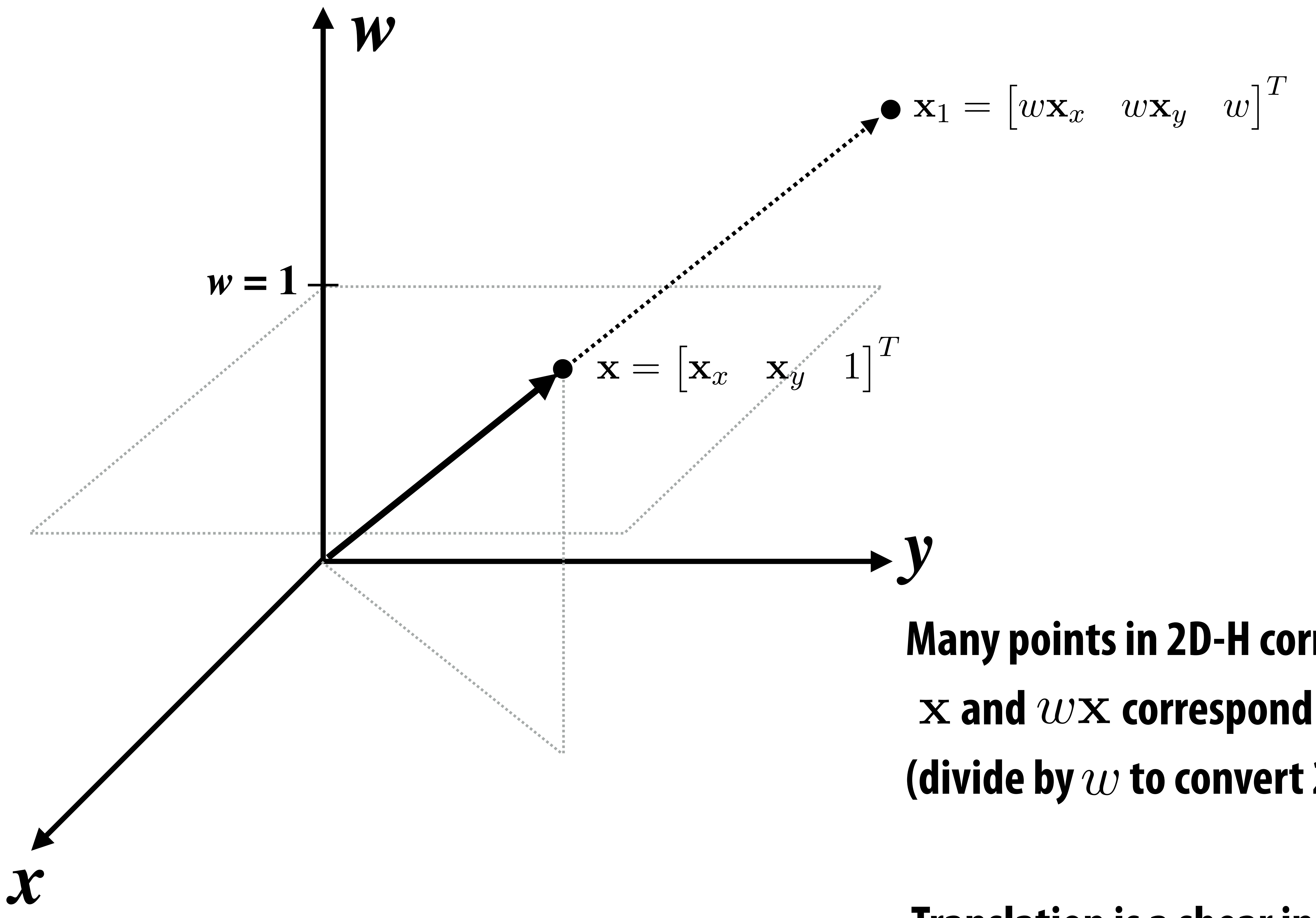
$$\mathbf{T_b} = \begin{bmatrix} 1 & 0 & \mathbf{b}_x \\ 0 & 1 & \mathbf{b}_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T_b}\mathbf{x} = \begin{bmatrix} 1 & 0 & \mathbf{b}_x \\ 0 & 1 & \mathbf{b}_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_x \\ \mathbf{x}_y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_x + \mathbf{b}_x \\ \mathbf{x}_y + \mathbf{b}_y \\ 1 \end{bmatrix}$$

(remember: just a linear combination of columns!)

Cool: homogeneous coordinates let us encode translations as *linear* transformations!

Homogeneous coordinates: some intuition



Many points in 2D-H correspond to same point in 2D

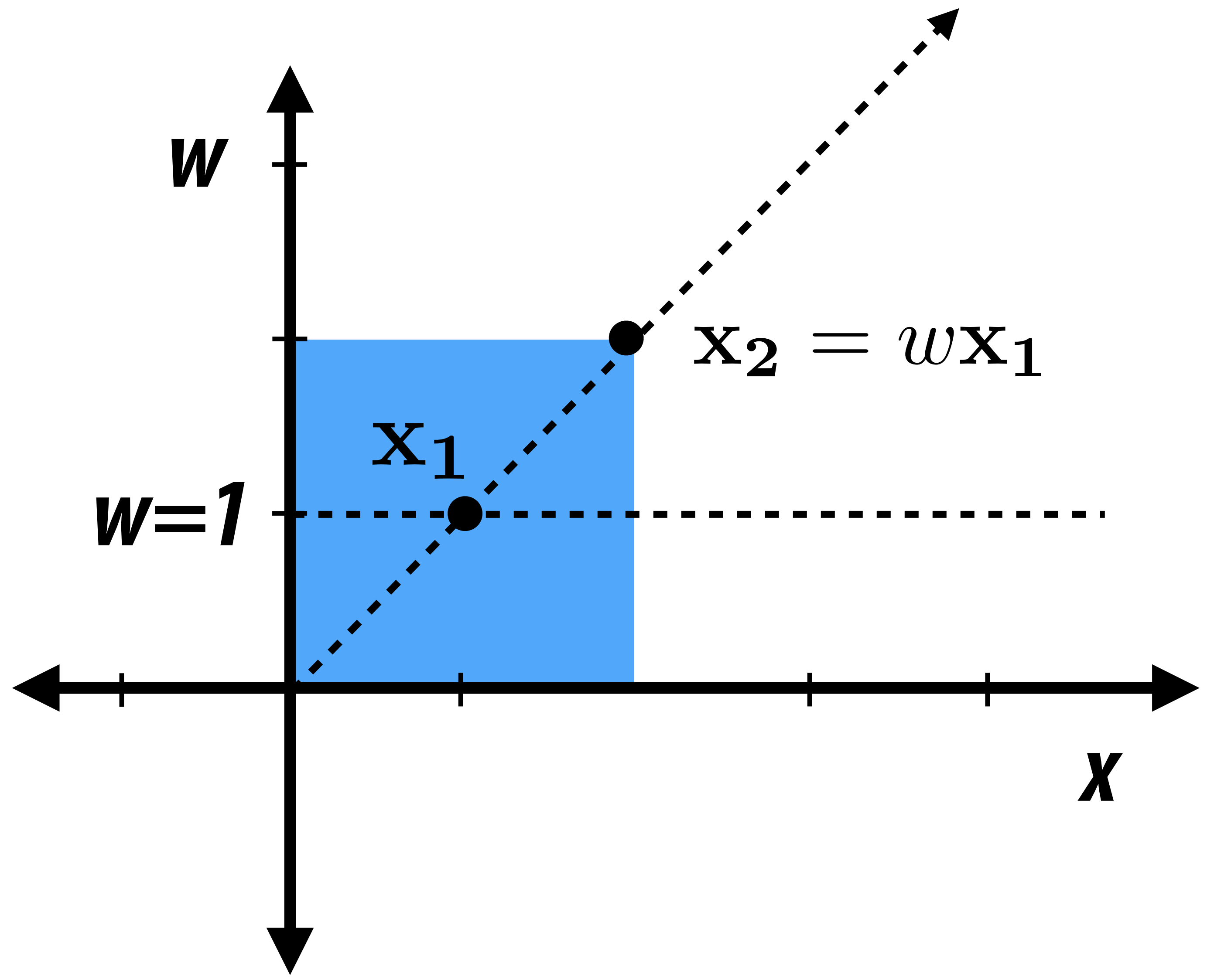
$\mathbf{x}$ and $w\mathbf{x}$ correspond to the same 2D point

(divide by w to convert 2D-H back to 2D)

Translation is a shear in x and y in 2D-H space

$$\mathbf{T_b}\mathbf{x} = \begin{bmatrix} 1 & 0 & \mathbf{b}_x \\ 0 & 1 & \mathbf{b}_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w\mathbf{x}_x \\ w\mathbf{x}_y \\ w \end{bmatrix} = \begin{bmatrix} w\mathbf{x}_x + w\mathbf{b}_x \\ w\mathbf{x}_y + w\mathbf{b}_y \\ w \end{bmatrix}$$

Translation = shear in homogeneous space

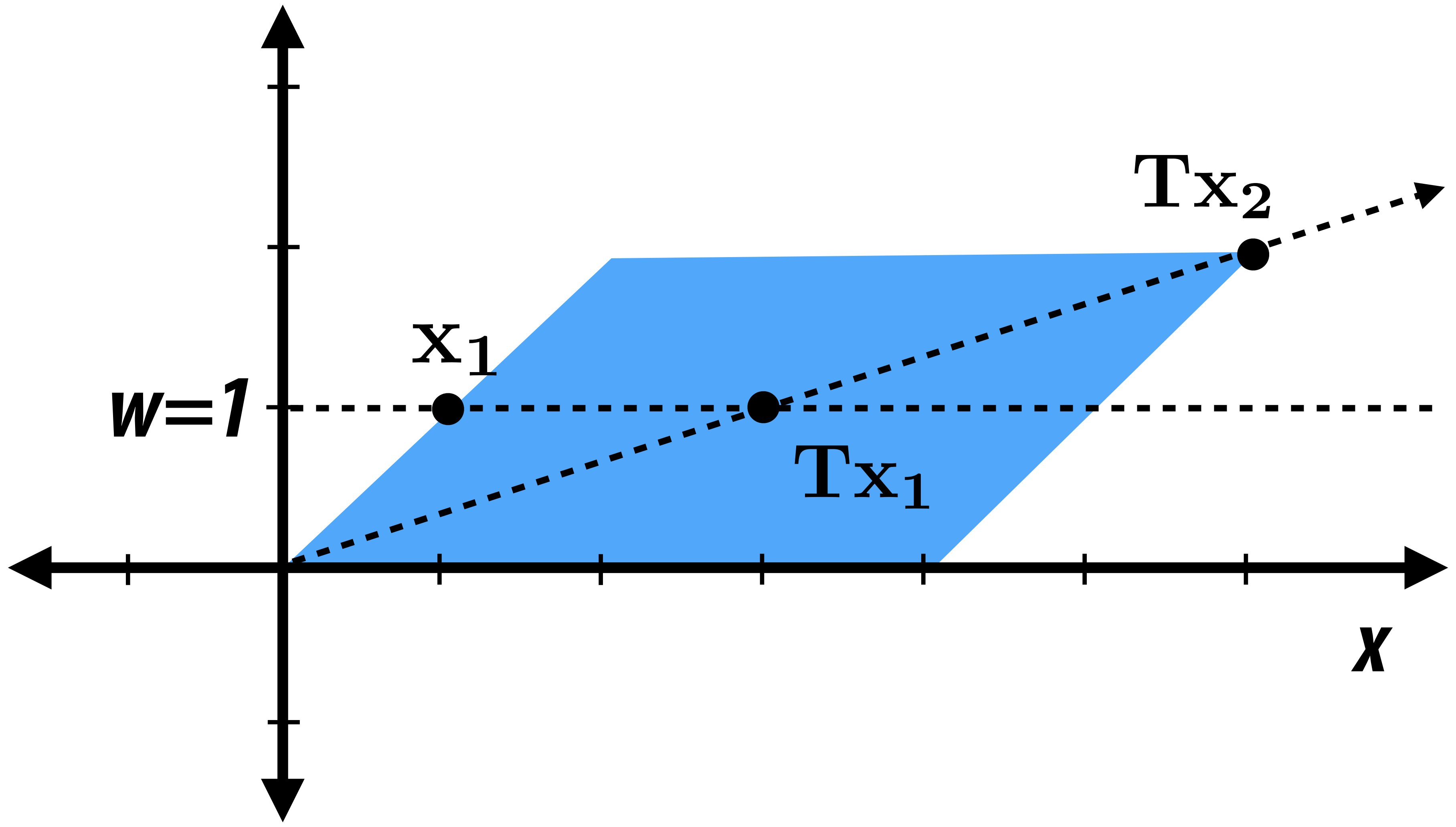


For simplicity, consider 1D-H:

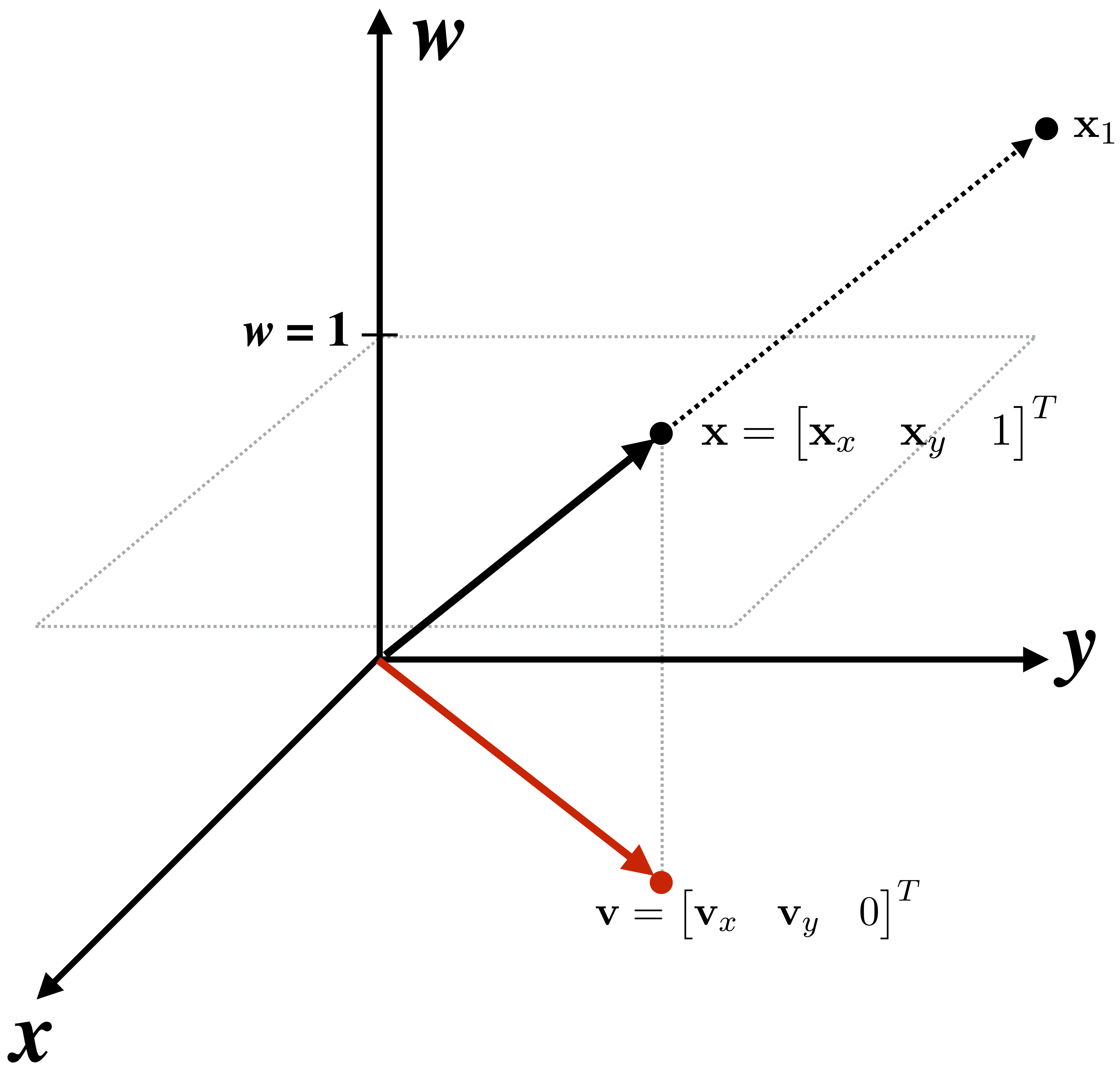
Translate by *t=2*: $\mathbf{T} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Recall: this is a shear in homogeneous *x*.

1D translation is affine in 1D ($x + t$),
but it is linear in 1D-H



Homogeneous coordinates: points vs. vectors



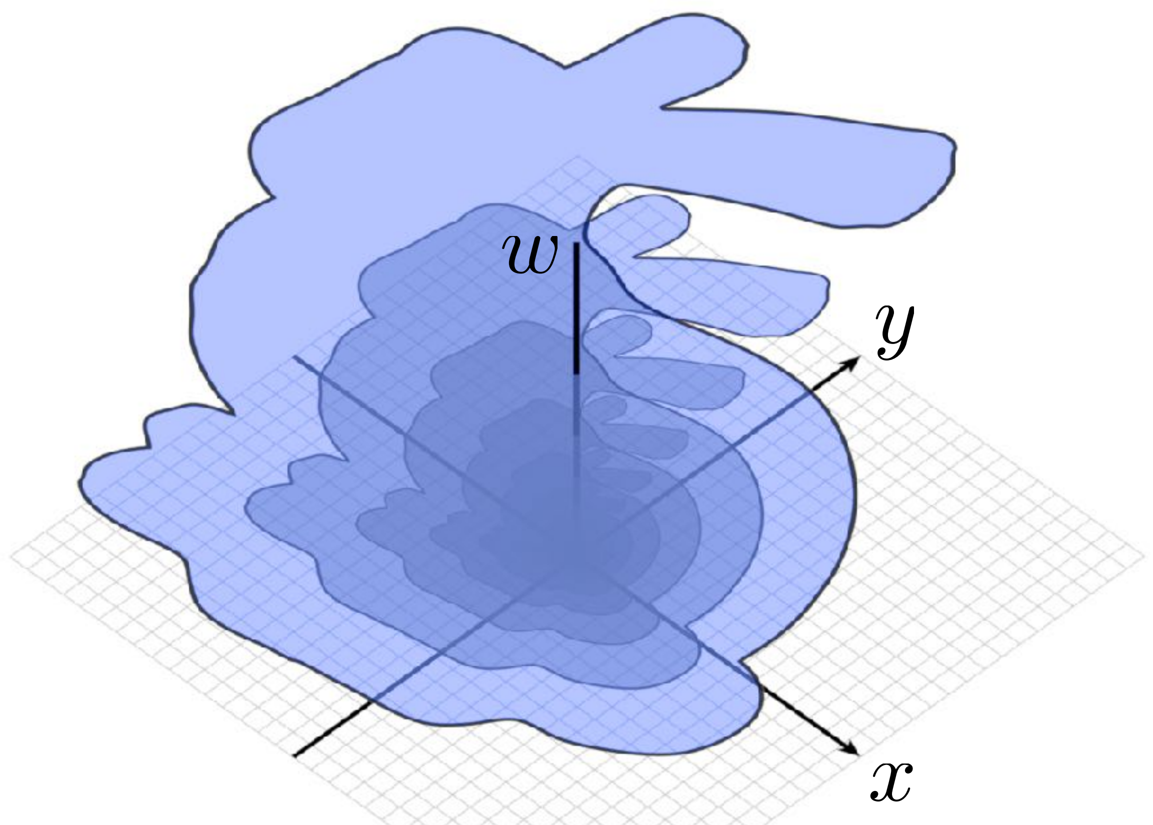
2D-H points with w=0 represent 2D vectors (think: directions are points at infinity)

Unlike 2D, points and directions are distinguishable by their representation in 2D-H

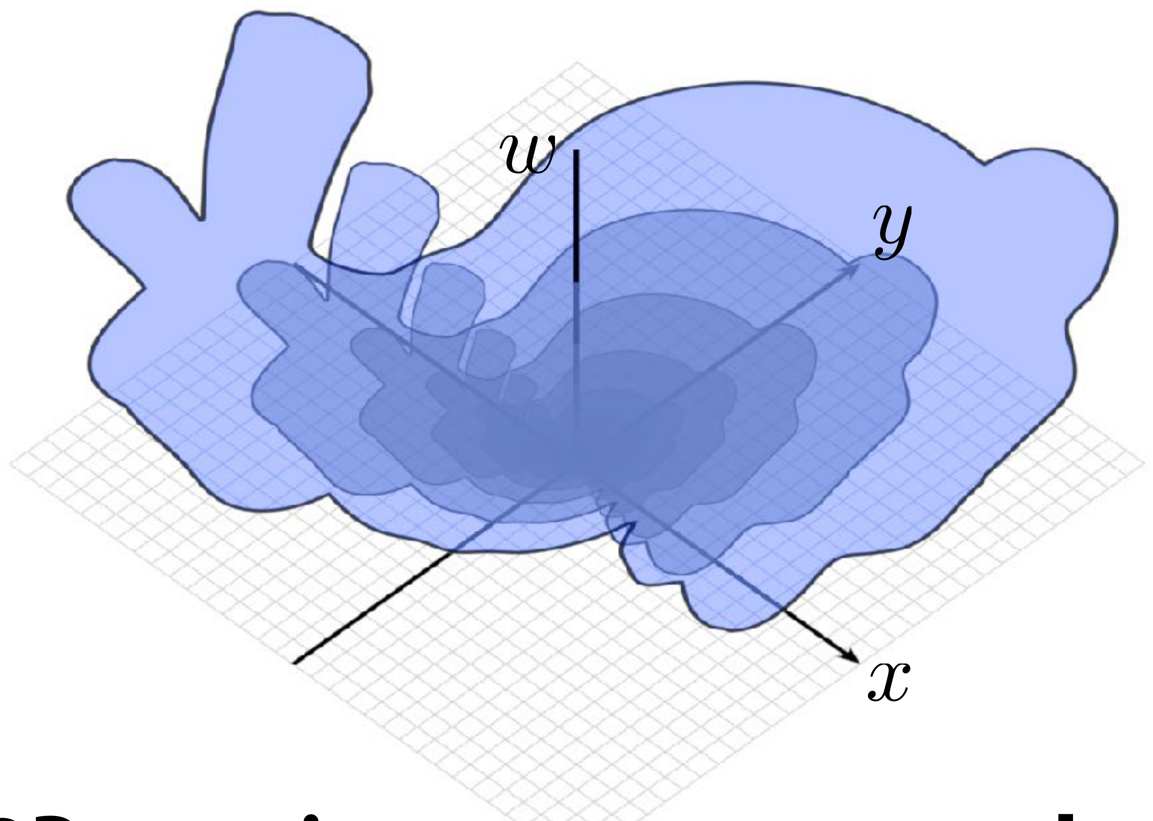
Note: translation does not modify directions:

$$\mathbf{T_b v} = \begin{bmatrix} 1 & 0 & \mathbf{b}_x \\ 0 & 1 & \mathbf{b}_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \\ 0 \end{bmatrix}$$

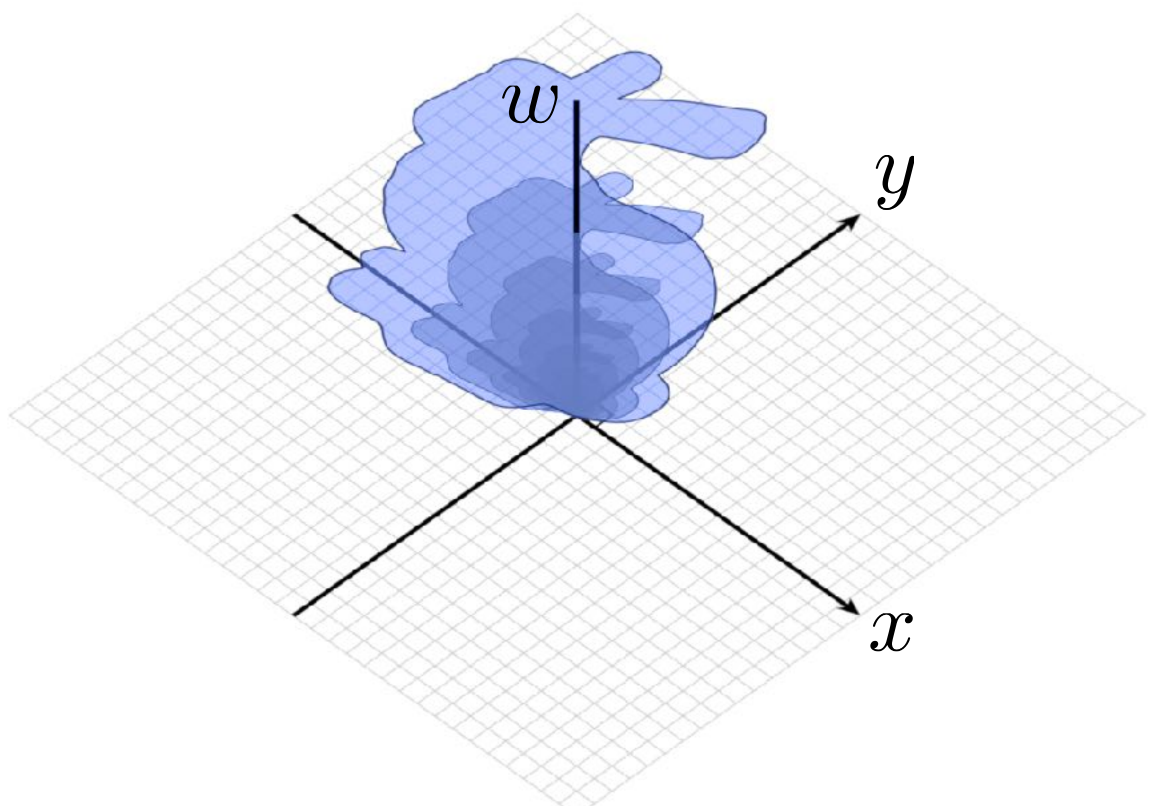
Visualizing 2D transformations in 2D-H



Original shape in 2D can be viewed as many copies, uniformly scaled by w.

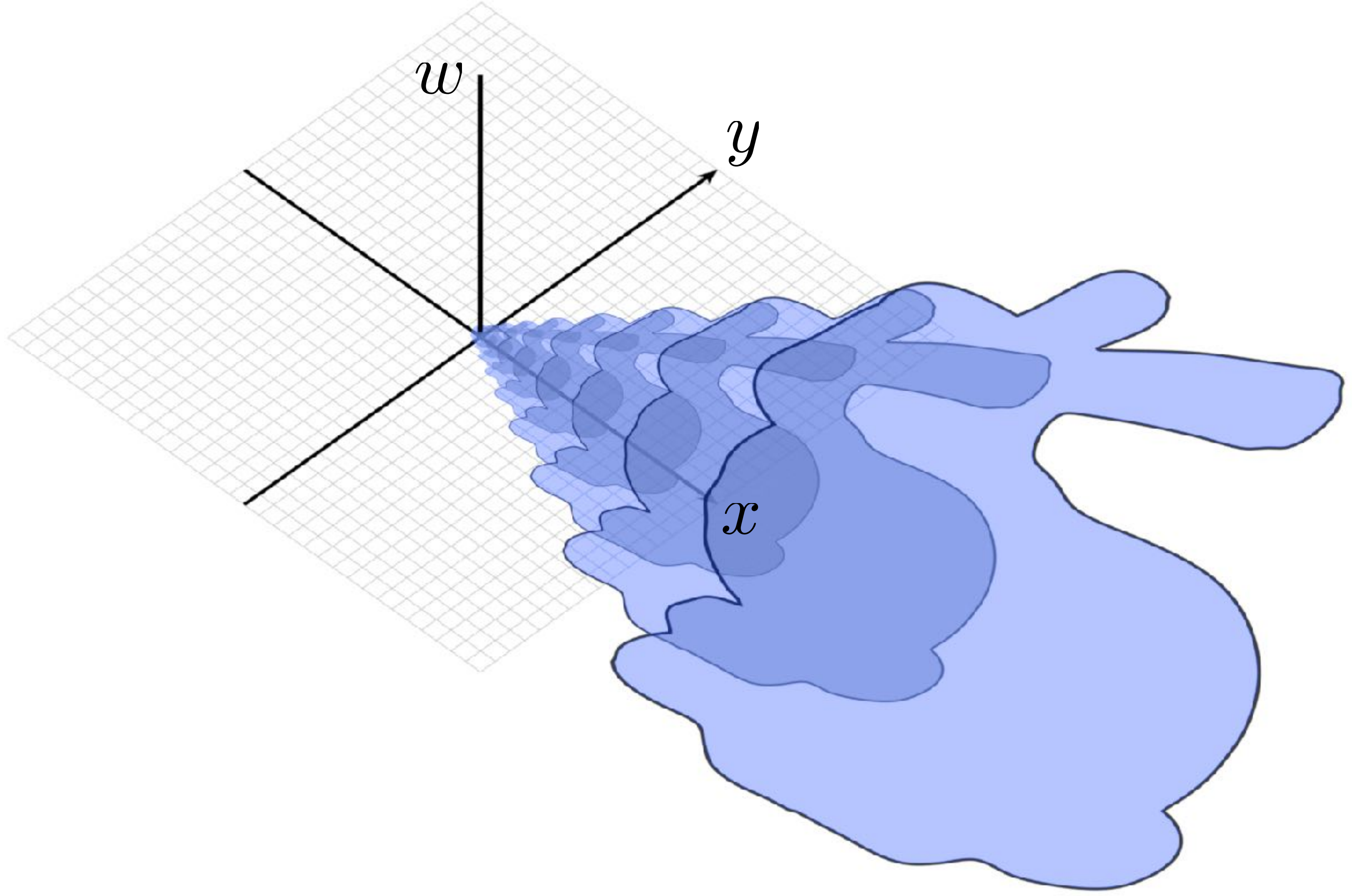


2D rotation ↔ rotate around w



2D scale ↔ scale x and y; preserve w

(Question: what happens to 2D shape if you scale x, y, and w uniformly?)



2D translate ↔ shear in 2D-H

(LINEAR!)

Moving to 3D (and 3D-H)

Represent 3D transformations as 3x3 matrices and 3D-H transformations as 4x4 matrices

Scale:

3D

$$\mathbf{S}_{\mathbf{s}} = \begin{bmatrix} \mathbf{S}_x & 0 & 0 \\ 0 & \mathbf{S}_y & 0 \\ 0 & 0 & \mathbf{S}_z \end{bmatrix}$$

3D-H

$$\mathbf{S}_{\mathbf{s}} = \begin{bmatrix} \mathbf{S}_x & 0 & 0 & 0 \\ 0 & \mathbf{S}_y & 0 & 0 \\ 0 & 0 & \mathbf{S}_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Shear (in x, based on y,z position):

$$\mathbf{H}_{x,\mathbf{d}} = \begin{bmatrix} 1 & \mathbf{d}_y & \mathbf{d}_z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{H}_{x,\mathbf{d}} = \begin{bmatrix} 1 & \mathbf{d}_y & \mathbf{d}_z & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

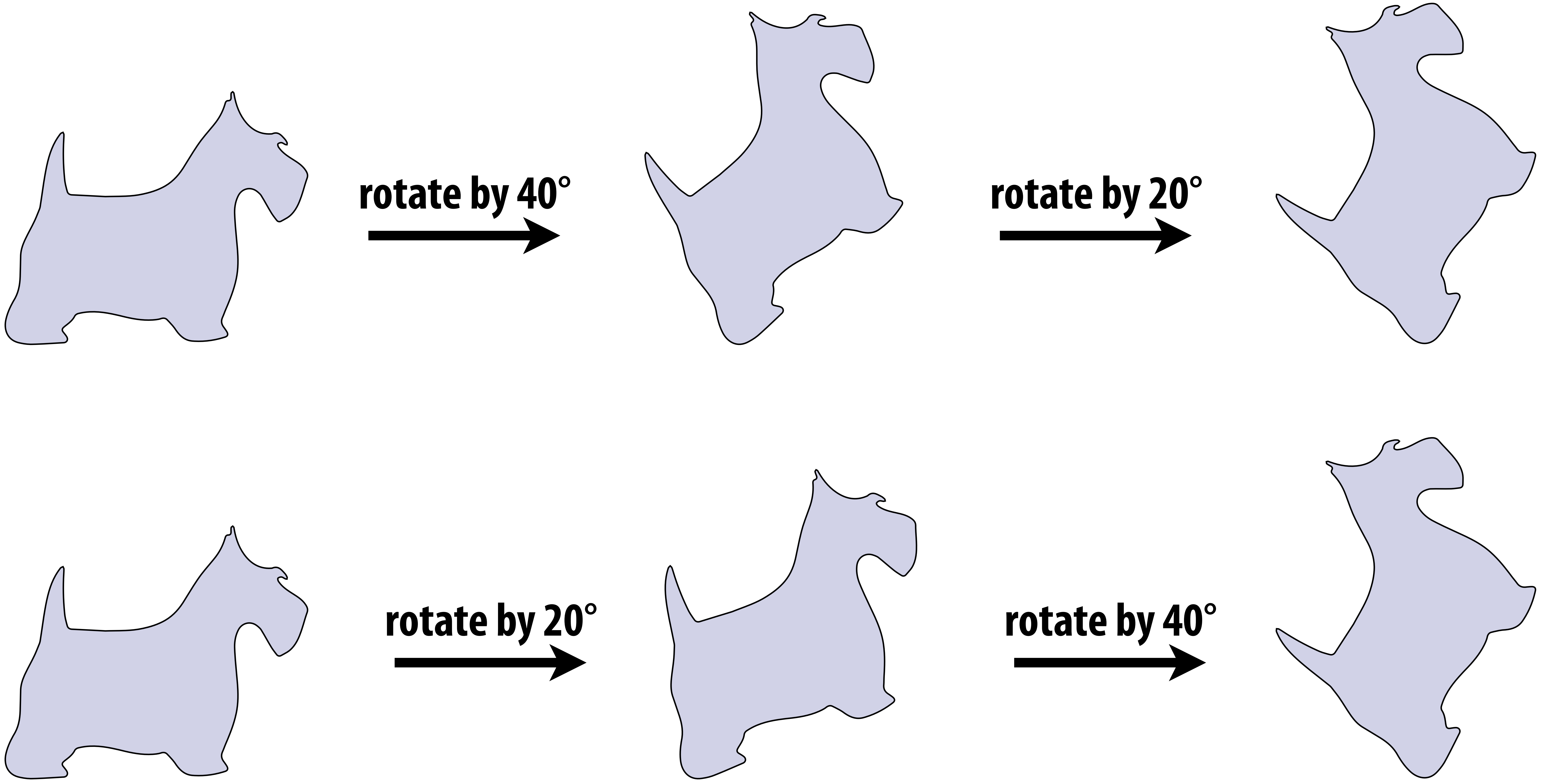
Translate:

3D-H

$$\mathbf{T}_{\mathbf{b}} = \begin{bmatrix} 1 & 0 & 0 & \mathbf{b}_x \\ 0 & 1 & 0 & \mathbf{b}_y \\ 0 & 0 & 1 & \mathbf{b}_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Commutativity of rotations—2D

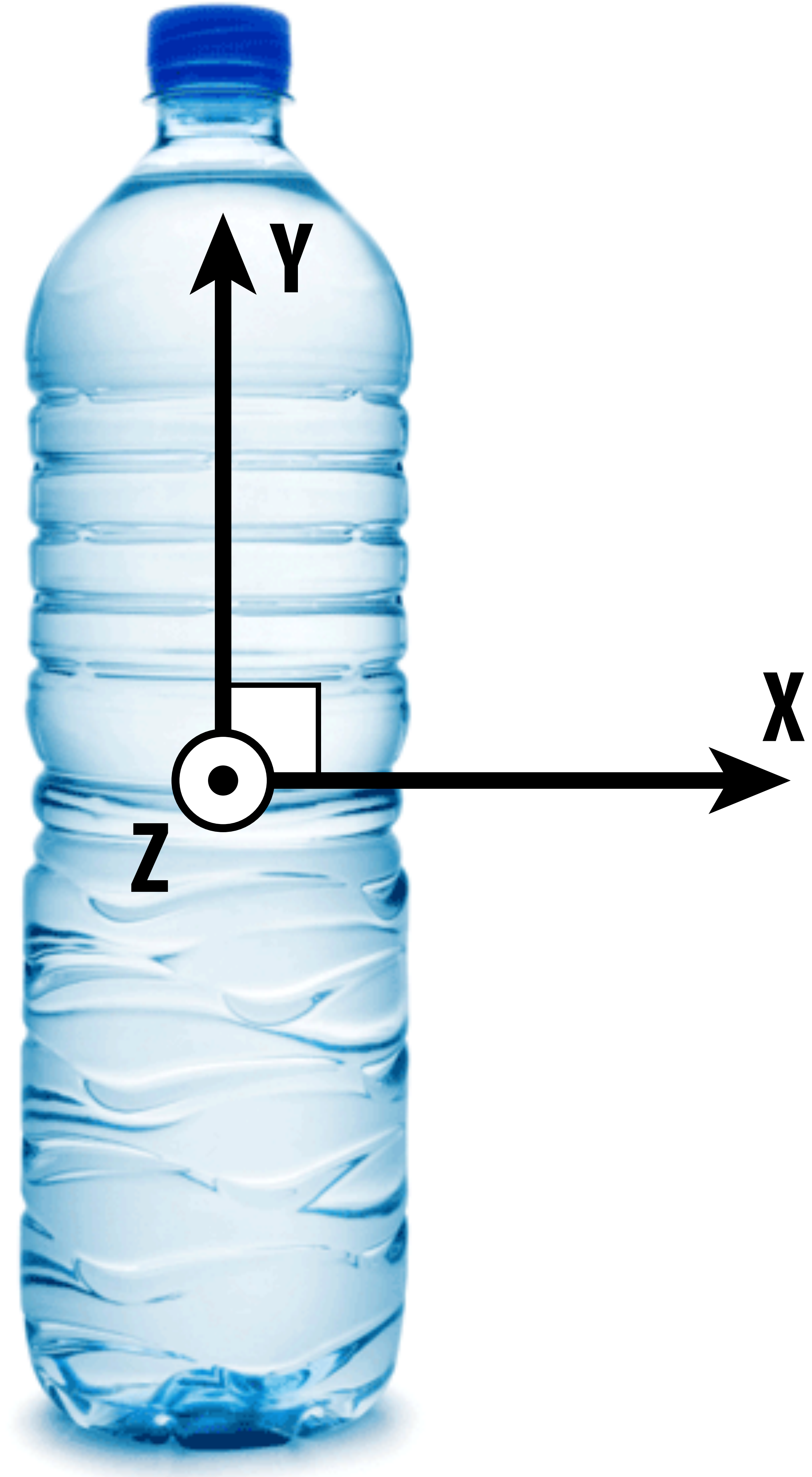
- In 2D, order of rotations doesn't matter:



Same result! ("2D rotations commute")

Commutativity of rotations—3D

- What about in 3D?
- IN-CLASS ACTIVITY:
  - Rotate 90° around Y, then 90° around Z, then 90° around X
  - Rotate 90° around Z, then 90° around Y, then 90° around X
  - (Was there any difference?)

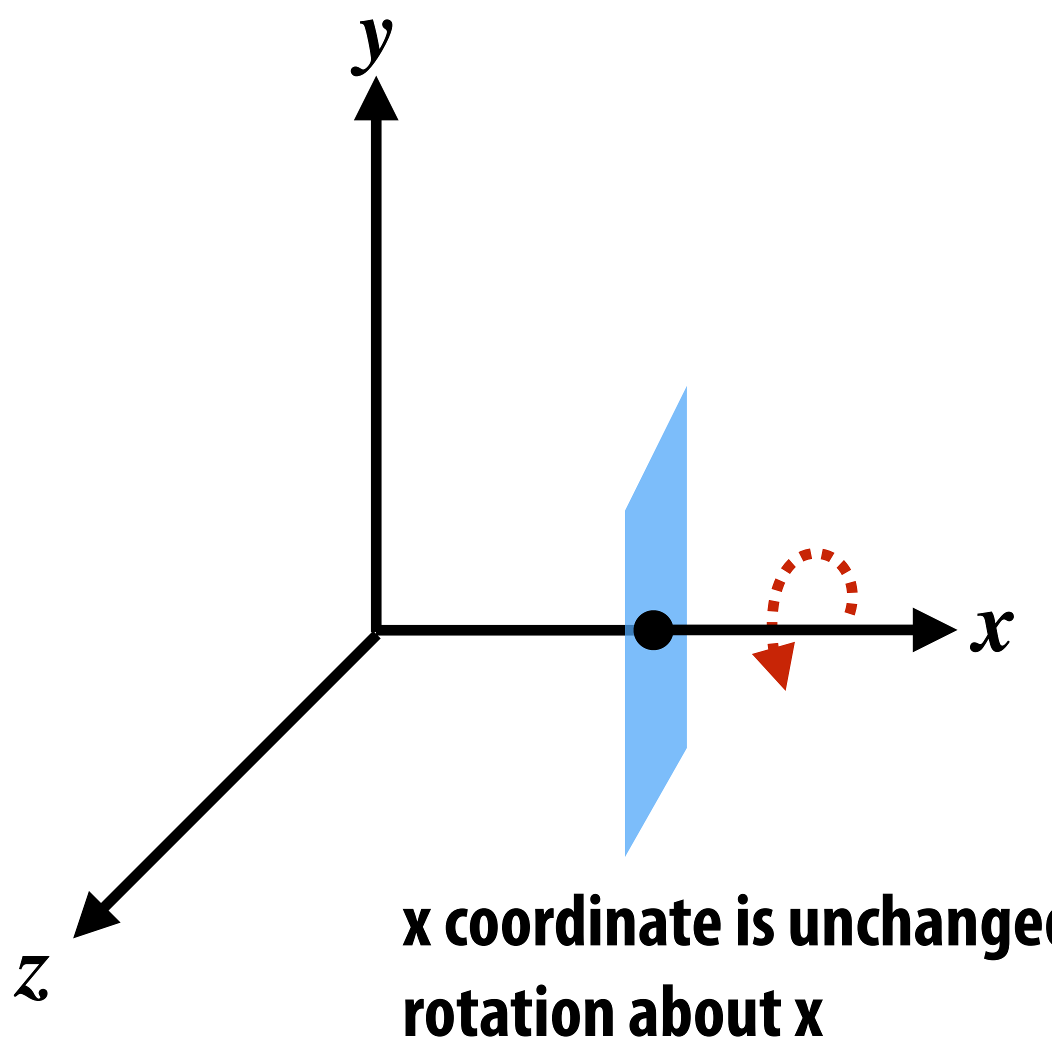


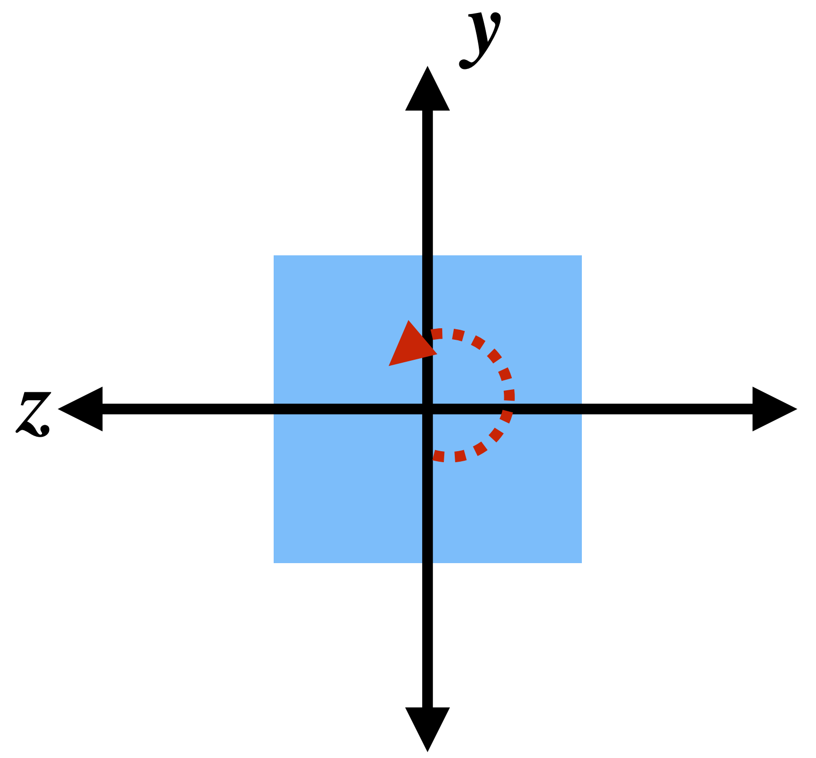
CONCLUSION: bad things can happen if we're not careful about the order in which we apply rotations!

Rotations in 3D

Rotation about x axis:

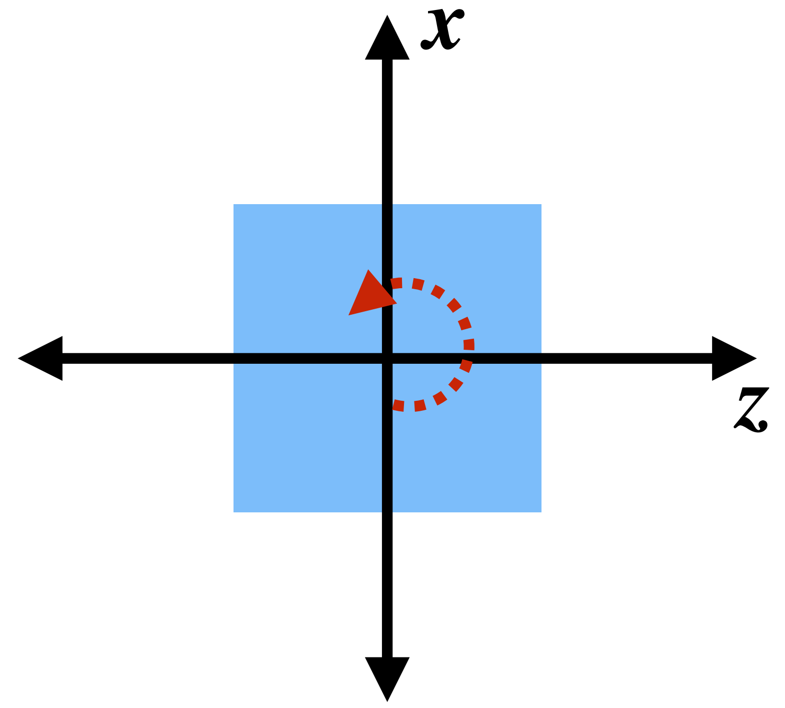
$$\mathbf{R}_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$





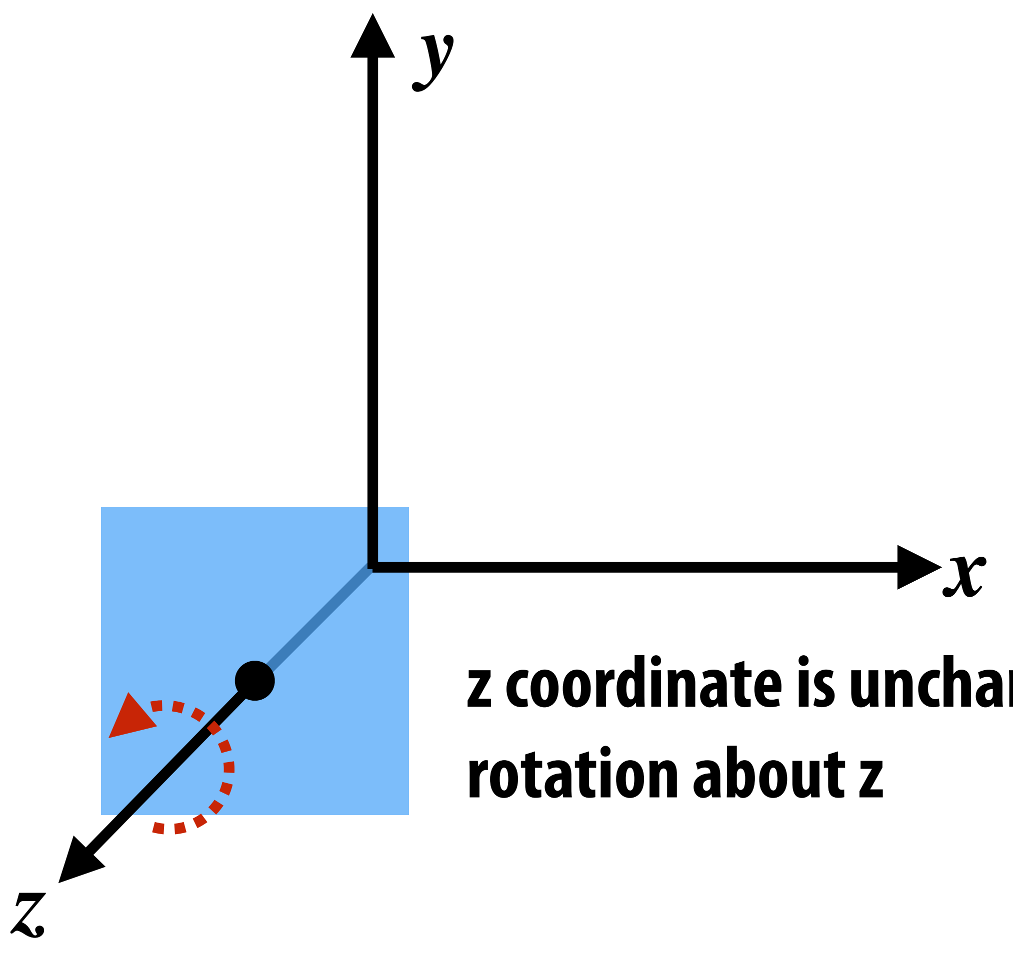
Rotation about y axis:

$$\mathbf{R}_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$



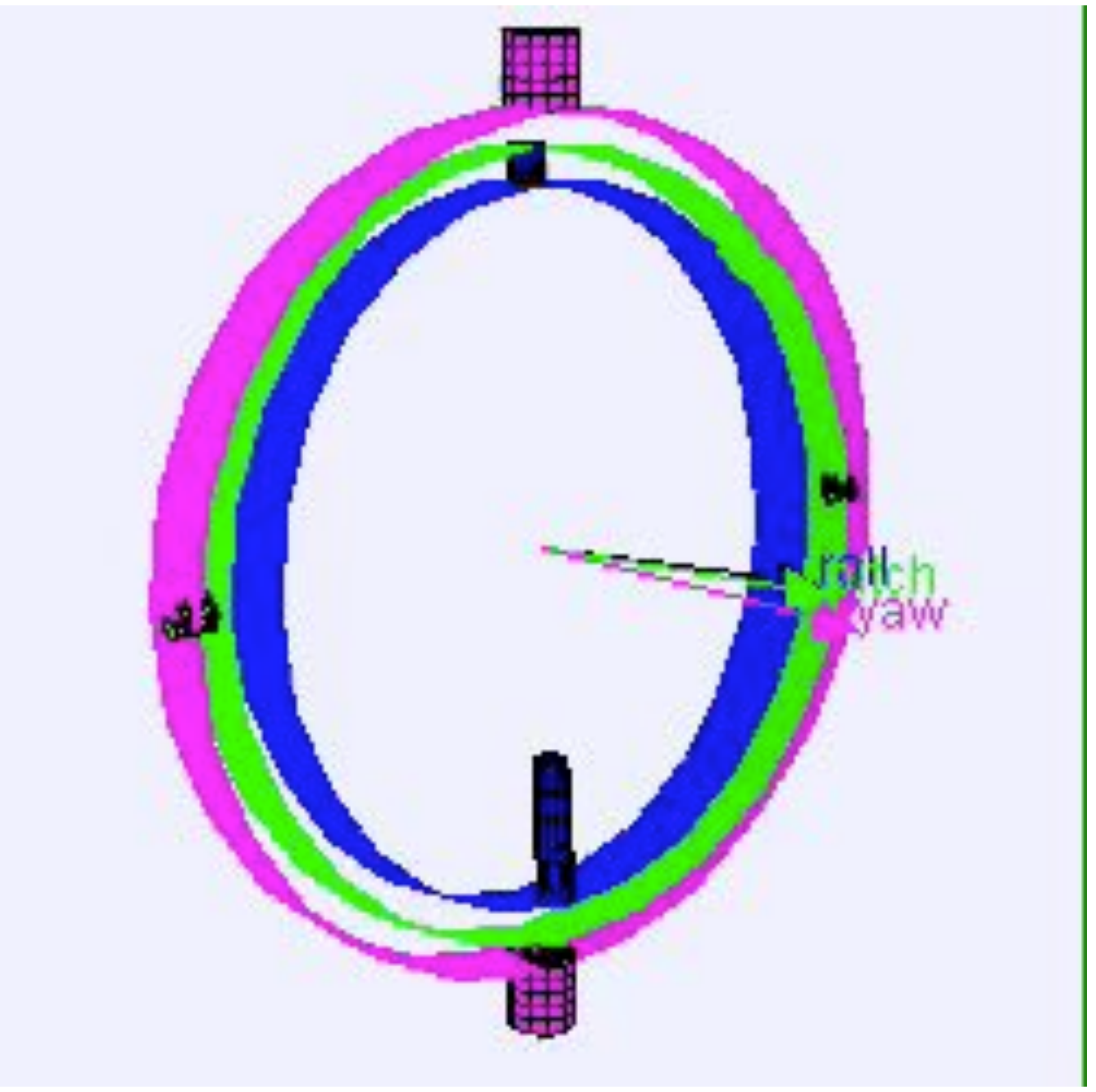
Rotation about z axis:

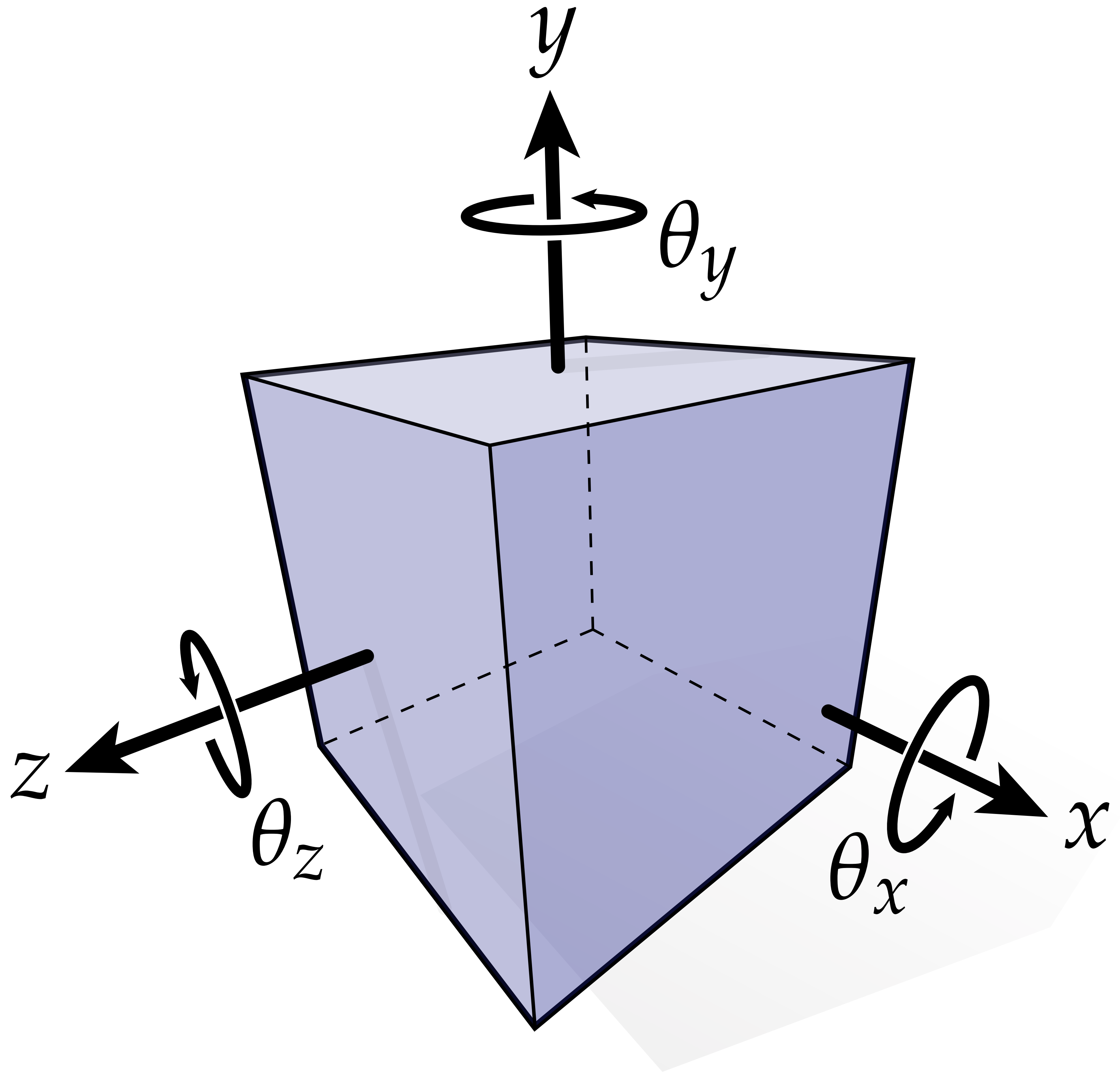
$$\mathbf{R}_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Representing rotations in 3D—euler angles

- How do we express rotations in 3D?
- One idea: we know how to do 2D rotations
- Why not simply apply rotations around the three axes? (X,Y,Z)
- Scheme is called *Euler angles*
- PROBLEM: “Gimbal Lock”



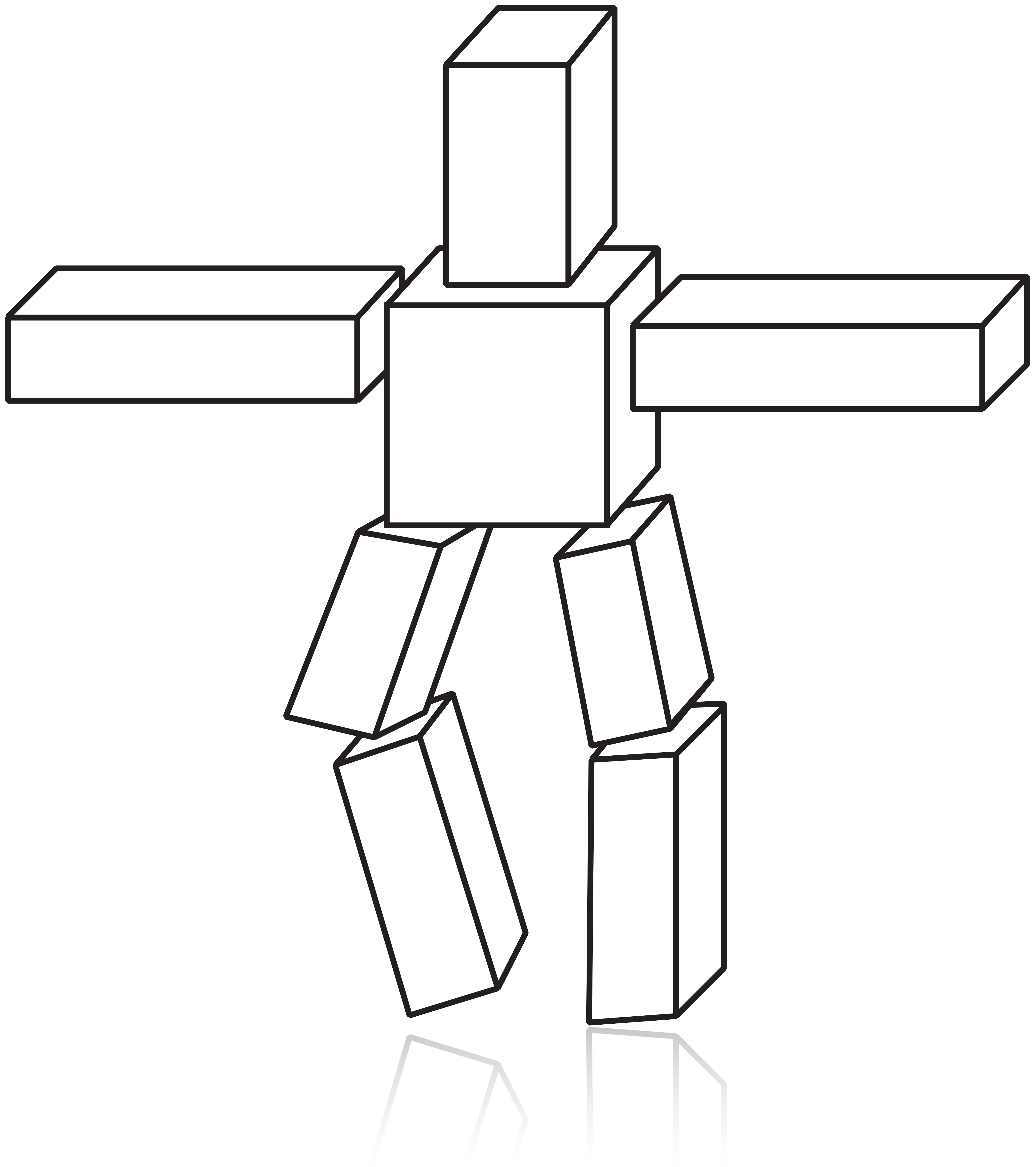


Alternative representations of 3D rotations

- Axis-angle rotations

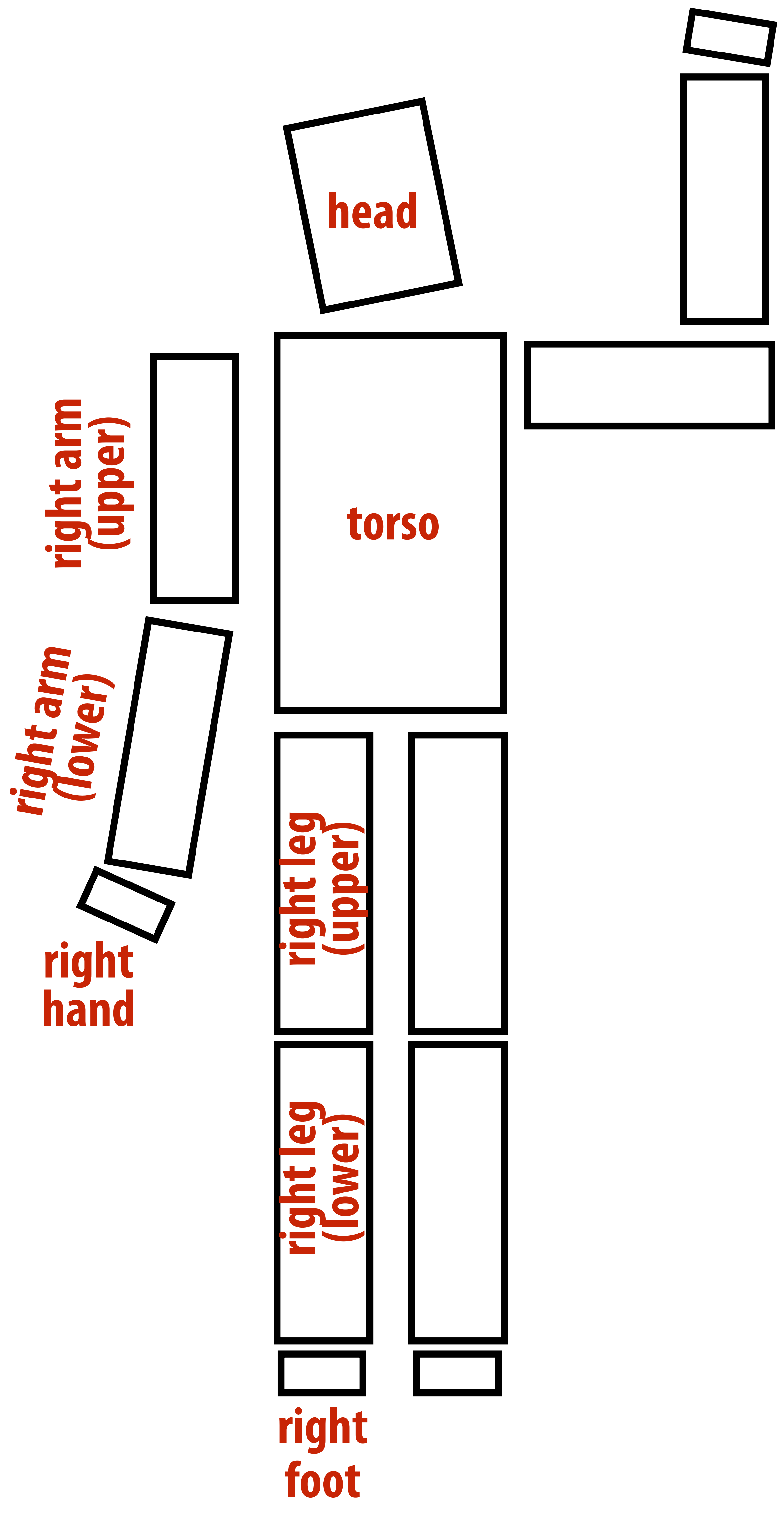
- Quaternions (not today)

Let's make that cube person…



Skeleton - hierarchical representation

- torso
  - head
  - right arm
    - upper arm
      - lower arm
        - hand
  - left arm
    - upper arm
      - lower arm
        - hand
  - right leg
    - upper leg
      - lower leg
        - foot
  - left leg
    - upper leg
      - lower leg
        - foot



Hierarchical representation

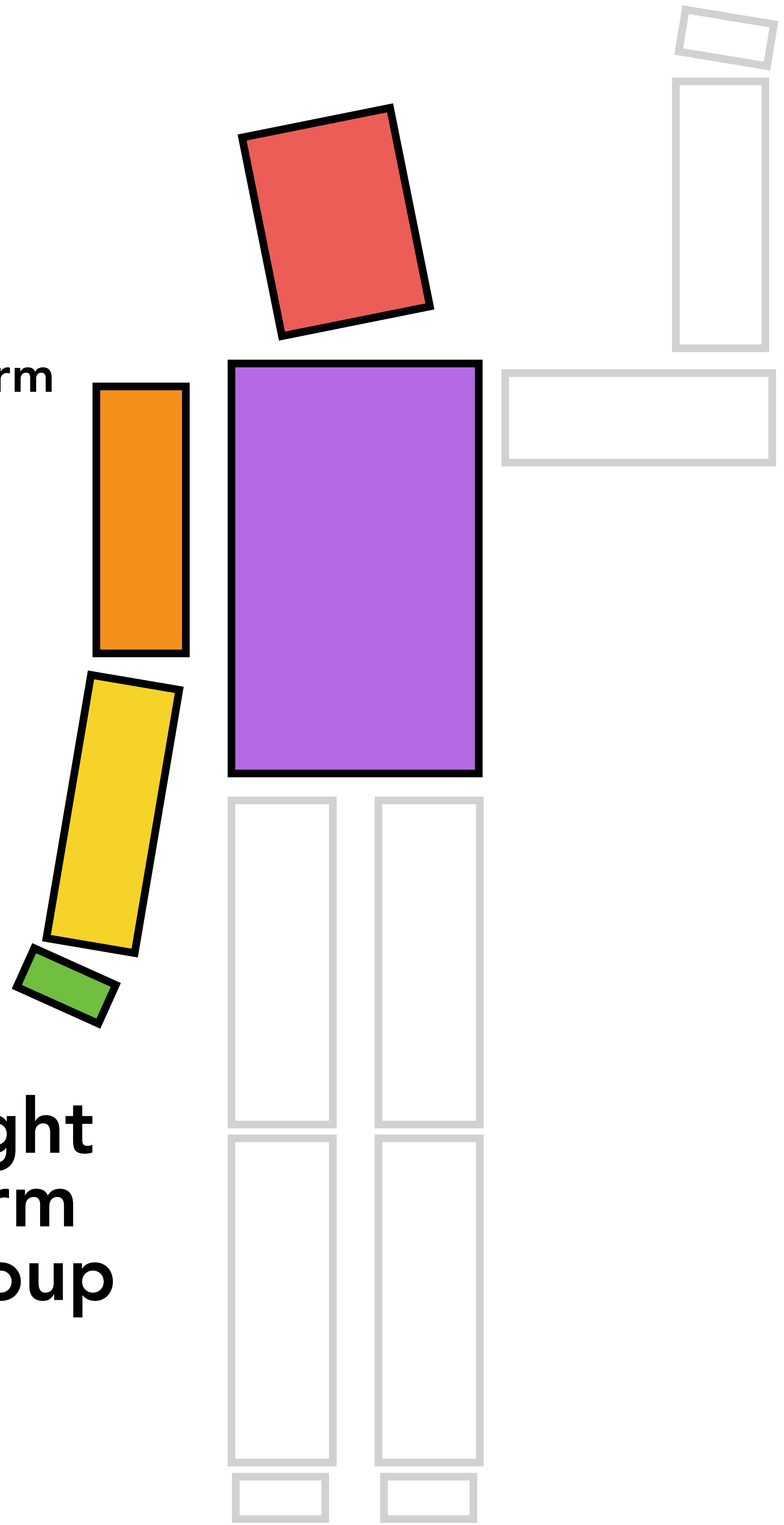
- Grouped representation (tree)
  - Each group contains subgroups and/or shapes
  - Each group is associated with a transform *relative to parent group*
  - Transform on leaf-node shape is concatenation of all transforms on path from root node to leaf
  - Changing a group's transform affects all descendent parts
    - Allows high level editing by changing only one node
    - E.g. raising left arm requires changing only one transform for that group

Skeleton - hierarchical representation

```
translate(0, 10);  // person centered at (0,10)

  drawTorso();

            pushmatrix();  // push a copy of transform onto stack
              translate(0, 5);  // right-multiply onto current transform
              rotate(headRotation);  // right-multiply onto current transform
              drawHead();
            popmatrix();  // pop current transform off stack
            pushmatrix();  ------------------------------------------ ┐
              translate(-2, 3);                                       │
              rotate(rightShoulderRotation);                          │
              drawUpperArm();                                         │
              pushmatrix();  --------------------------- ┐            │
                translate(0, -3);                        │            │
                rotate(elbowRotation);                   │            │
                drawLowerArm();                          │            │
                pushmatrix();  --------- ┐               │            │
                  translate(0, -3);      │               │            │
                  rotate(wristRotation); │  right hand   │  right     │  right arm
                  drawHand();            │               │  lower arm │  group
                popmatrix();  ---------- ┘               │  group     │
              popmatrix();  ---------------------------- ┘            │
            popmatrix();  ------------------------------------------- ┘
            ....
```



Skeleton - hierarchical representation

```
translate(0, 10);

    drawTorso();

pushmatrix();  // push a copy of transform onto stack
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    rotate(headRotation);  // right-multiply onto current transform
    drawHead();
popmatrix();  // pop current transform off stack
pushmatrix();
    translate(-2, 3);
    rotate(rightShoulderRotation);
    drawUpperArm();
    pushmatrix();
        translate(0, -3);
        rotate(elbowRotation);
        drawLowerArm();
        pushmatrix();
            translate(0, -3);
            rotate(wristRotation);
            drawHand();
        popmatrix();
    popmatrix();
popmatrix();
....
```
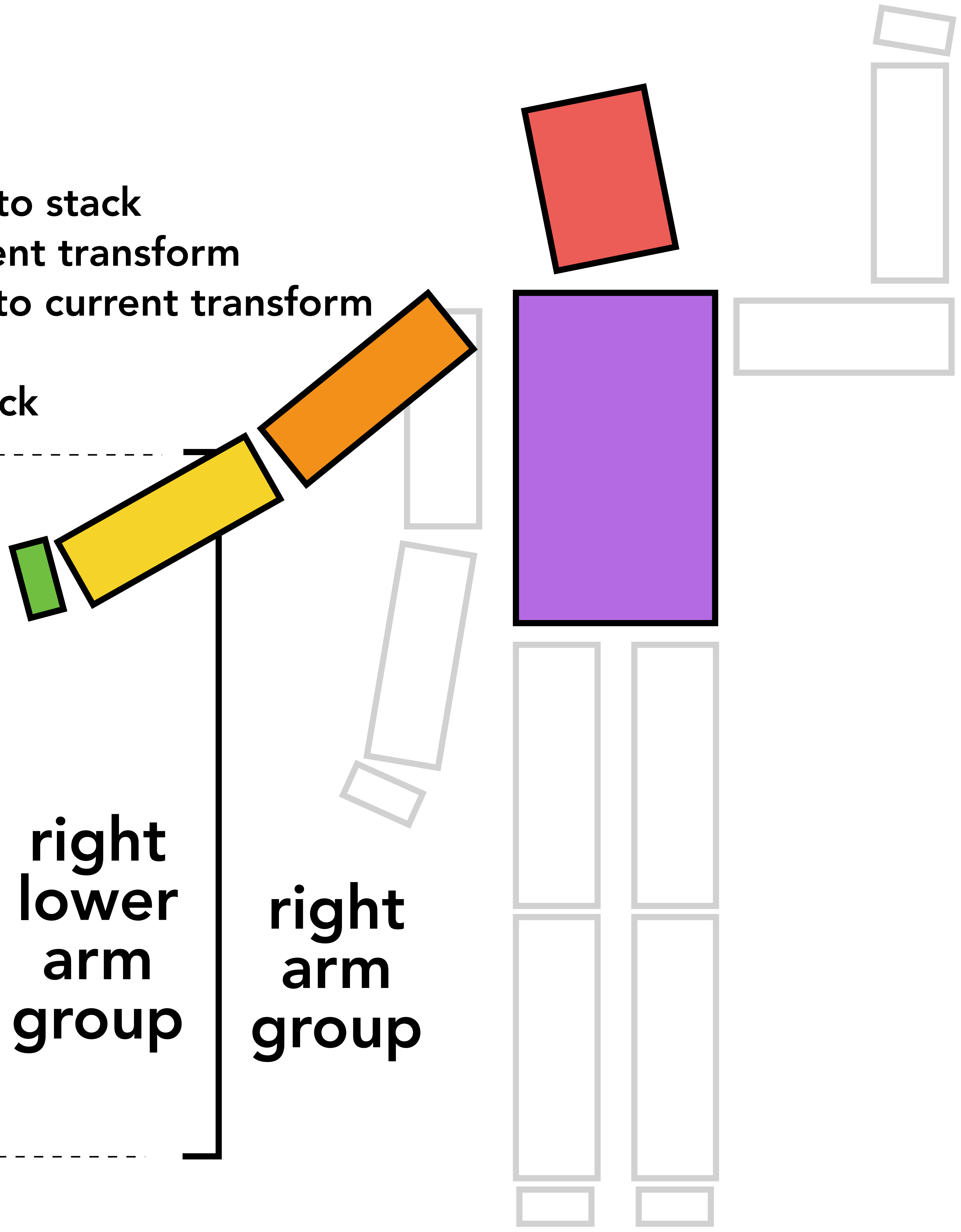

right hand

right lower arm group

right arm group

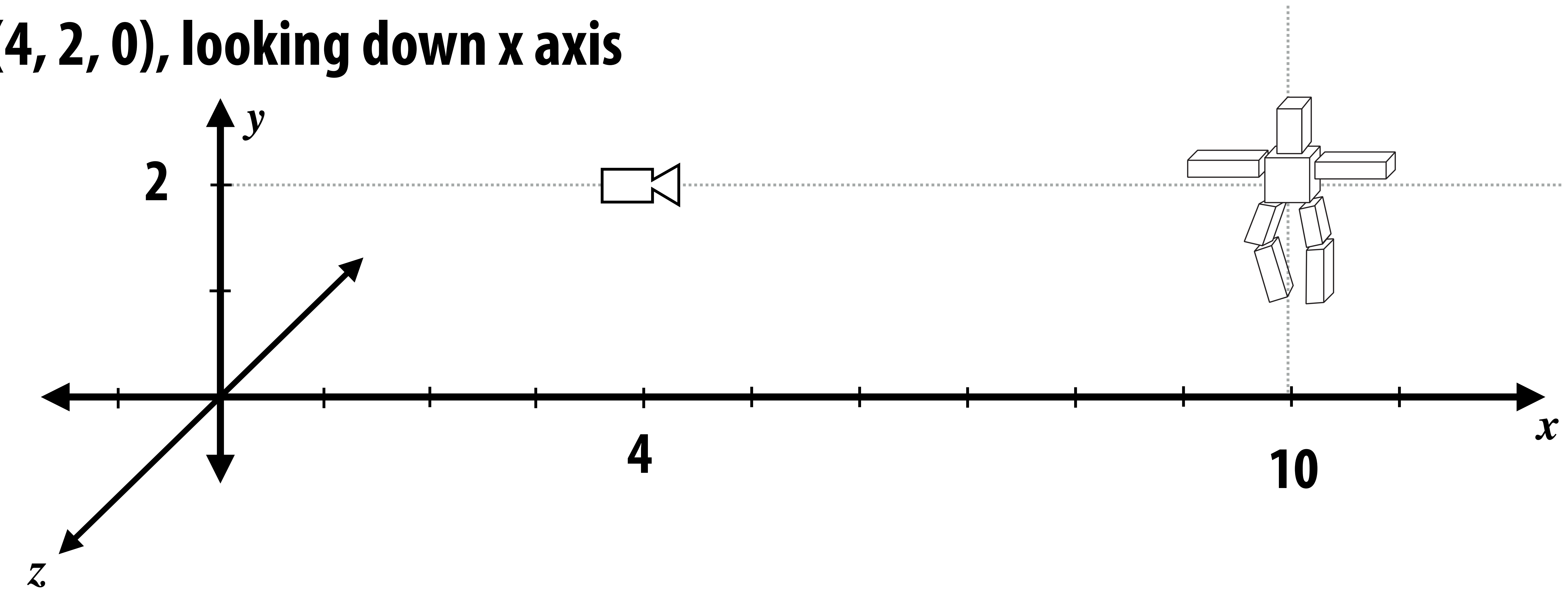


Transforming points into camera-relative coordinates

Example: simple camera transform

Consider object positioned in world at (10, 2, 0)

Consider camera at (4, 2, 0), looking down x axis



What transform places in the object in a coordinate space where the camera is at the origin and the camera is looking directly down the -z axis?

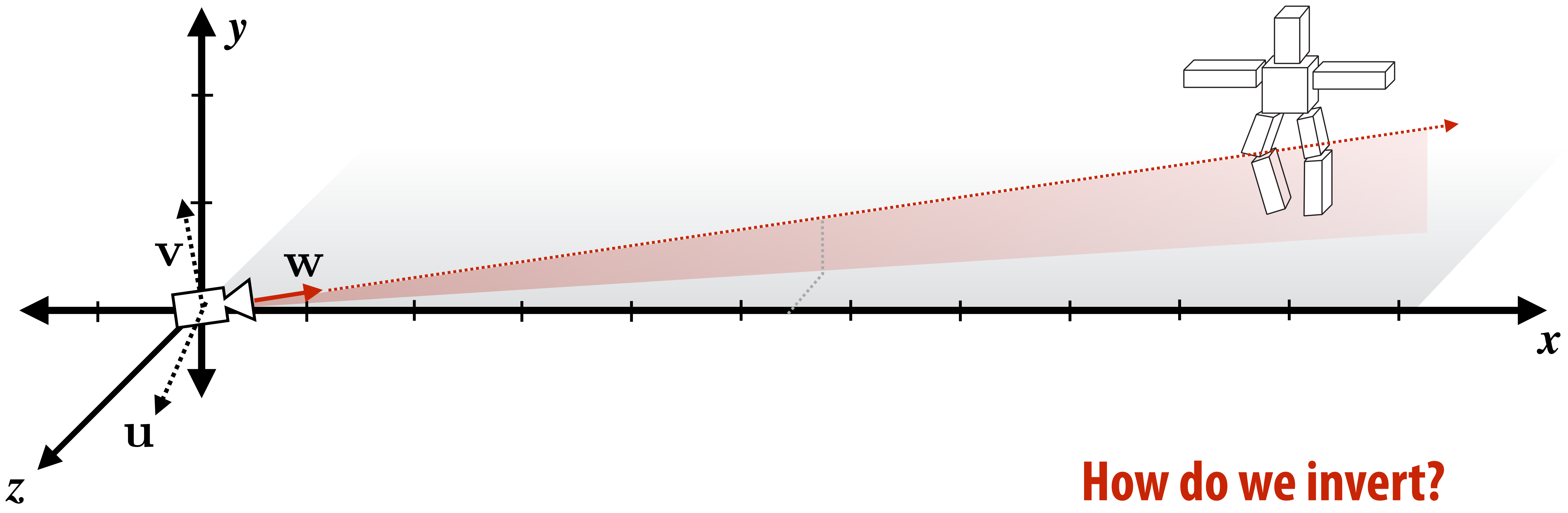
- Translating object vertex positions by (-4, -2, 0) yields position relative to camera
- Rotation about y by $\pi/2$ gives position of object in new coordinate system where camera's view direction is aligned with the -z axis *

* The convenience of such a coordinate system will become clear when we talk about projection!

Camera looking in a different direction

Consider camera at origin looking in direction $\mathbf{w}$

What transform places in the object in a coordinate space where the camera is at the origin and the camera is looking directly down the -z axis?



Form orthonormal basis around $\mathbf{w}$: (see $\mathbf{u}$ and $\mathbf{v}$)

Consider orthogonal matrix: $\mathbf{R}$

$$\mathbf{R} = \begin{bmatrix} \mathbf{u}_x & \mathbf{v}_x & -\mathbf{w}_x \\ \mathbf{u}_y & \mathbf{v}_y & -\mathbf{w}_y \\ \mathbf{u}_z & \mathbf{v}_z & -\mathbf{w}_z \end{bmatrix}$$

$\mathbf{R}$ maps x-axis to $\mathbf{u}$, y-axis to $\mathbf{v}$, z axis to $-\mathbf{w}$

How do we invert?

$$\mathbf{R}^{-1} = \mathbf{R}^T = \begin{bmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \\ \mathbf{v}_x & \mathbf{v}_y & \mathbf{v}_z \\ -\mathbf{w}_x & -\mathbf{w}_y & -\mathbf{w}_z \end{bmatrix}$$

Why is that the inverse?

$$\mathbf{R}^T\mathbf{u} = \begin{bmatrix} \mathbf{u}\cdot\mathbf{u} & \mathbf{v}\cdot\mathbf{u} & -\mathbf{w}\cdot\mathbf{u} \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$$

$$\mathbf{R}^T\mathbf{v} = \begin{bmatrix} \mathbf{u}\cdot\mathbf{v} & \mathbf{v}\cdot\mathbf{v} & -\mathbf{w}\cdot\mathbf{v} \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$$

$$\mathbf{R}^T\mathbf{w} = \begin{bmatrix} \mathbf{u}\cdot\mathbf{w} & \mathbf{v}\cdot\mathbf{w} & -\mathbf{w}\cdot\mathbf{w} \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}^T$$

Self-check exercise (for home)

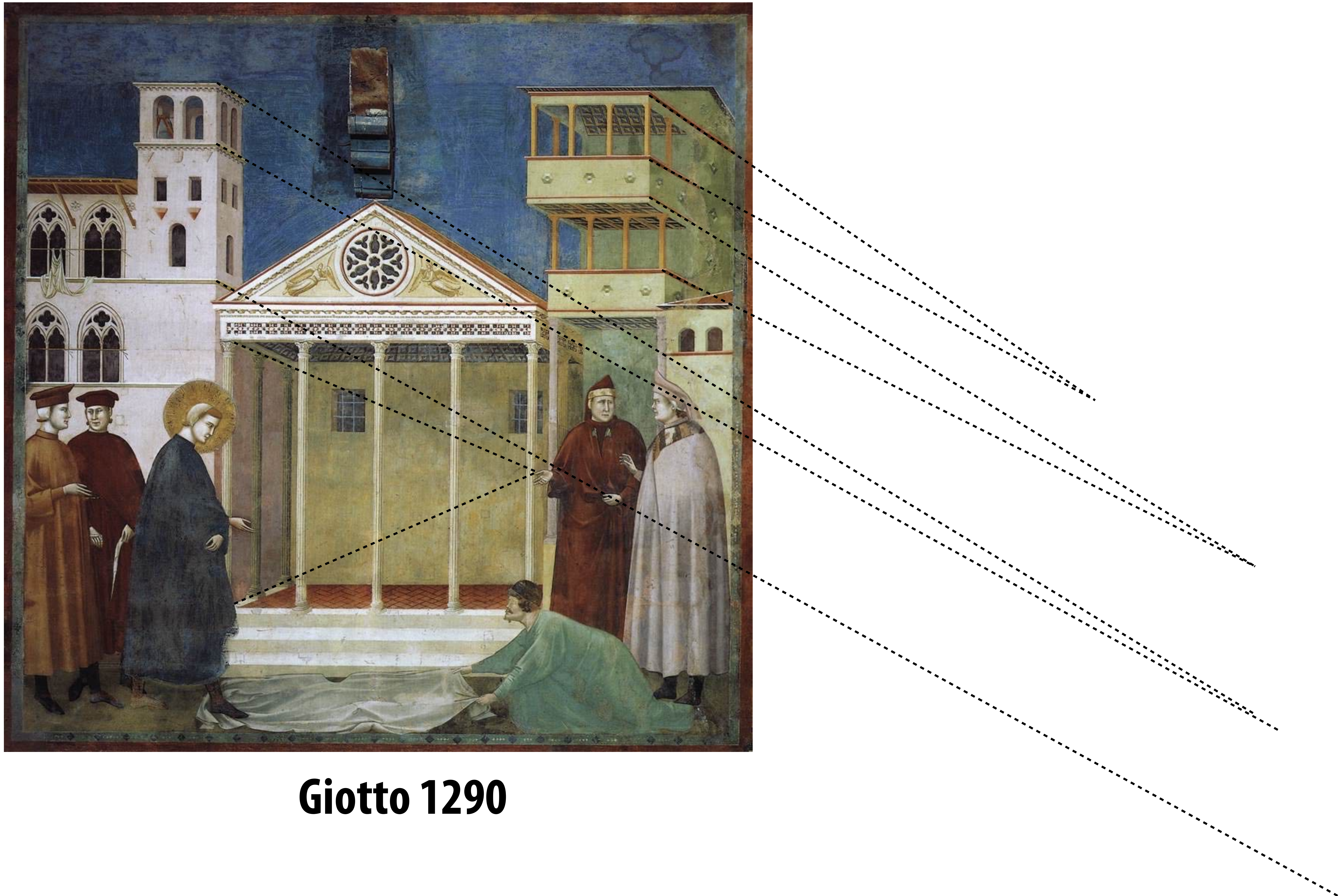
- Given a camera position *P*
- And a camera orientation given by orthonormal basis *u*,*v*,*w* (camera looking in *w*)
- What is a transformation matrix that places the scene in a coordinate space where…
  - The camera is at the origin
  - The camera is looking down -*z*.

Perspective projection

Early painting: incorrect perspective

Carolingian painting from the 8-9th century

Perspective in art



Giotto 1290

Evolution toward correct perspective

Ambrogio Lorenzetti
Annunciation, 1344

First known perspective painting by Fillipo Brunelleshi

Brunelleschi, elevation of Santo Spirito, 1434-83, Florence

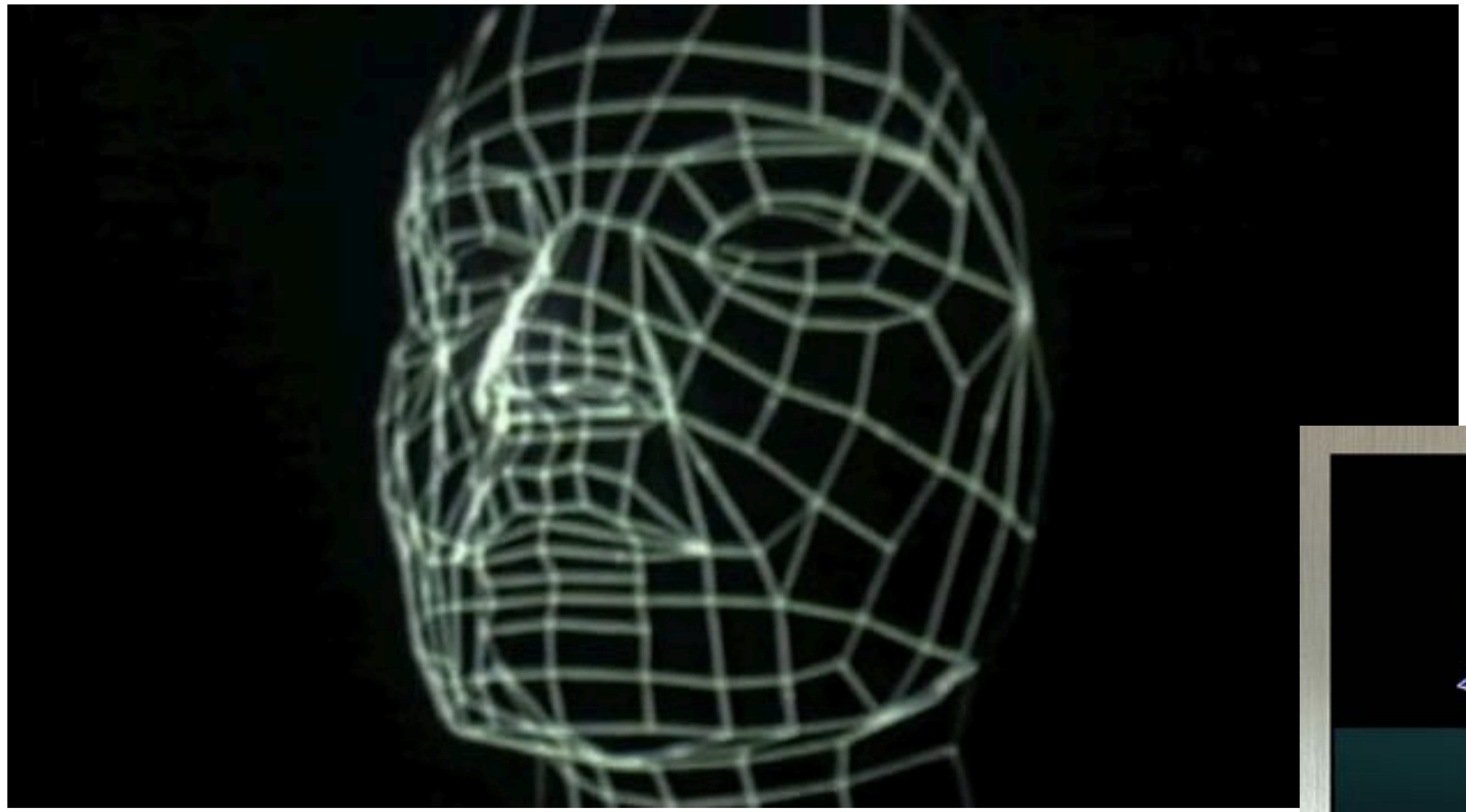
Masaccio – The Tribute Money c.1426-27
Fresco, The Brancacci Chapel, Florence

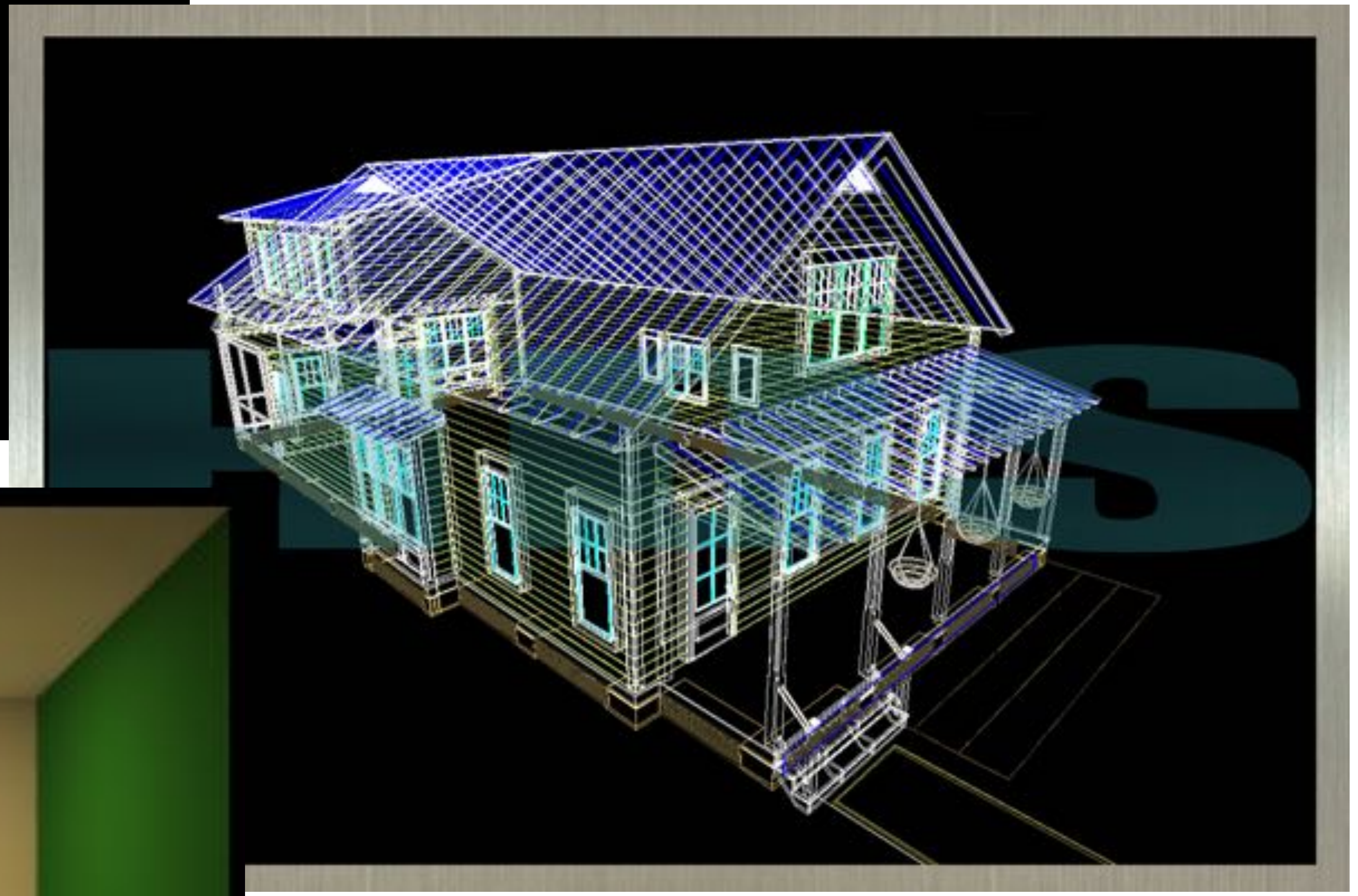
Perspective in art

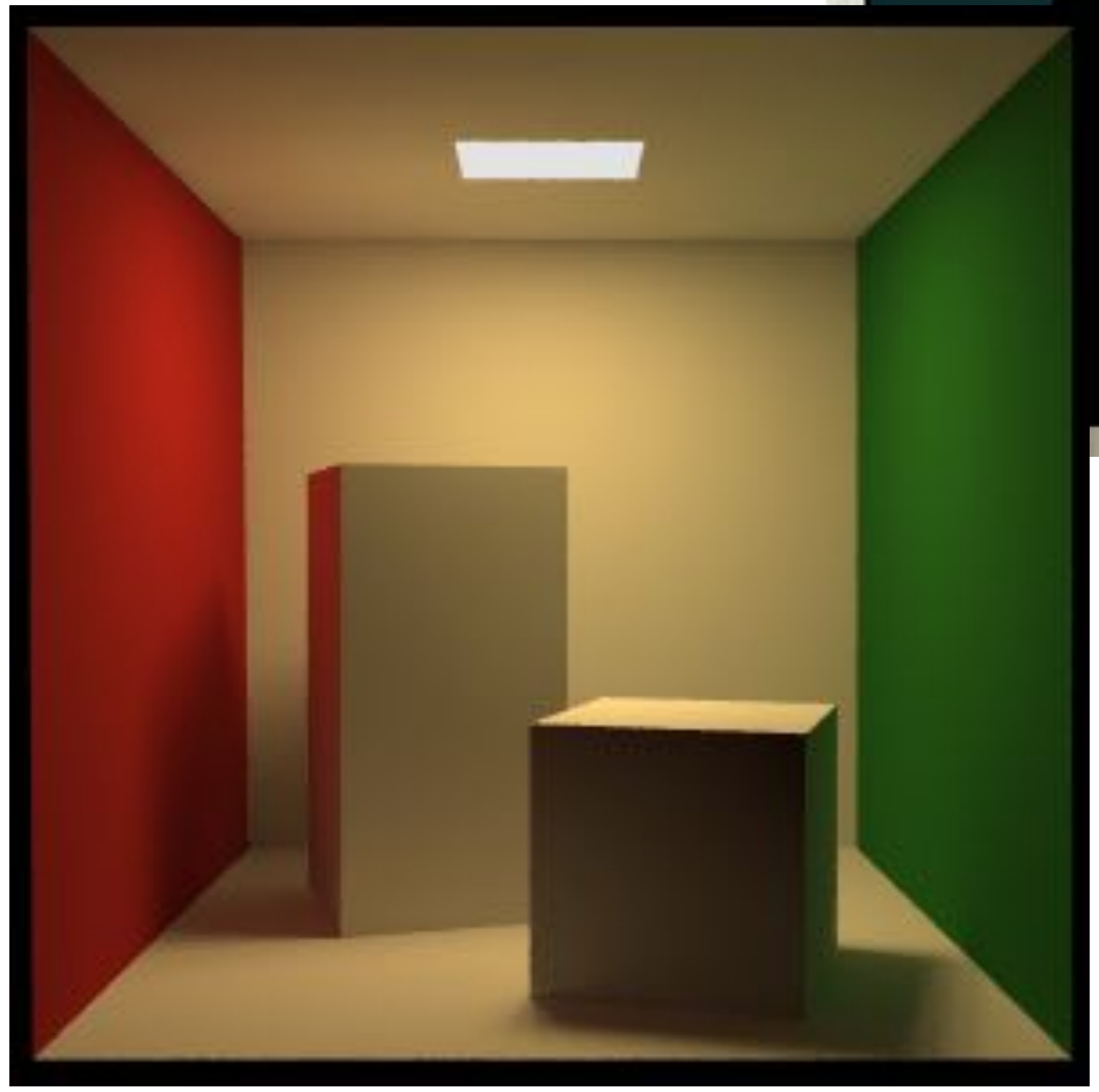
Delivery of the Keys (Sistine Chapel), Perugino, 1482

Later… rejection of proper perspective projection

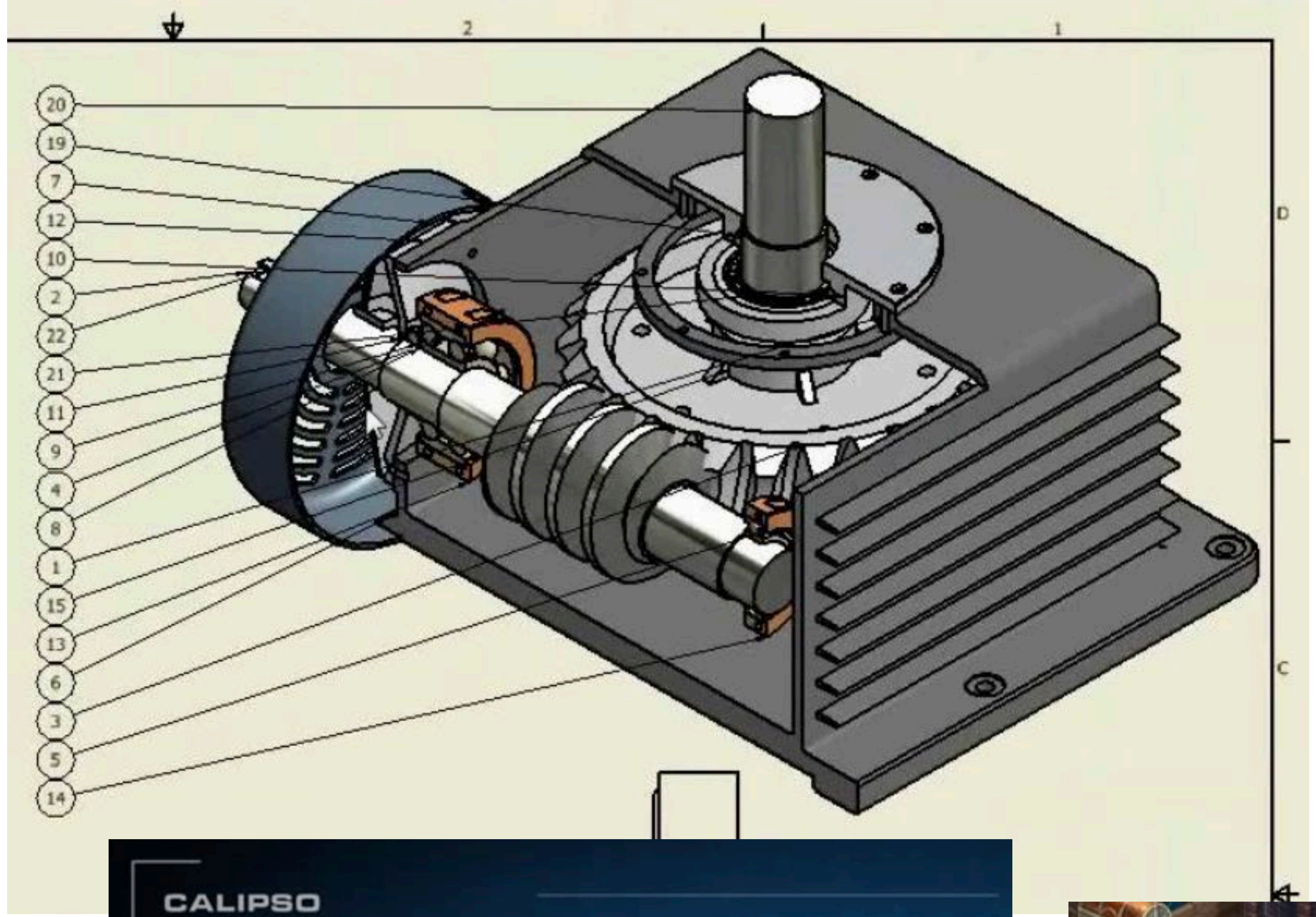
Correct perspective in computer graphics

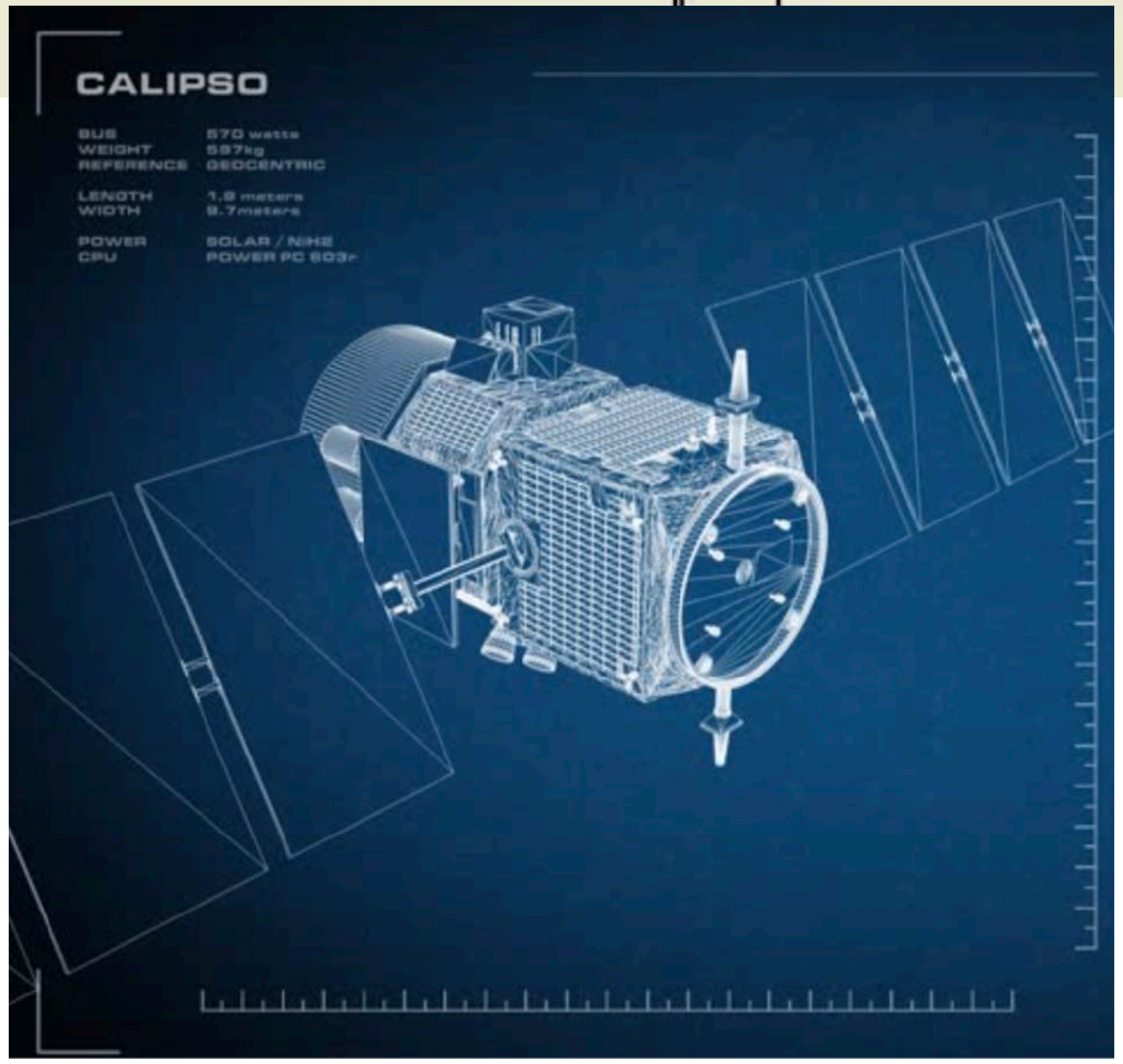




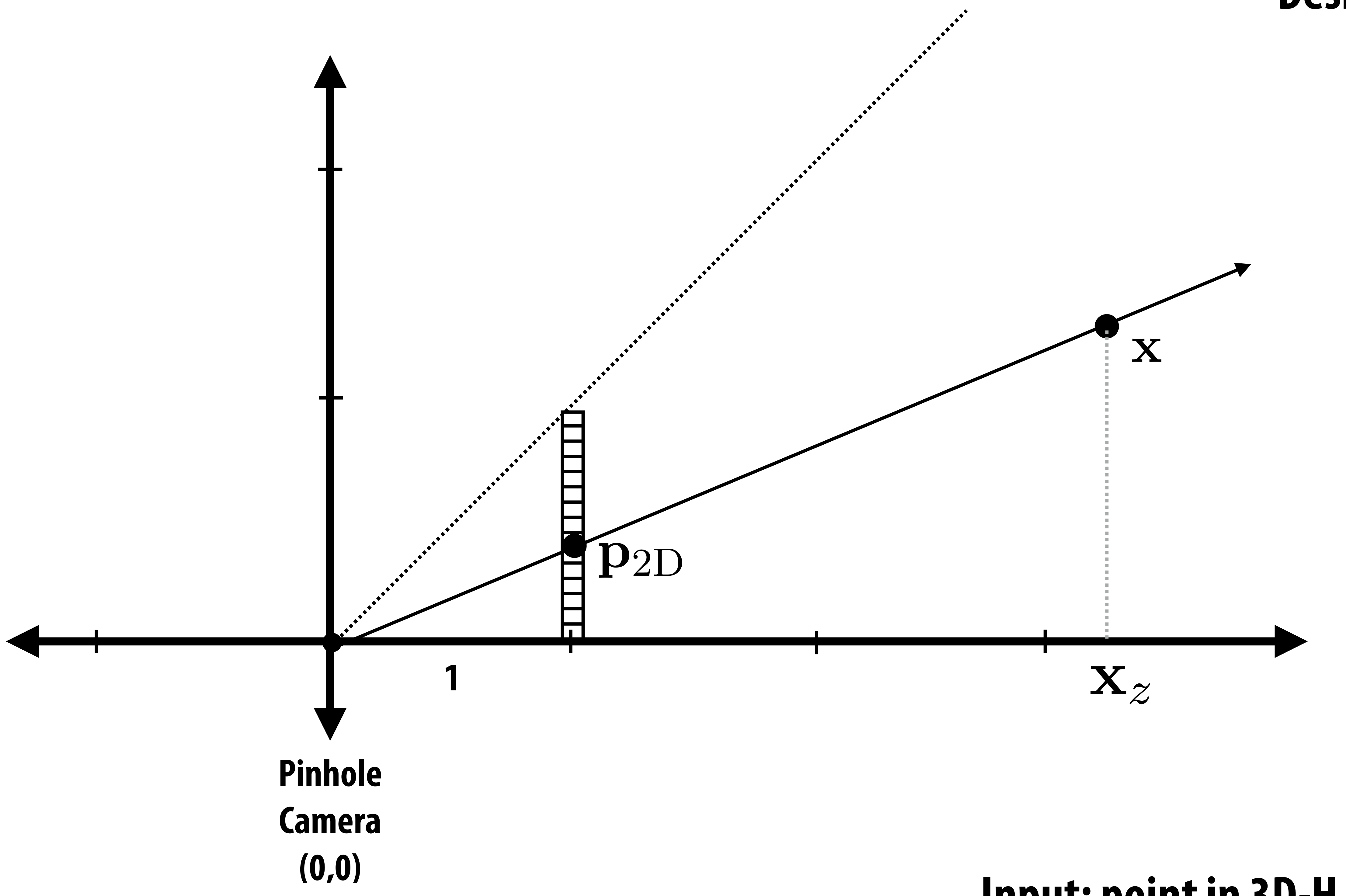


Rejection of perspective in computer graphics





Basic perspective projection



Desired perspective projected result (2D point):

$$\mathbf{p}_{2\text{D}} = \begin{bmatrix}\mathbf{x}_x/\mathbf{x}_z & \mathbf{x}_y/\mathbf{x}_z\end{bmatrix}^T$$

$$\mathbf{P} = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 1 & 0\end{bmatrix}$$

Input: point in 3D-H $\mathbf{x} = \begin{bmatrix}\mathbf{x}_x & \mathbf{x}_y & \mathbf{x}_z & 1\end{bmatrix}$

After applying $\mathbf{P}$: point in 3D-H $\mathbf{P}\mathbf{x} = \begin{bmatrix}\mathbf{x}_x & \mathbf{x}_y & \mathbf{x}_z & \mathbf{x}_z\end{bmatrix}^T$

After homogeneous divide: $\begin{bmatrix}\mathbf{x}_x/\mathbf{x}_z & \mathbf{x}_y/\mathbf{x}_z & 1\end{bmatrix}^T$

(throw out third component to get 2D)

Assumption:
Pinhole camera at (0,0) looking down z

Perspective vs. orthographic projection

- **Most basic version of perspective projection matrix:**

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ z \end{bmatrix} \longmapsto \begin{bmatrix} x/z \\ y/z \\ 1 \\ 1 \end{bmatrix}$$

objects shrink in distance

- **Most basic version of orthographic projection matrix:**

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \longmapsto \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

objects stay the same size

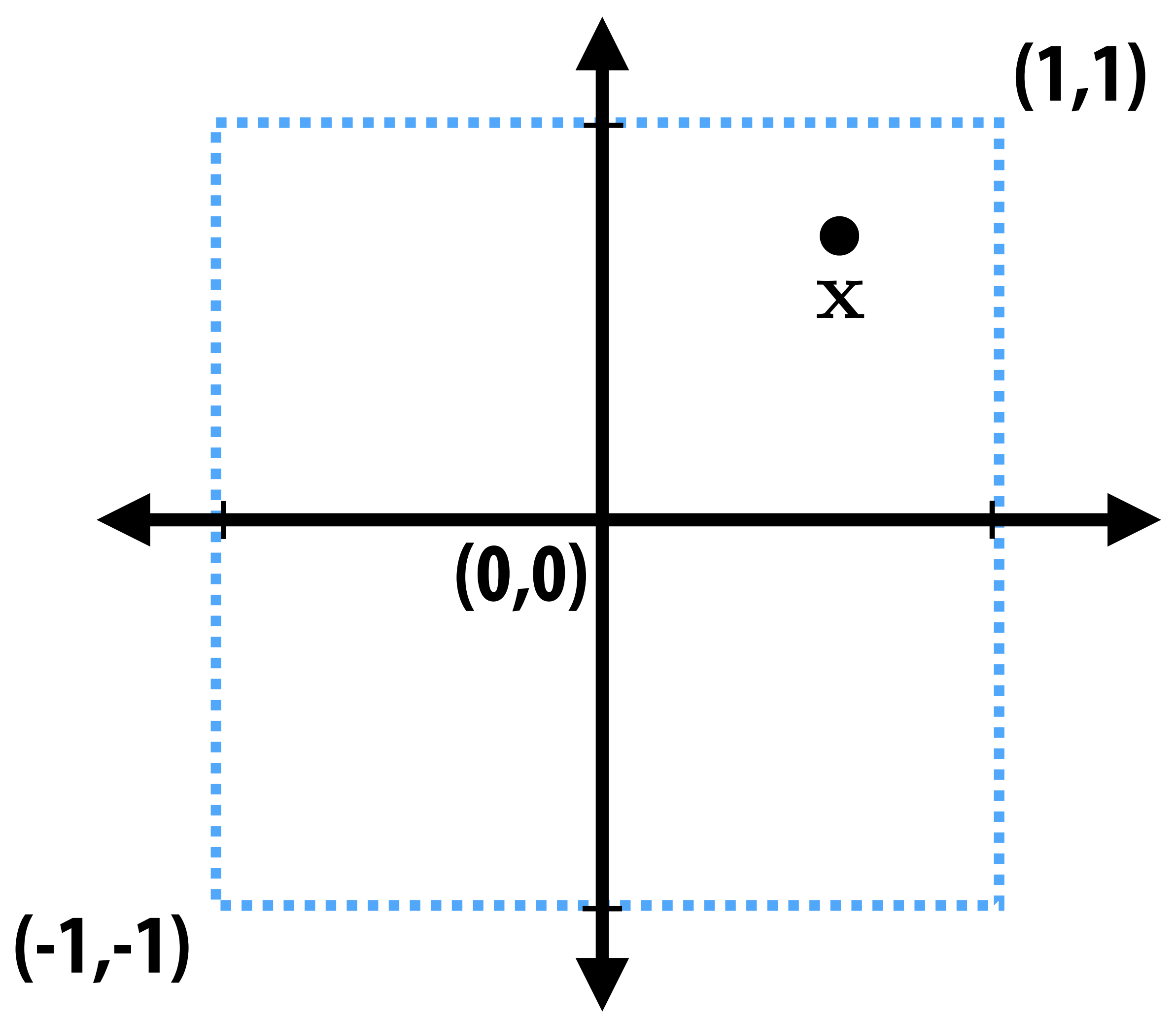
A good exercise:
Transforming points into screen-relative coordinates

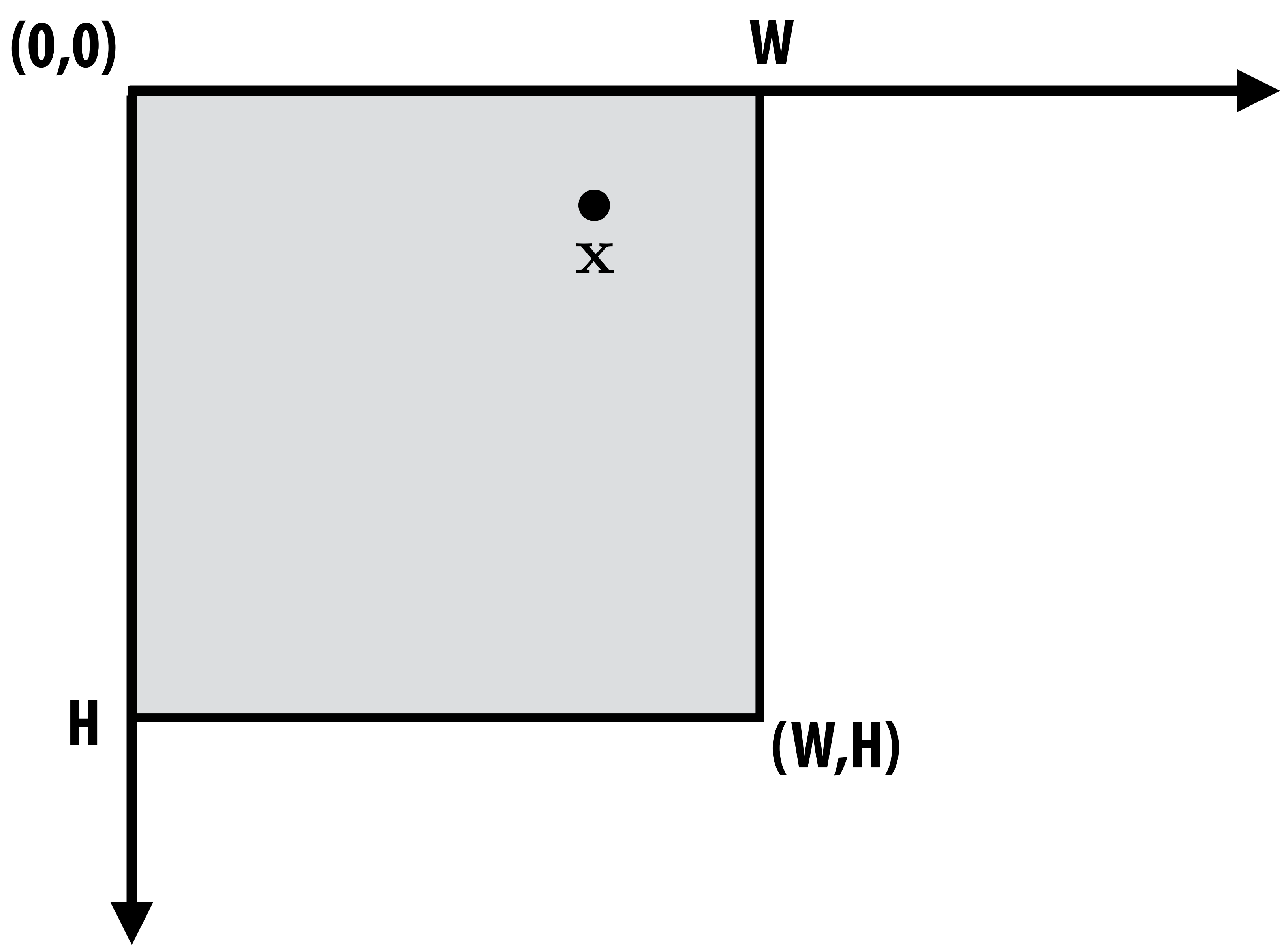
Screen transformation *

Convert points in normalized coordinate space to screen pixel coordinates

Example: all points within (-1,1) to (1,1) region are on screen

- **(1,1) in normalized space maps to (W,0) in screen space**
- **(-1,-1) in normalized space maps to (0,H) in screen space**



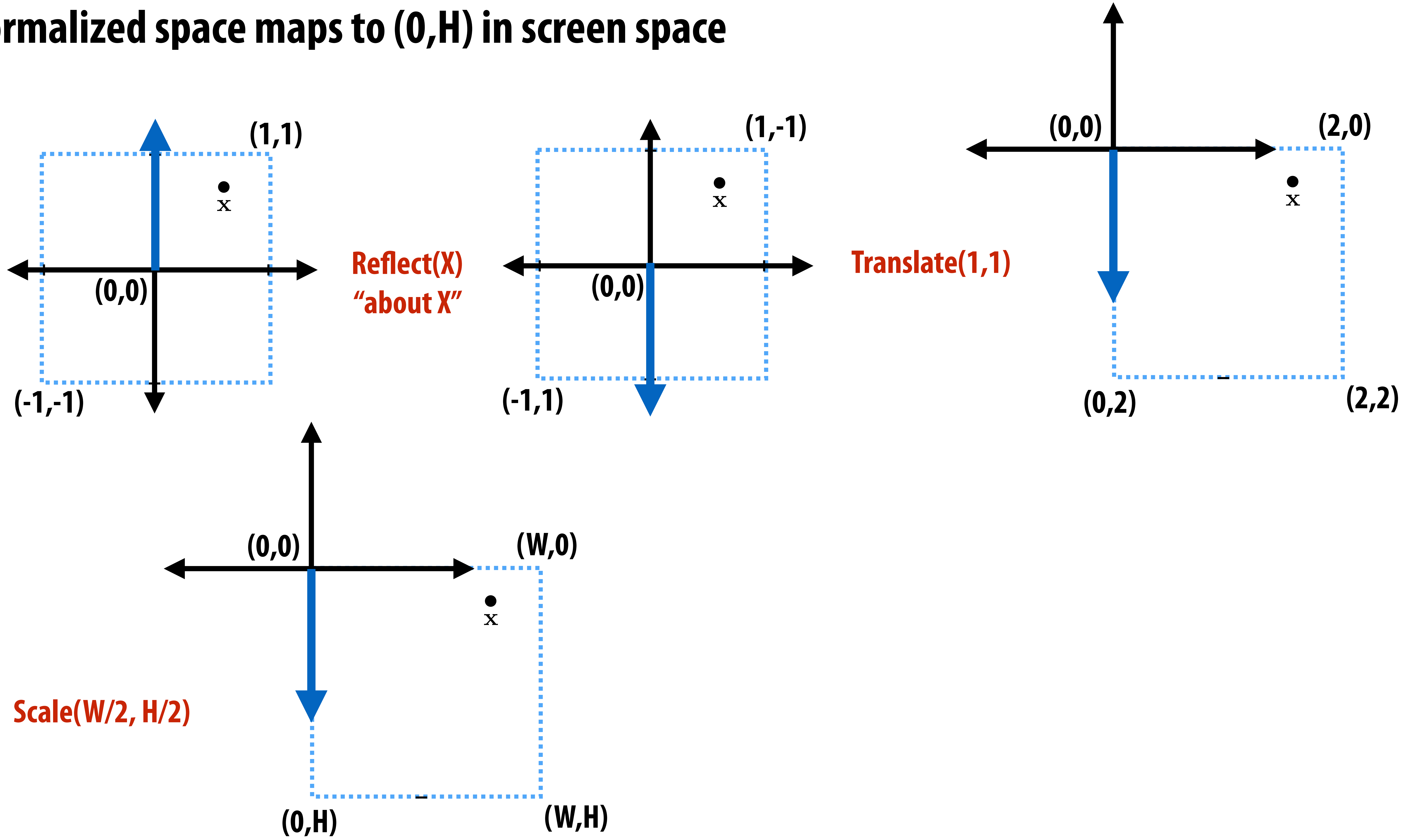


* This slide adopts convention that top-left of screen is (0,0) to match SVG convention in Assignment 1.
Many 3D graphics systems like OpenGL place (0,0) in bottom-left. In this case what would the transform be?

Screen transformation

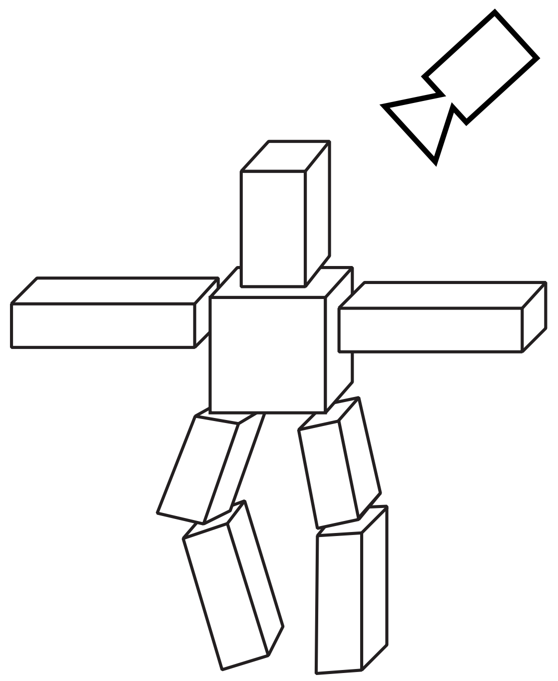
Example: all points within (-1,1) to (1,1) region are on screen

- **(1,1) in normalized space maps to (W,0) in screen space**
- **(-1,-1) in normalized space maps to (0,H) in screen space**



Transformations: from objects in 3D to their 2D screen positions

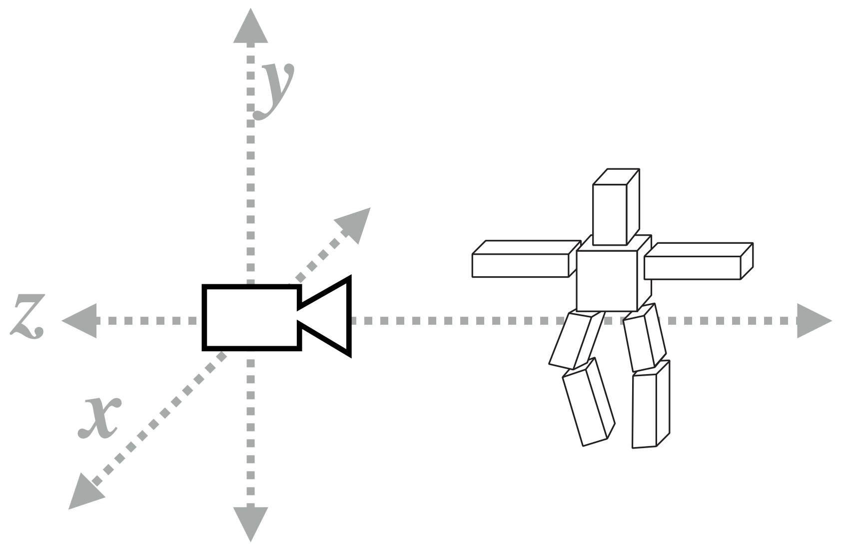
[WORLD COORDINATES]



original description
of objects

view transform

[VIEW COORDINATES]

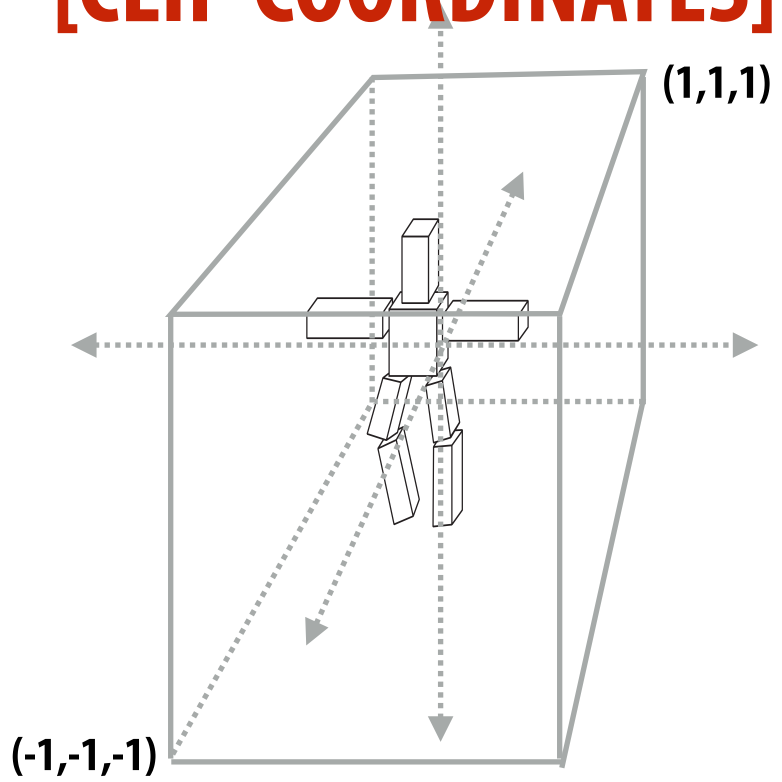


vertex positions now expressed relative to camera; camera is sitting at origin looking down -z direction
(Canonical frame of reference allows for use of canonical projection matrix)

projection transform

[CLIP COORDINATES]

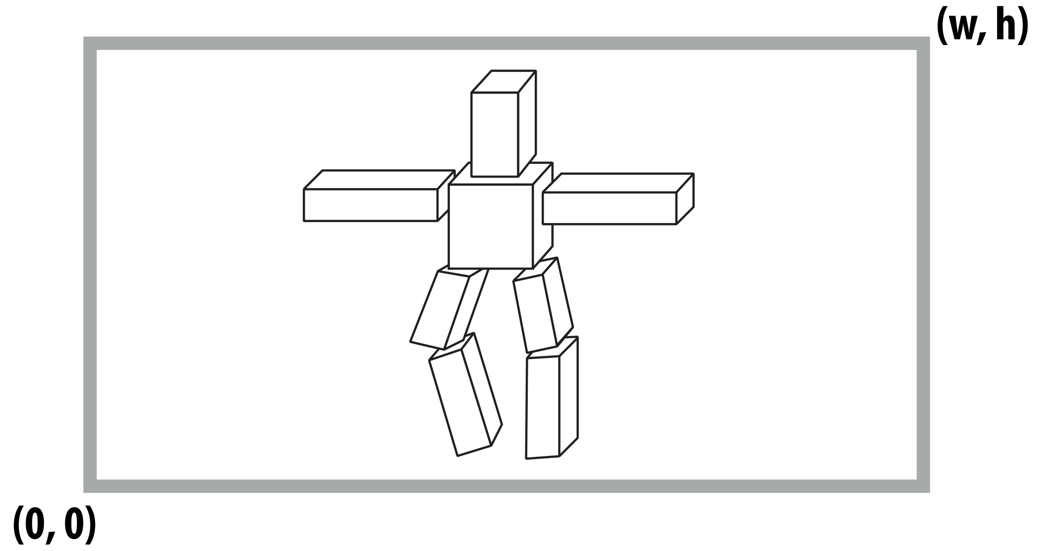
(Also called "normalized device coordinates")



everything visible to the camera is mapped to unit cube for easy triangle "clipping"

screen transform

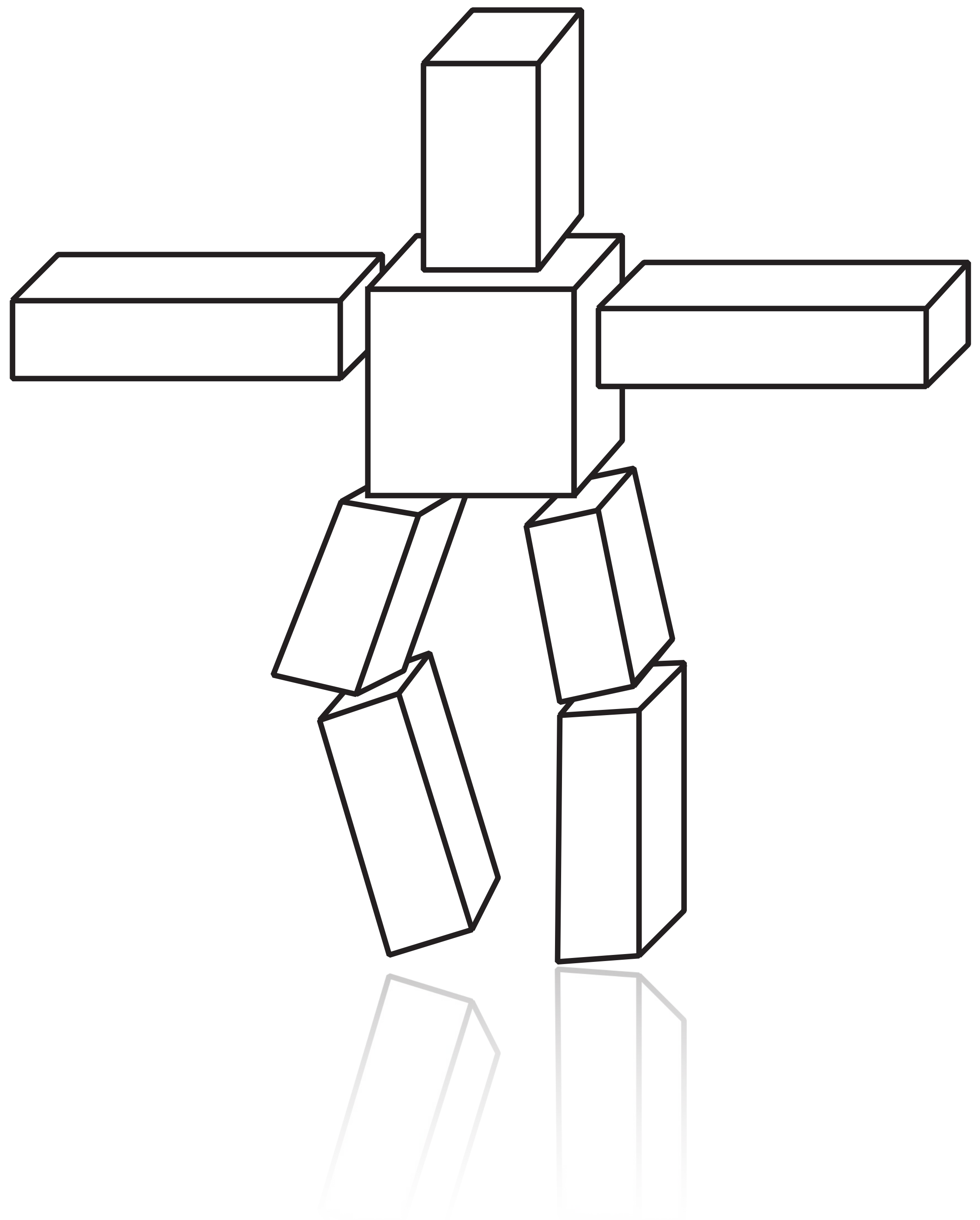
[WINDOW COORDINATES]



objects now in
2D screen coordinates

primitives are now 2D and can be drawn via rasterization

Transformations summary



- Transformations can be interpreted as operations that move points in space
  - e.g., for modeling, animation
- Or as a change of coordinate system
  - e.g., screen and view transforms
- Construct complex transformations as compositions of basic transforms
- Homogeneous coordinate representation allows for expression of non-linear transforms (e.g., translation, perspective projection) as matrix operations (linear transforms) in higher-dimensional space
  - Matrix representation affords simple implementation and efficient composition