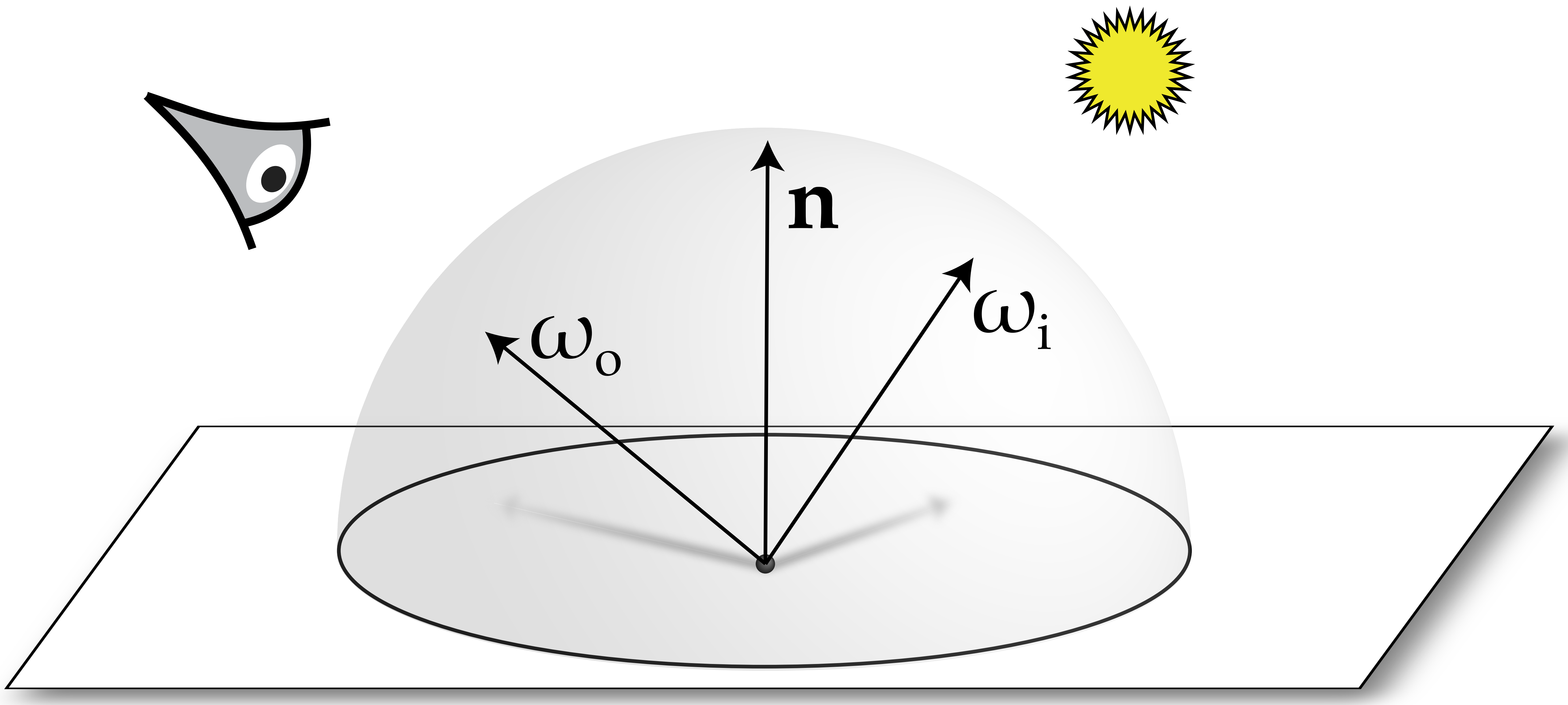
Direct Lighting I

Today

- MC estimation of the reflection equation
- Sampling shapes
- Importance sampling, variance and efficiency
- Stratified sampling

The Reflection Equation

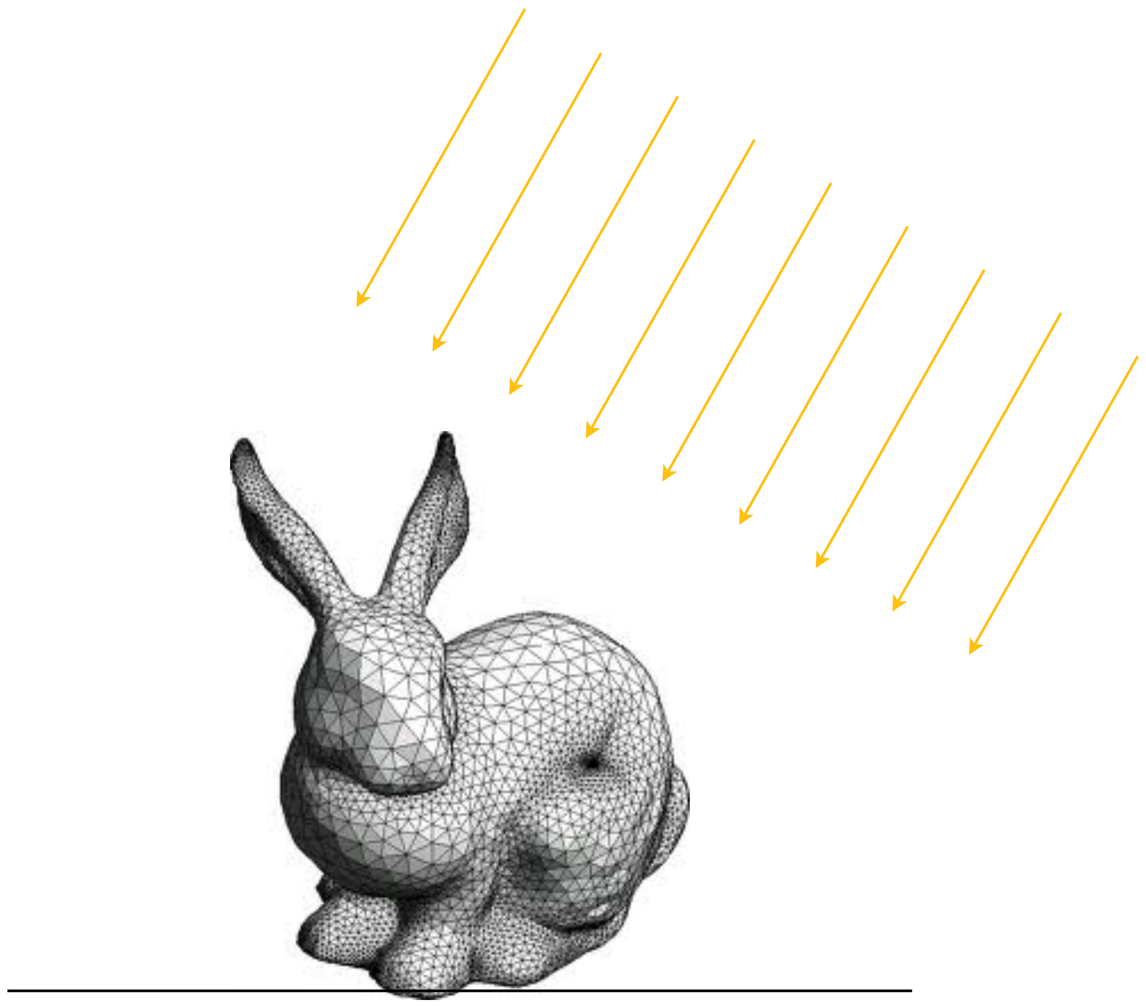


$$L_o(p, \omega_o) = \int_{\Omega^2} \underbrace{f_r(p, \omega_i \to \omega_o)}_{\textbf{BRDF}} \underbrace{L_i(p, \omega_i) \cos\theta_i}_{\textbf{Illumination}} \, d\omega_i$$

Directional Light Sources

Incident radiance in parallel rays

$$L_i(\omega) = L\,\delta(\omega - \omega_{\text{light}})$$



$$L_{\text{o}}(\text{p}, \omega_{\text{o}}) = \int_{H^2} f_r(\text{p}, \omega_{\text{i}} \to \omega_{\text{o}})\, L_{\text{i}}(\text{p}, \omega_{\text{i}})\, \cos\theta_{\text{i}}\, \text{d}\omega_{\text{i}}$$

$$= \int_{H^2} f_r(\text{p}, \omega_{\text{i}} \to \omega_{\text{o}})\, L\,\delta(\omega - \omega_{\text{light}})\, \cos\theta_{\text{i}}\, \text{d}\omega_{\text{i}}$$

$$= f_r(\text{p}, \omega_{\text{light}} \to \omega_{\text{o}})\, L\, V(\text{p}, \omega_{\text{light}})\, \cos\theta_{\text{light}}$$

Binary visibility function

Specular BRDFs

$$L_{\mathrm{o}}(\mathrm{p}, \omega_{\mathrm{o}}) = \int_{H^2} f_r(\mathrm{p}, \omega_{\mathrm{i}} \rightarrow \omega_{\mathrm{o}}) \, L_{\mathrm{i}}(\mathrm{p}, \omega_{\mathrm{i}}) \, \cos\theta_{\mathrm{i}} \, \mathrm{d}\omega_{\mathrm{i}}$$

Perfect Specular BRDF:

$$f_r(\omega_{\mathrm{i}} \rightarrow \omega_{\mathrm{o}}) = \frac{\delta(\omega_{\mathrm{i}} - R(\omega_{\mathrm{o}}, \vec{\mathrm{n}}))}{\cos\theta_{\mathrm{i}}}$$

$R(\omega_r, \vec{\mathrm{n}})$ **is specular direction direction**

↓

$$L_{\mathrm{o}}(\mathrm{p}, \omega_{\mathrm{o}}) = L_{\mathrm{i}}(\mathrm{p}, R(\omega_{\mathrm{o}}, \vec{\mathrm{n}}))$$

Review: The Monte Carlo Estimator

$$\int f(x)\,\mathrm{d}x \approx \frac{1}{n}\sum_{i=1}^{n}\frac{f(x_i)}{p(x_i)} \qquad\qquad x_i \sim p$$

Non-Uniform Monte Carlo Sampling

$$\int f(x)\,\mathrm{d}x \approx \frac{1}{n}\sum_{i=1}^{n}\frac{f(x_i)}{p(x_i)} \qquad\qquad x_i \sim p$$

Key idea: make up for non-uniform sampling density by increasing the weight of low probability samples.

Monte Carlo Estimate

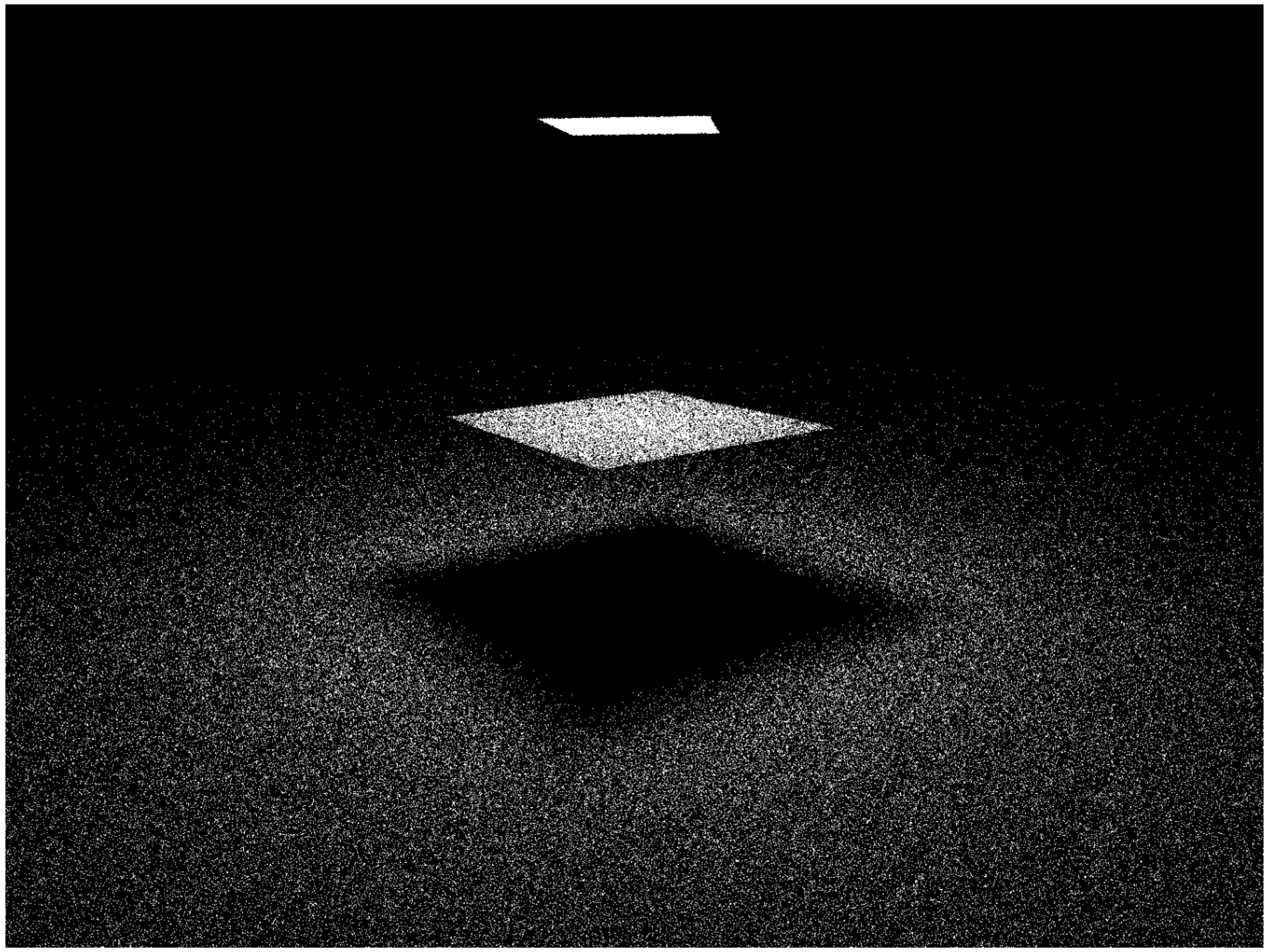
$$L_{\mathrm{o}}(\mathrm{p}, \omega_{\mathrm{o}}) = \int_{H^2} f_r(\mathrm{p}, \omega_{\mathrm{i}} \to \omega_{\mathrm{o}})\, L_{\mathrm{i}}(\mathrm{p}, \omega_{\mathrm{i}})\, \cos\theta_{\mathrm{i}}\, \mathrm{d}\omega_{\mathrm{i}}$$

Basic Monte Carlo estimate:

- **Generate directions $\omega_j \sim p(\omega)$ sampled from some distribution**
- **Compute estimator**

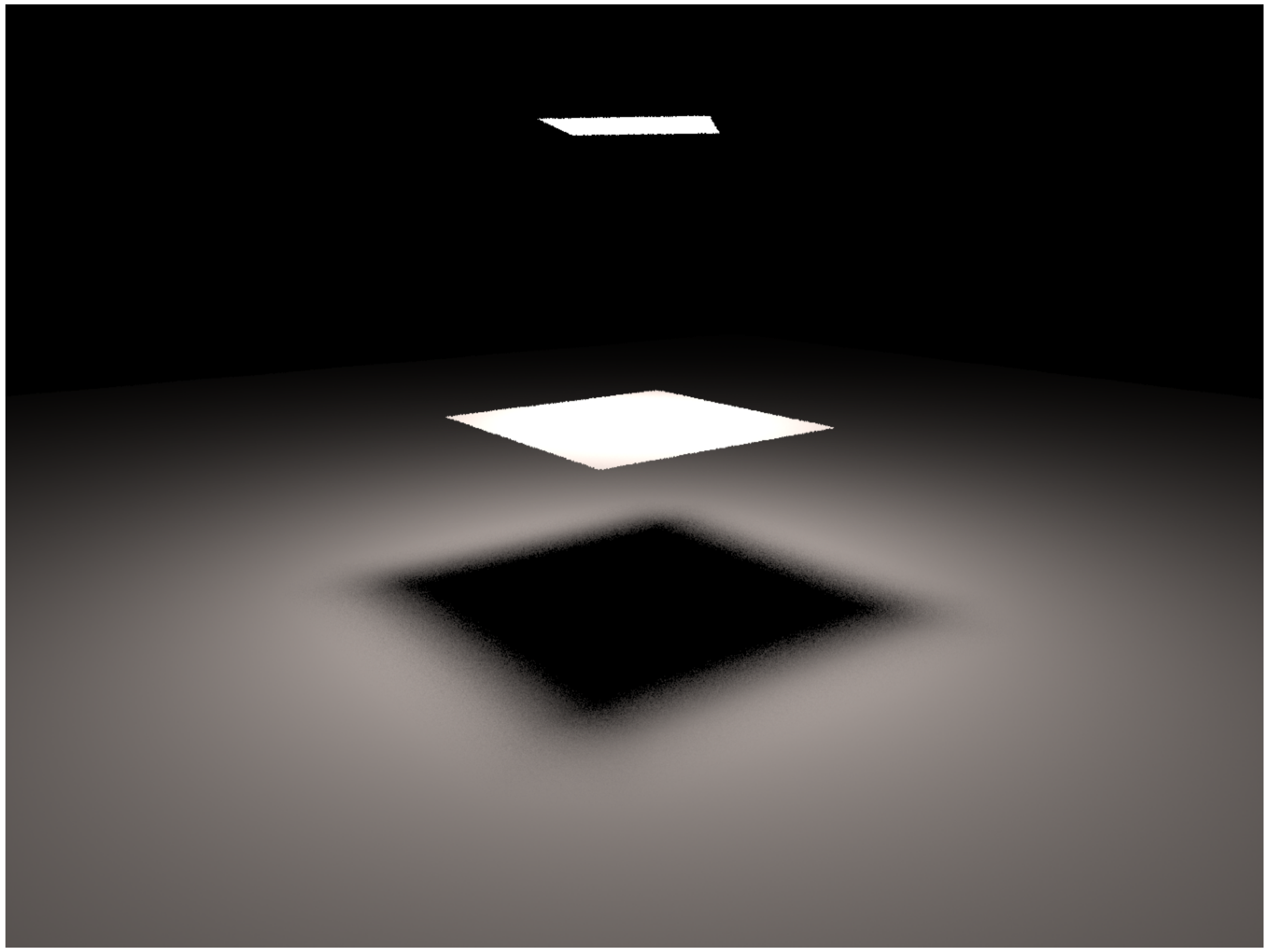
$$\frac{1}{N} \sum_{j=1}^{N} \frac{f_r(\mathrm{p}, \omega_j \to \omega_{\mathrm{o}})\, L_{\mathrm{i}}(\mathrm{p}, \omega_j)\, \cos\theta_j}{p(\omega_j)}$$

Why is Area Better than Hemisphere?



Hemisphere

16 shadow rays



Area

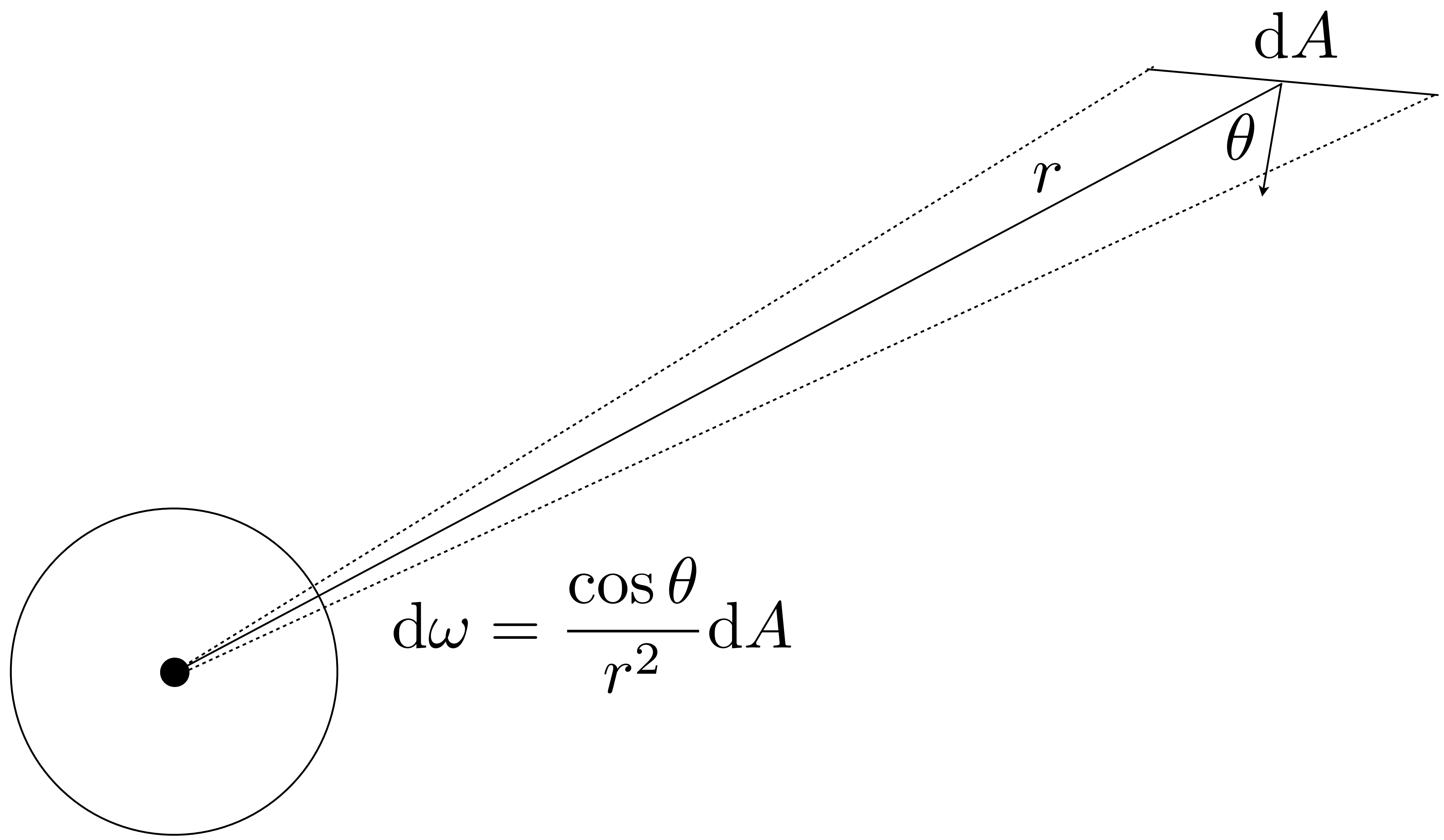
16 shadow rays

Sampling Lights

Sampling Area Light Sources

Sample uniformly by area on the light's surface

Convert to solid angle, compute estimator

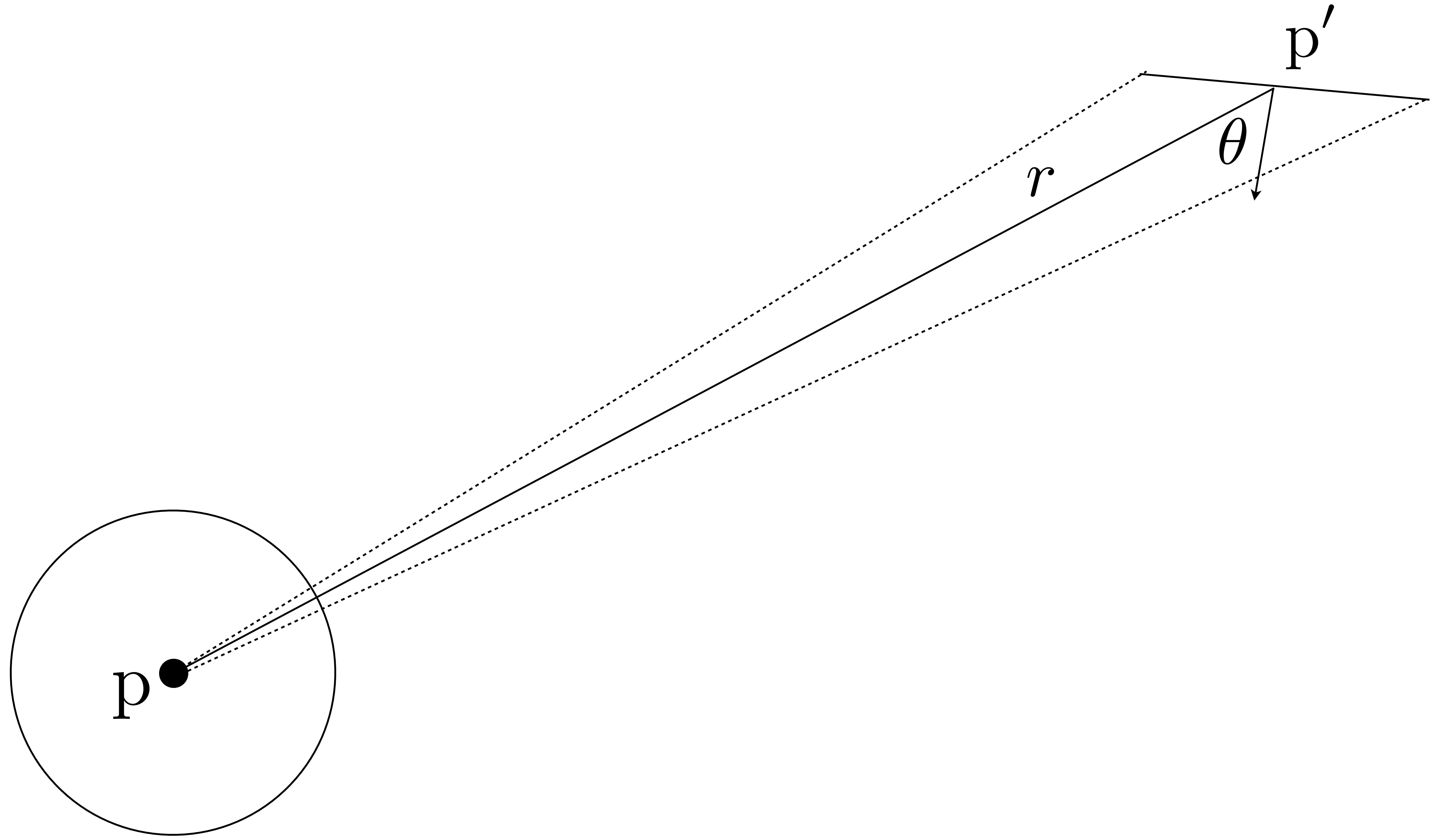


$$L_o(\mathrm{p}, \omega_o) = \int_A f_r(\mathrm{p}, \omega_i \to \omega_o)\, L_i(\mathrm{p}, \omega_i)\, \cos\theta_i\, \frac{\cos\theta}{r^2}\mathrm{d}A$$

Sampling Area Light Sources

Sample uniformly by area on the light's surface

Convert to solid angle, compute estimator

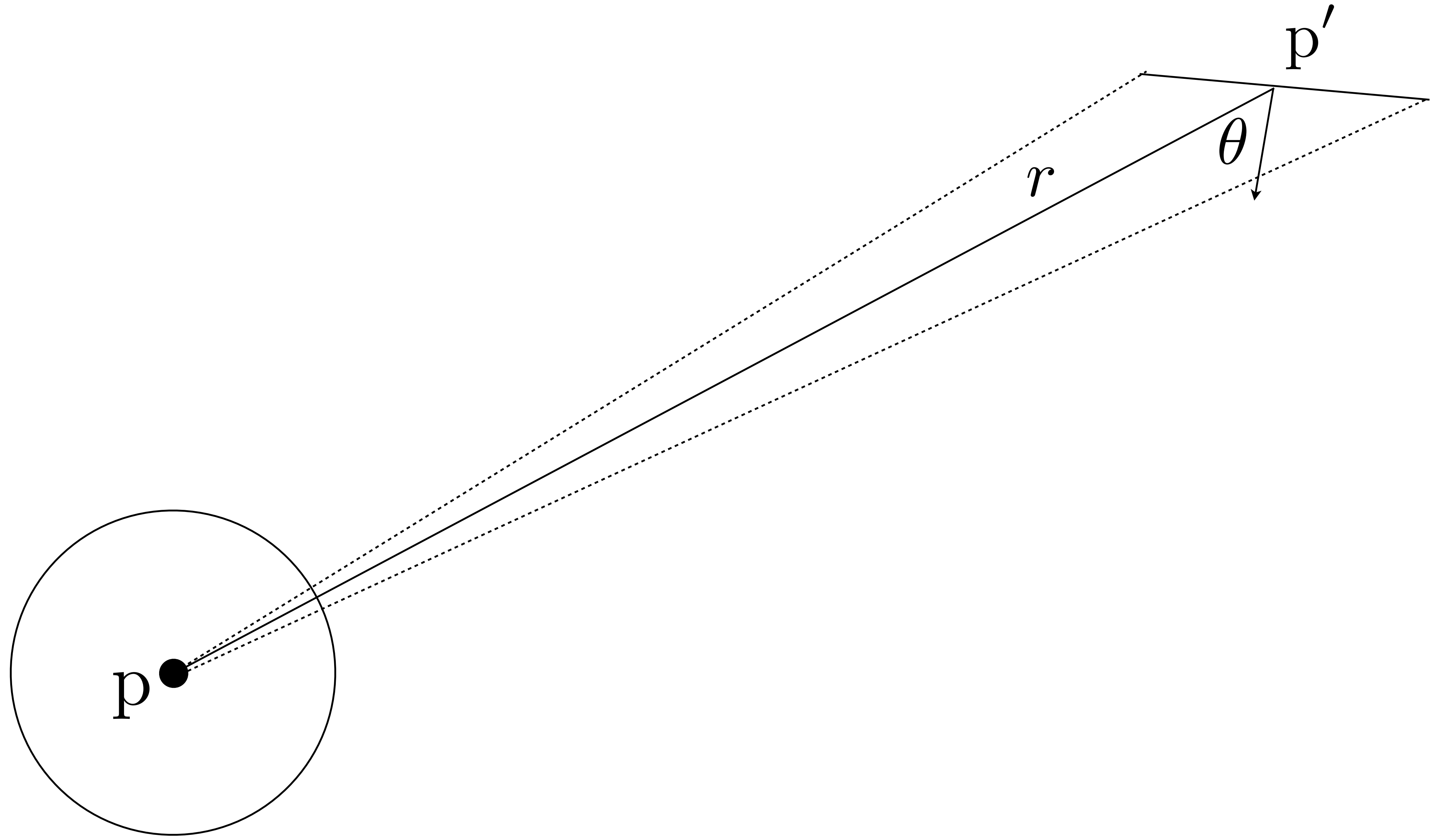


$$L_{\mathrm{o}}(\mathrm{p}, \omega_{\mathrm{o}}) = \int_A f_r(\mathrm{p}, \omega_{\mathrm{i}} \to \omega_{\mathrm{o}})\, L_{\mathrm{e}} V(\mathrm{p}, \mathrm{p}')\, \cos\theta_{\mathrm{i}}\, \frac{\cos\theta}{r^2} \mathrm{d}A$$

Sampling Area Light Sources

Sample uniformly by area on the light's surface

Convert to solid angle, compute estimator



$$L_{\mathrm{o}}(\mathrm{p}, \omega_{\mathrm{o}}) \approx \frac{f_r(\mathrm{p}, \omega_{\mathrm{i}} \to \omega_{\mathrm{o}})\, L_{\mathrm{e}}\, V(\mathrm{p}, \mathrm{p}')\, \cos\theta_{\mathrm{i}}\, \cos\theta}{p(\mathrm{p}')\, r^2}$$

Sampling via Spherical Coordinates

Differential measure on the sphere:

$$\mathrm{d}\omega = \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi$$

Integrate to get normalization term:

$$\int_{S^2} \mathrm{d}\omega = 4\pi$$

PDF for directions:

$$p(\omega) = \frac{1}{4\pi}$$

PDF w.r.t. (θ, ϕ)**:**

$$p(\theta, \phi) = \frac{\sin\theta}{4\pi}$$

Sampling a Unit Sphere

PDF w.r.t. (θ, ϕ)**:** $$p(\theta, \phi) = \frac{\sin\theta}{4\pi} = p(\theta)p(\phi) = \frac{\sin\theta}{2}\frac{1}{2\pi}$$

CDF for phi: $$P(\phi) = \int_0^\phi \frac{1}{2\pi}\,\mathrm{d}\phi = \frac{\phi}{2\pi}$$

Sampling phi: $$\xi_1 = P(\phi) = \frac{\phi}{2\pi}$$

$$\boxed{\phi = 2\pi\xi_1}$$

Sampling a Unit Sphere

PDF w.r.t. (θ, ϕ)**:** $$p(\theta, \phi) = \frac{\sin\theta}{4\pi} = p(\theta)p(\phi) = \frac{\sin\theta}{2}\frac{1}{2\pi}$$

CDF for theta: $$P(\theta) = \int_0^\theta \frac{\sin\theta}{2}\,\mathrm{d}\theta = \frac{1-\cos\theta}{2}$$

Sampling theta: $$\xi_2 = \frac{1-\cos\theta}{2}$$

$$\begin{aligned}\cos\theta =& 1 - 2\xi_2 \\ \theta =& \arccos(1 - 2\xi_2)\end{aligned}$$

Sampling a Unit Sphere

Recipes:

$$\cos\theta = 1 - 2\xi$$
$$\theta = \arccos(1 - 2\xi)$$

$$\phi = 2\pi\xi$$

Spherical coordinates: $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

Sample locations:

$$x = \cos(2\pi\xi_2)\sqrt{1 - z^2}$$
$$y = \sin(2\pi\xi_2)\sqrt{1 - z^2}$$
$$z = 1 - 2\xi_1$$

Direct Lighting With Spherical Lights

1. Choose point randomly on sphere surface

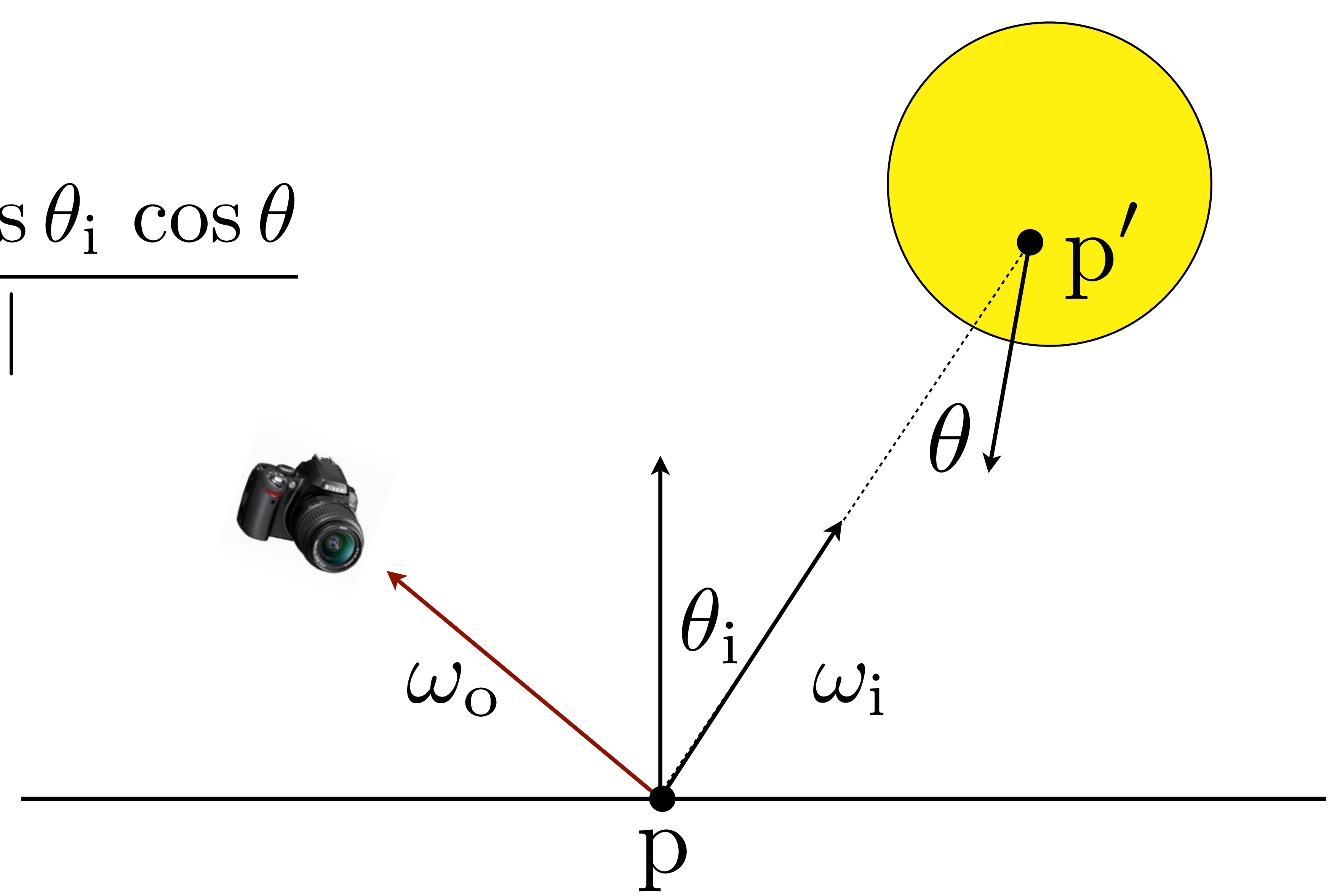
2. PDF over area of sphere:

$$p_A(x, y, z) = \frac{1}{4\pi r^2}$$

3. Trace ray to see if p **and** p' **are mutually visible**

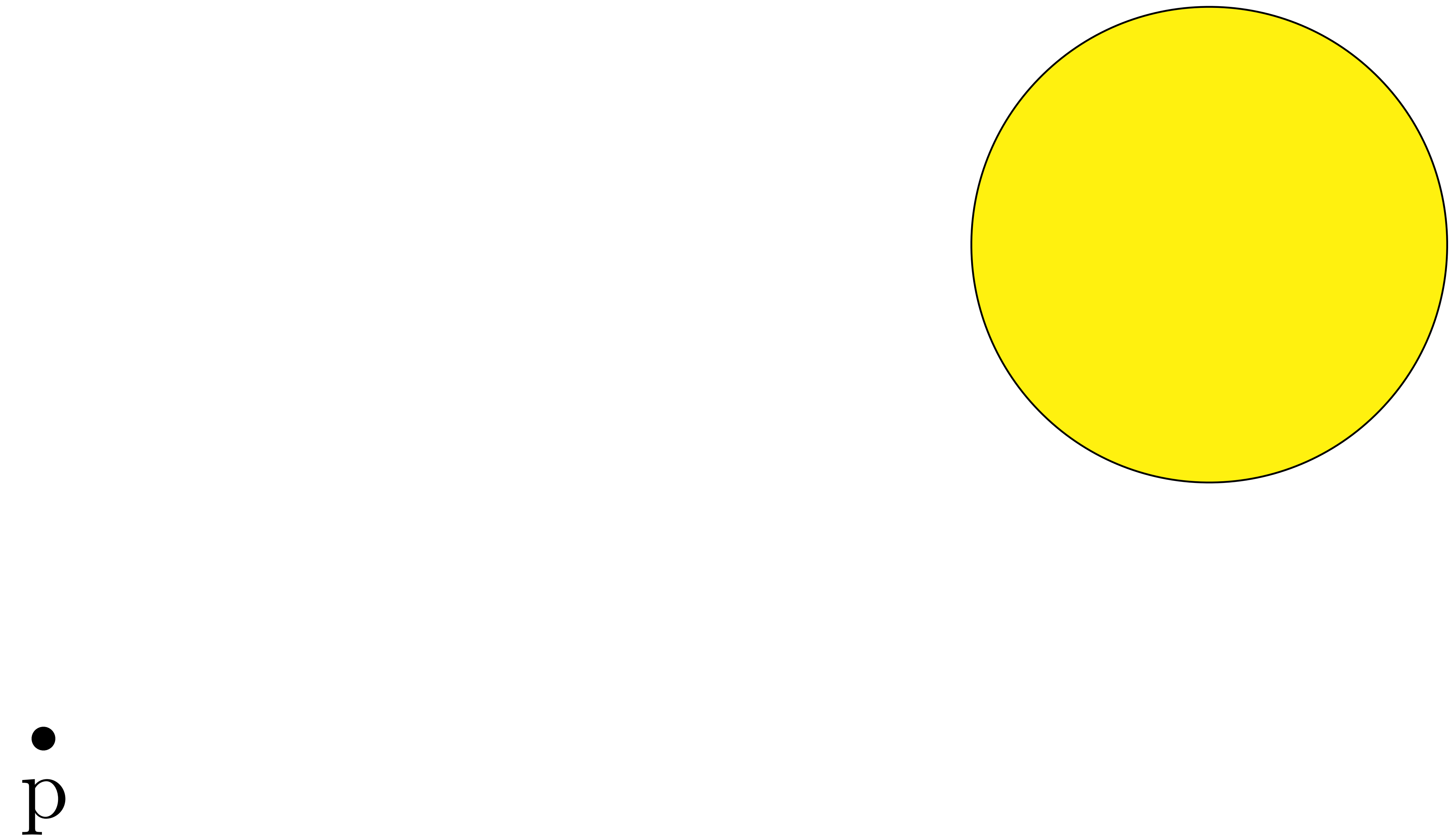
4. Evaluate estimator:

$$L_{\mathrm{o}}(\mathrm{p}, \omega_{\mathrm{o}}) \approx \frac{f_r(\mathrm{p}, \omega_{\mathrm{i}} \to \omega_{\mathrm{o}})\, L_{\mathrm{e}}\, \cos\theta_{\mathrm{i}}\, \cos\theta}{p(A)\, \|\mathrm{p} - \mathrm{p}'\|}$$



Sampling Spherical Lights: Directions

Problems with sampling uniformly by area?



Sampling Spherical Lights: Directions

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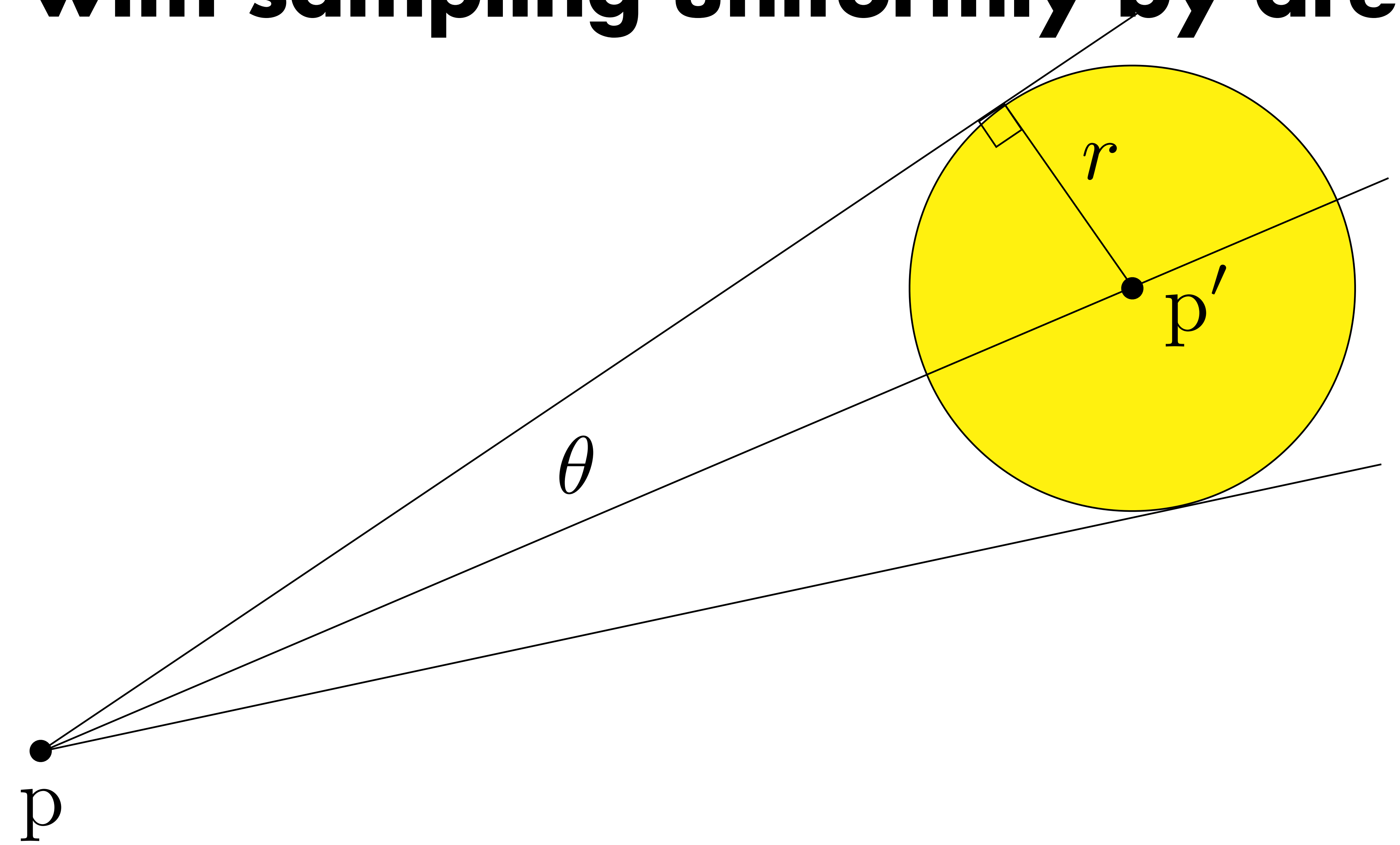
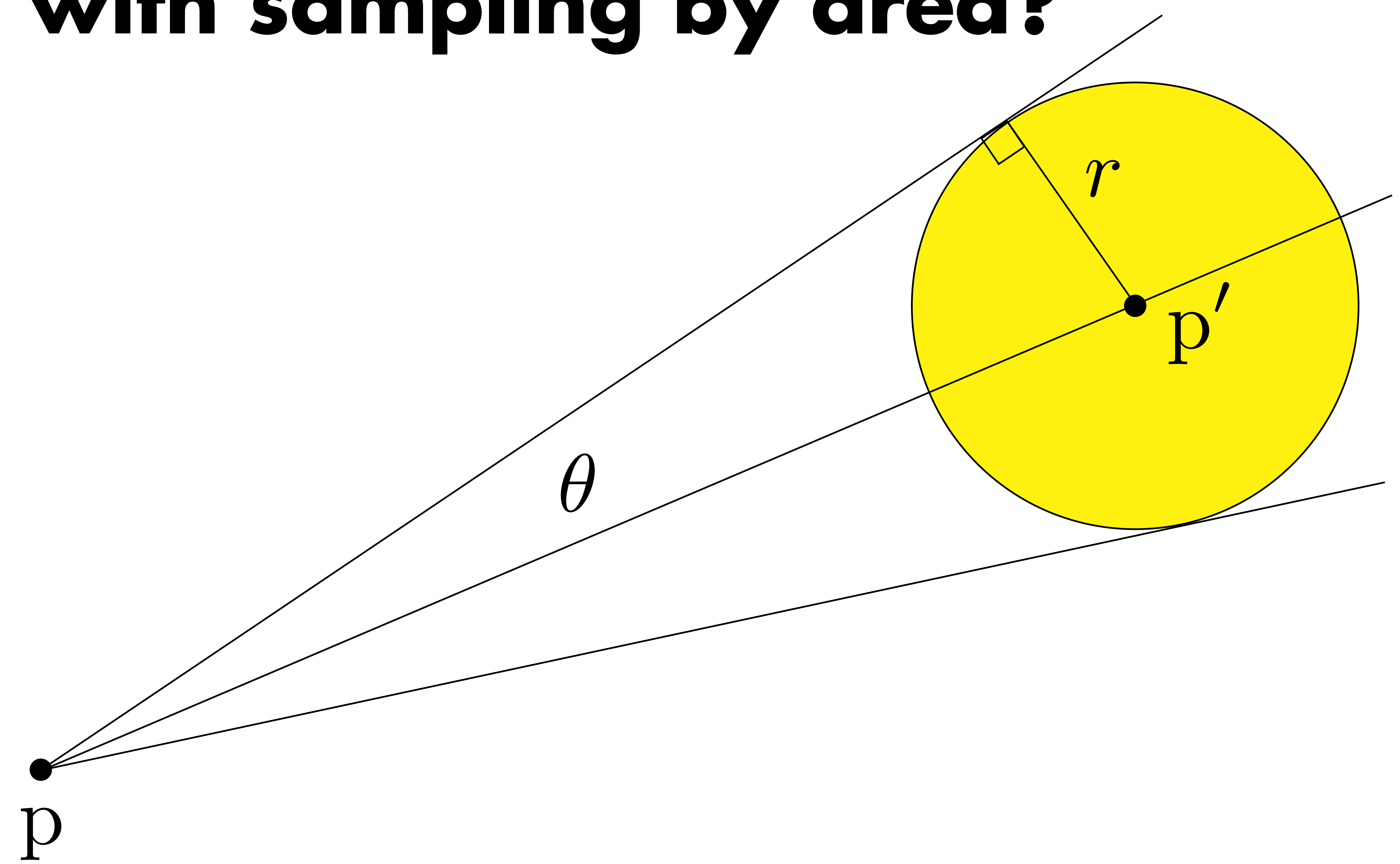


Uniform Cone Sampling

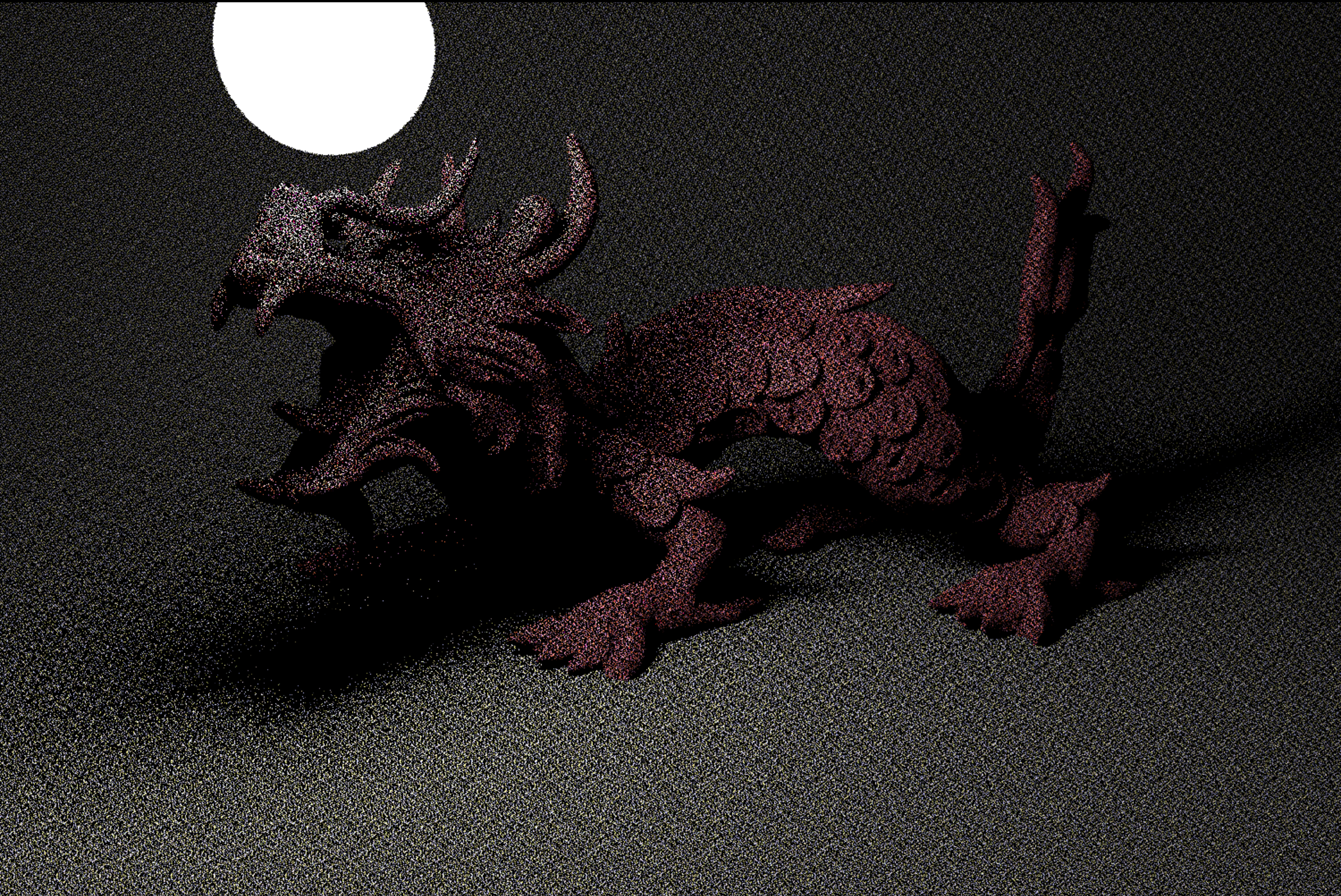
Problems with sampling by area?



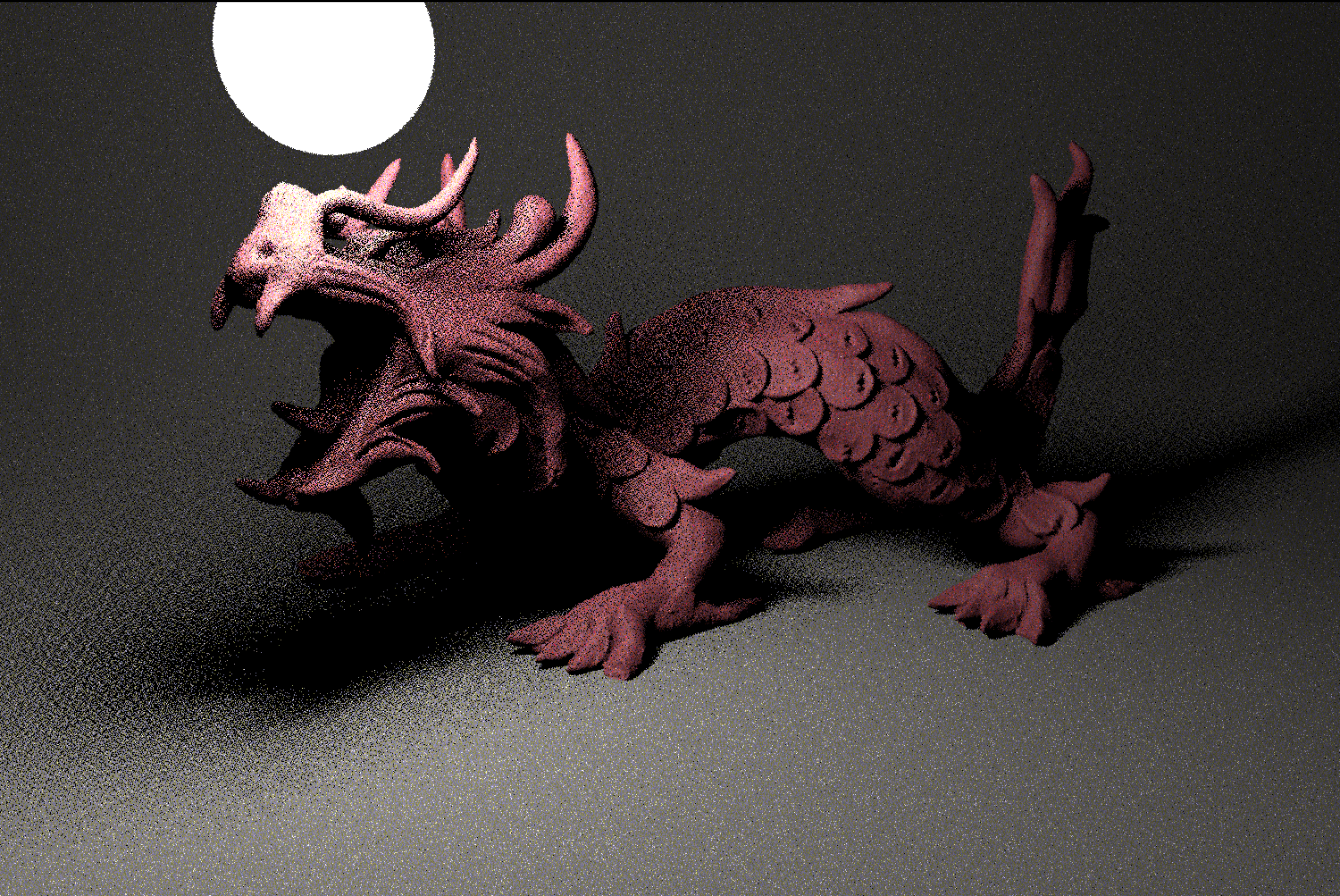
$$\cos\theta' = (1-\xi) + \xi\cos\theta \qquad \phi = 2\pi\xi$$

$$p(\omega) = \frac{1}{2\pi(1-\cos\theta)}$$

Uniform Area Sampling



Uniform Angle Sampling



Sampling Area vs. Solid Angle

Solid angle reflection equation:

$$L_{\mathrm{o}}(\mathrm{p}, \omega_{\mathrm{o}}) = \int_{H^2} f_r(\mathrm{p}, \omega_{\mathrm{i}} \rightarrow \omega_{\mathrm{o}})\, L_{\mathrm{i}}(\mathrm{p}, \omega_{\mathrm{i}})\, \cos\theta_{\mathrm{i}}\, \mathrm{d}\omega_{\mathrm{i}}$$

Area reflection equation:

$$L_{\mathrm{o}}(\mathrm{p}, \omega_{\mathrm{o}}) = \int_{A} f_r(\mathrm{p}, \omega_{\mathrm{i}} \rightarrow \omega_{\mathrm{o}})\, L_{\mathrm{e}} V(\mathrm{p}, \mathrm{p}')\, \cos\theta_{\mathrm{i}}\, \frac{\cos\theta}{r^2} \mathrm{d}A$$

Sampling Area vs. Solid Angle

Solid angle reflection estimator:

$$L_{\mathrm{o}}(\mathrm{p}, \omega_{\mathrm{o}}) \approx \frac{f_r(\mathrm{p}, \omega_{\mathrm{i}} \rightarrow \omega_{\mathrm{o}})\, L_{\mathrm{i}}(\mathrm{p}, \omega_{\mathrm{i}})\ \cos\theta_{\mathrm{i}}}{p(\omega_{\mathrm{i}})}$$

Area reflection estimator:

$$L_{\mathrm{o}}(\mathrm{p}, \omega_{\mathrm{o}}) \approx \frac{f_r(\mathrm{p}, \omega_{\mathrm{i}} \rightarrow \omega_{\mathrm{o}})\, L_{\mathrm{e}}\ \cos\theta_{\mathrm{i}}\ \cos\theta}{p(A)\, \|\mathrm{p} - \mathrm{p}'\|}$$

Sampling Area vs. Solid Angle

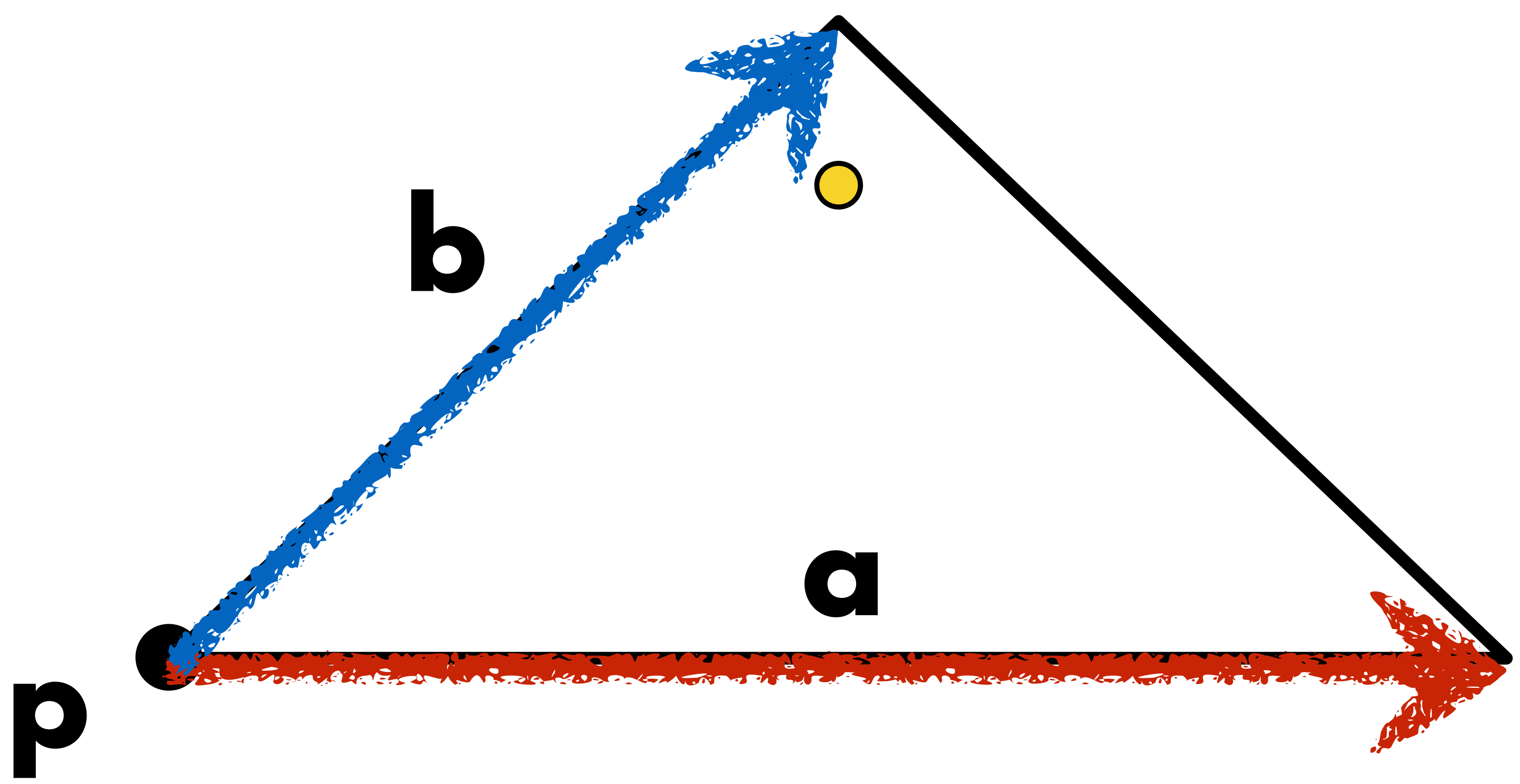
Solid angle estimator (diffuse, uniform sample dir.)

$$L_\mathrm{o}(\mathrm{p}, \omega_\mathrm{o}) \approx \left[\frac{f_r \, L_i}{p}\right] V(\mathrm{p}, \omega_\mathrm{i}) \, \cos\theta_\mathrm{i}$$

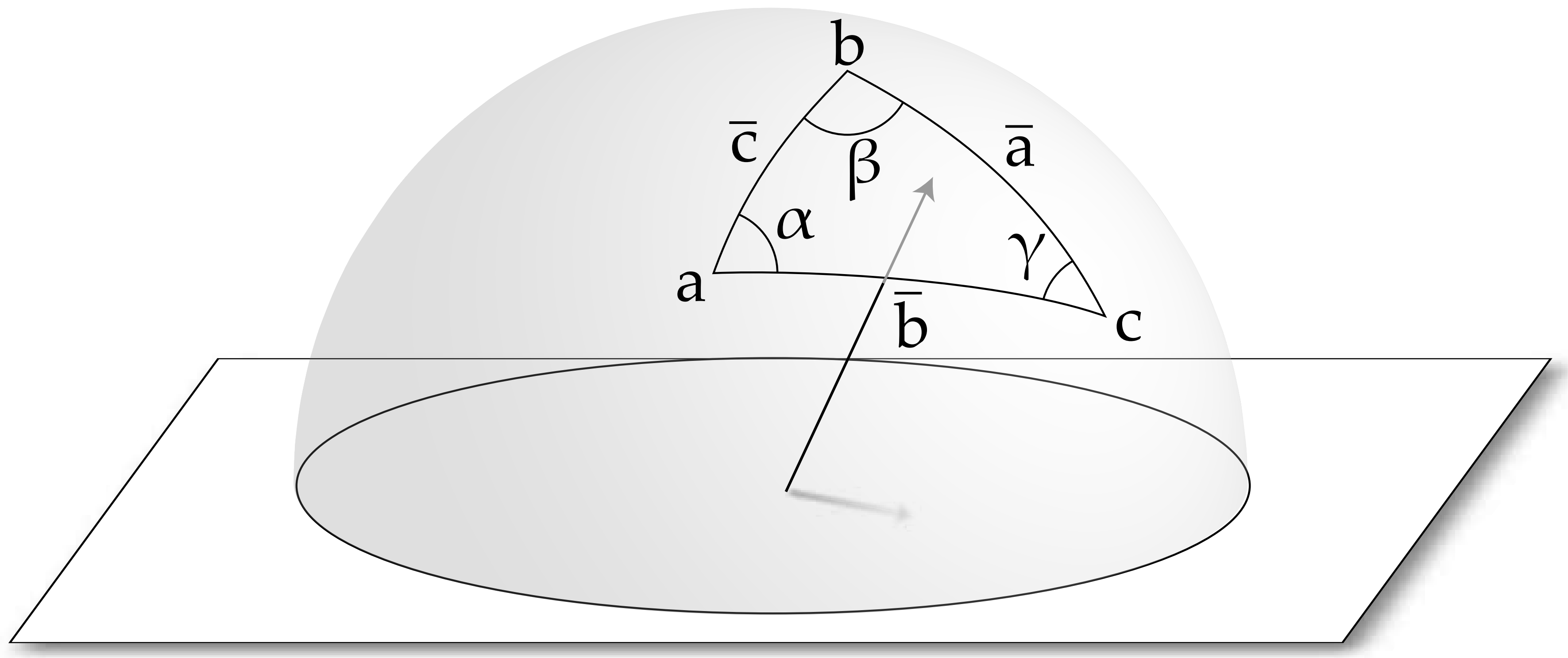
Area estimator (diffuse, uniform sample area)

$$L_\mathrm{o}(\mathrm{p}, \omega_\mathrm{o}) \approx \left[\frac{f_r \, L_i}{p}\right] \frac{V(\mathrm{p}, \omega_\mathrm{i}) \, \cos\theta_\mathrm{i} \, \cos\theta}{\|\mathrm{p} - \mathrm{p}'\|}$$

Triangle Sampling

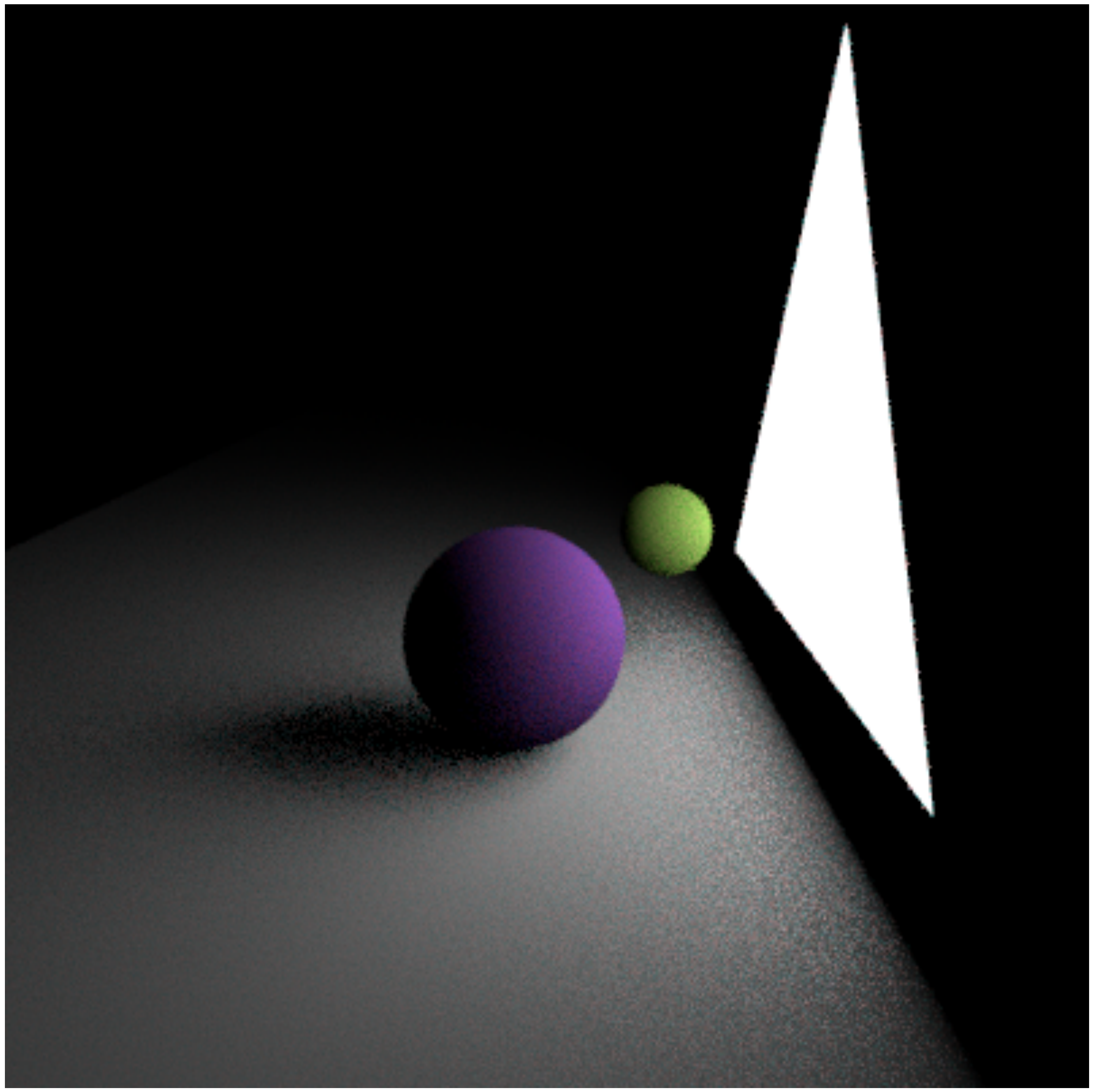


Area

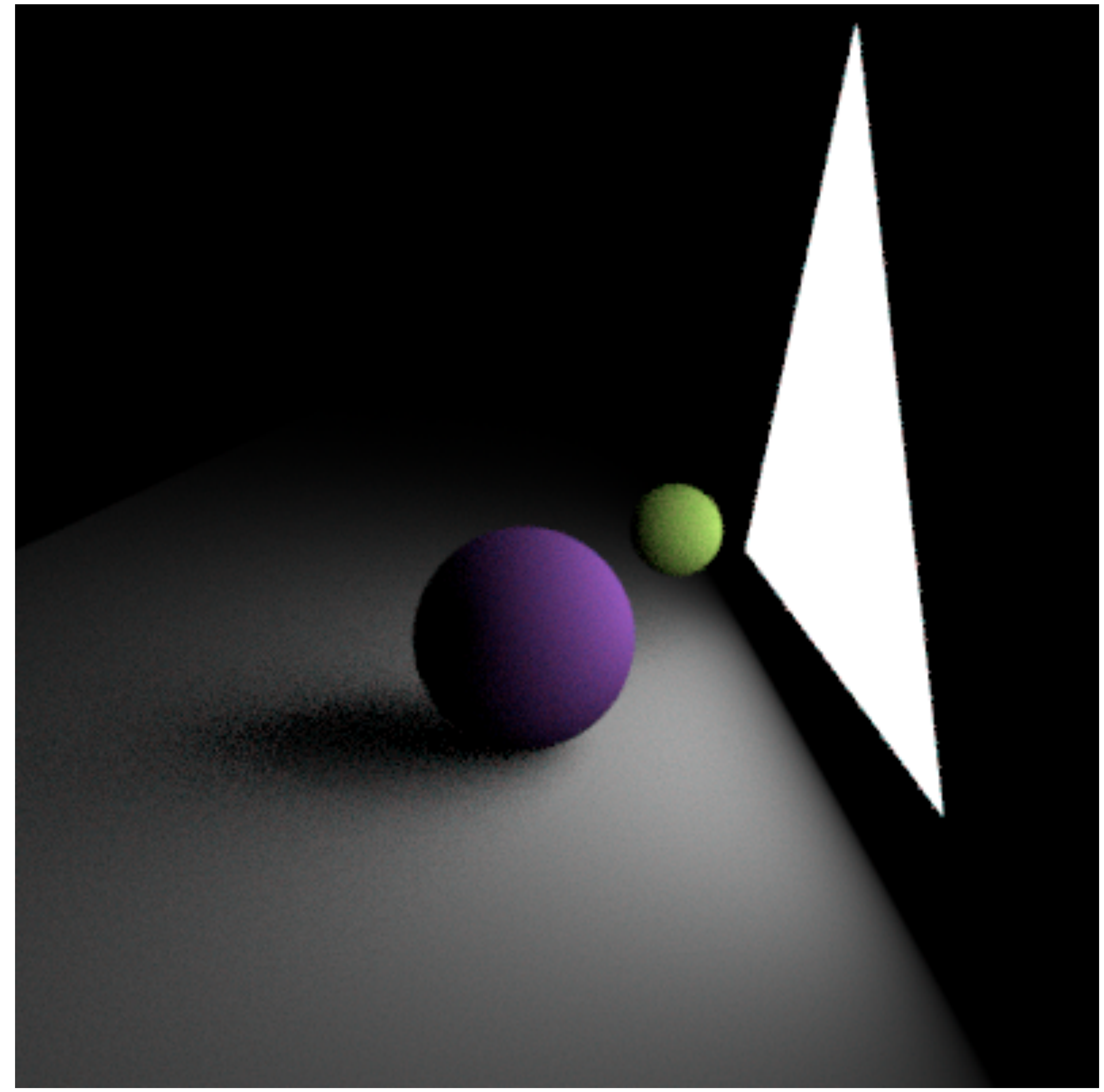


Solid Angle

Area vs. Solid Angle Sampling

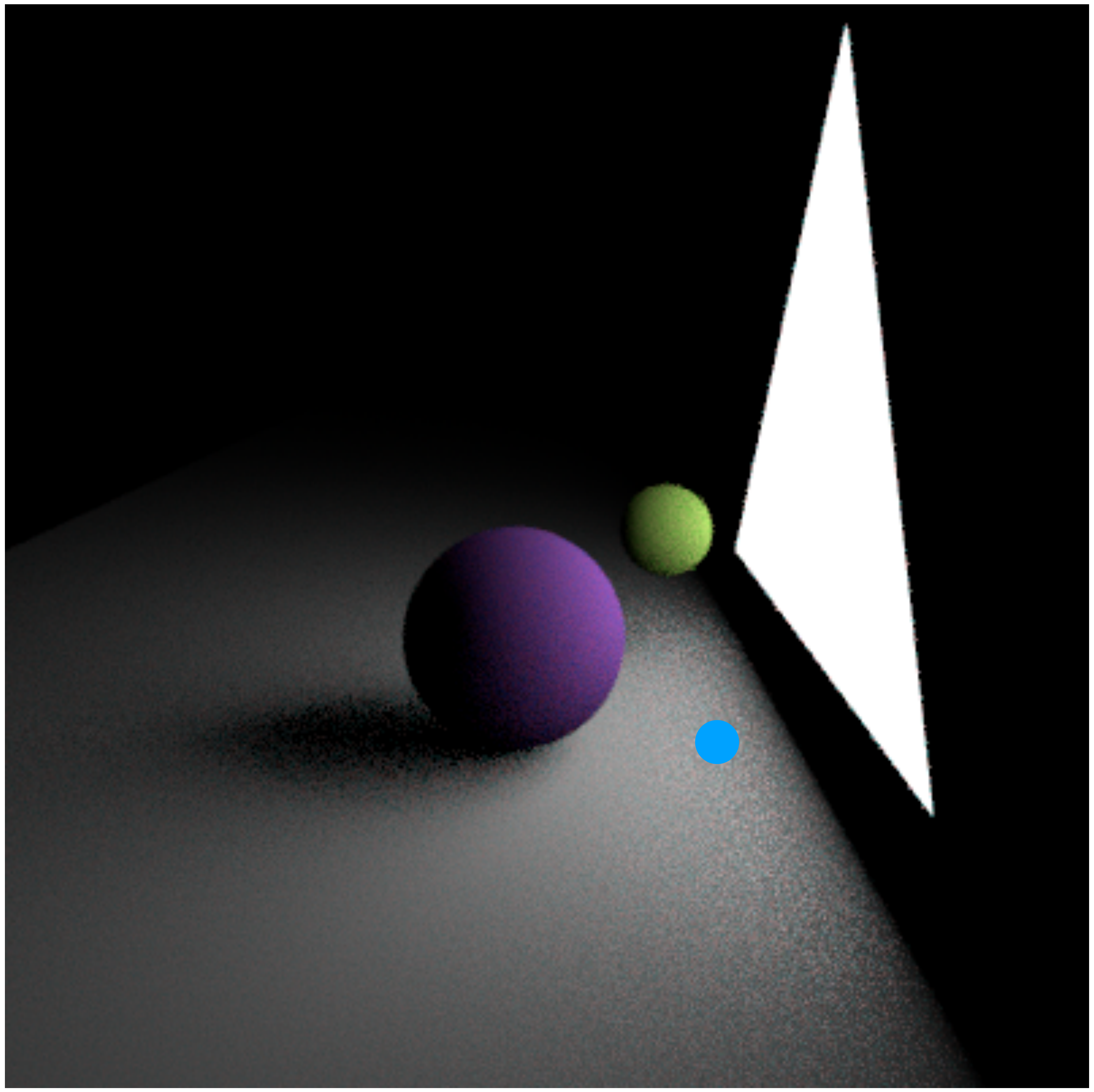


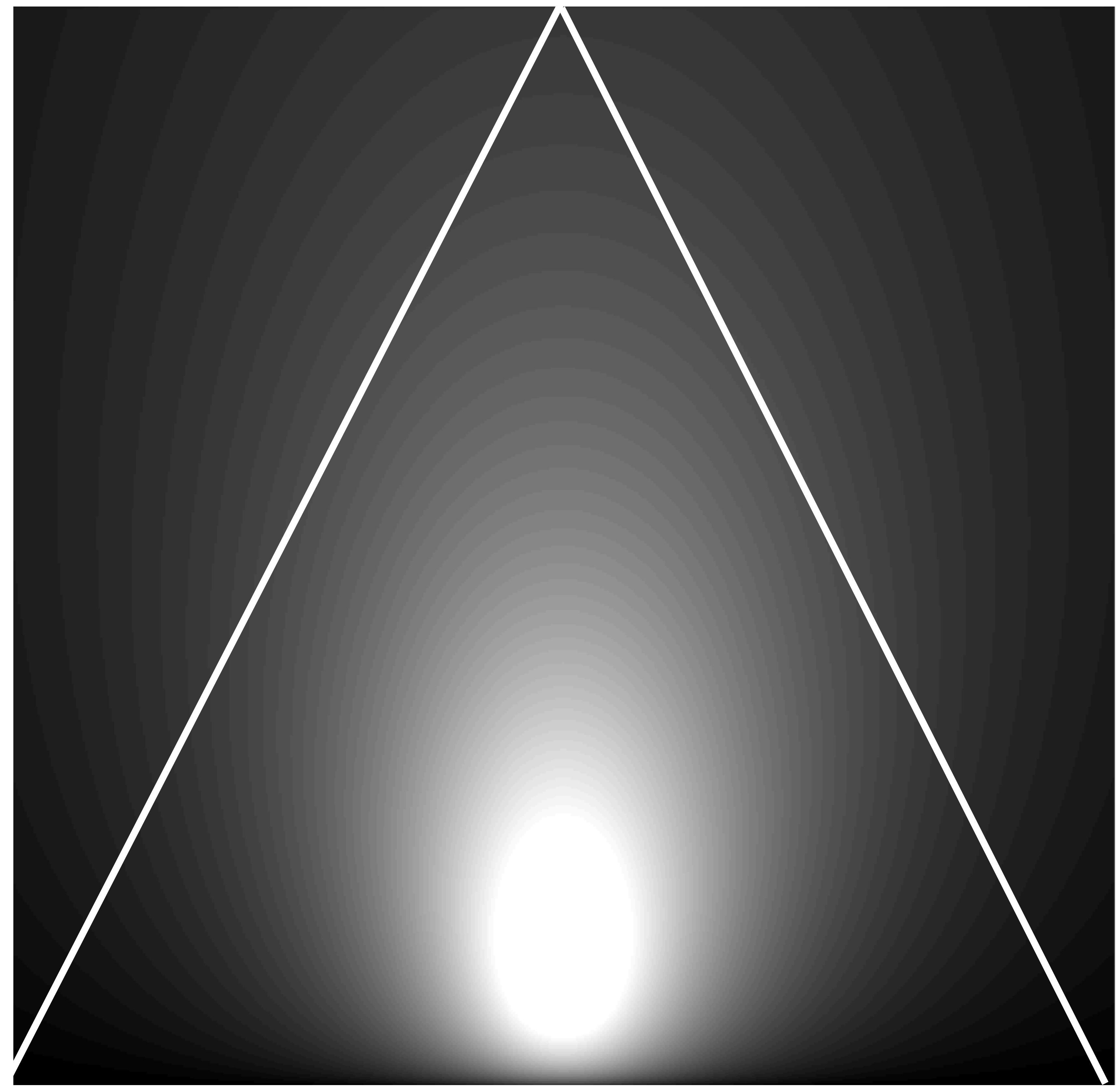
Area



Solid Angle
(1.81x MSE reduction)

Variation of Area Factors over Triangle

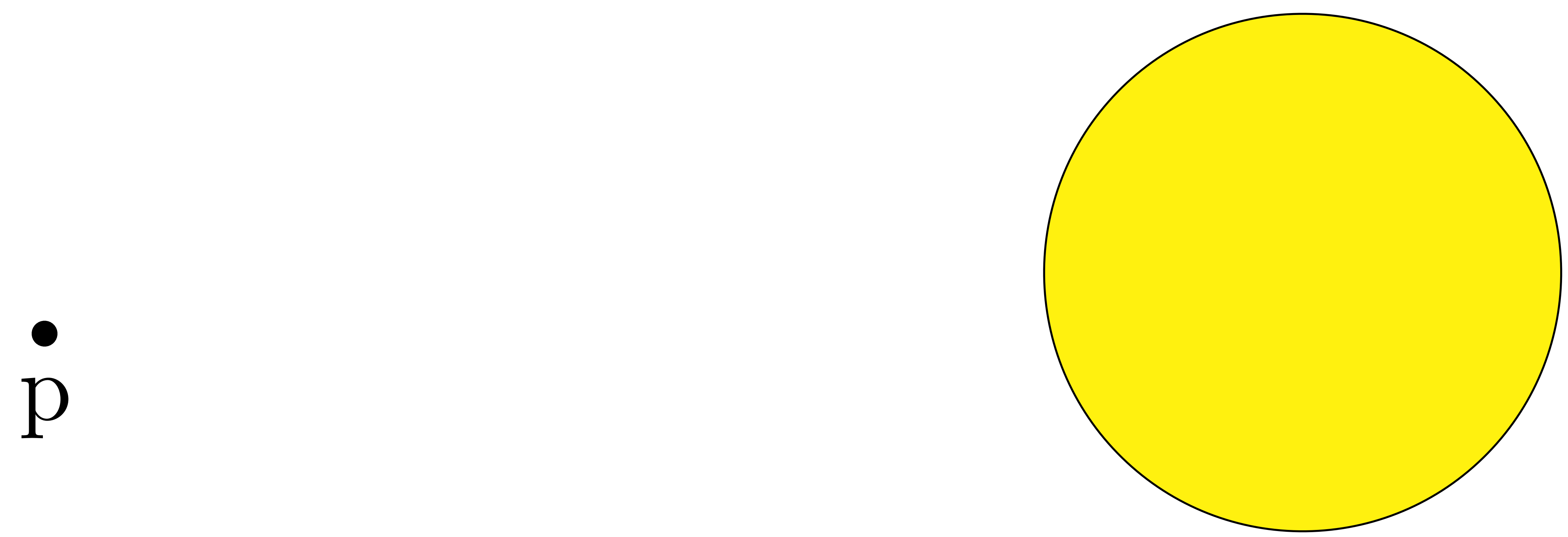




$$\frac{\cos\theta_i \cos\theta_l}{r^2}$$

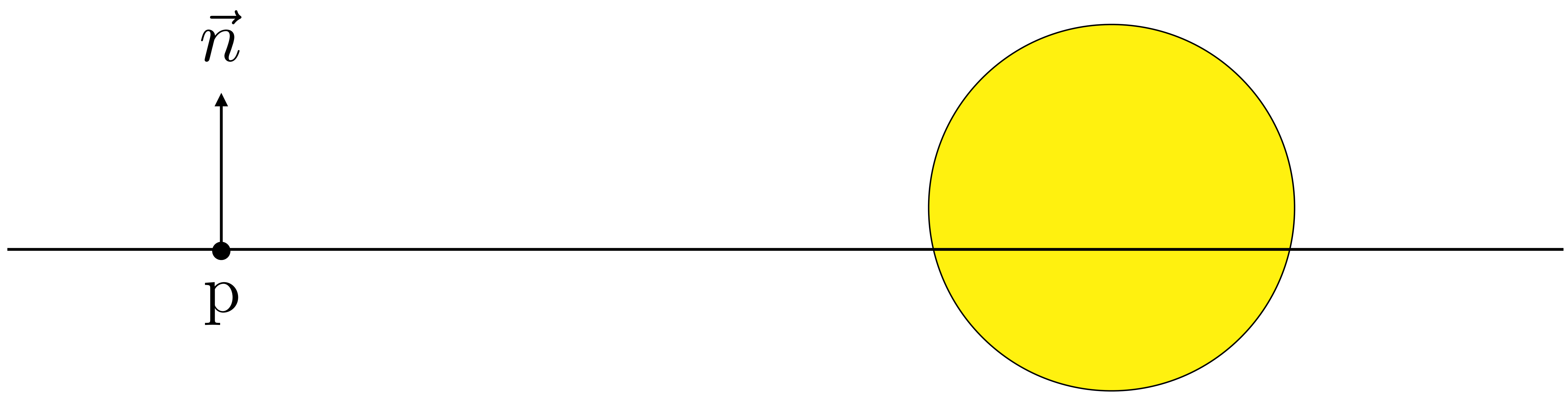
Sampling Spherical Lights: Directions

Problems with sampling uniformly by solid angle?

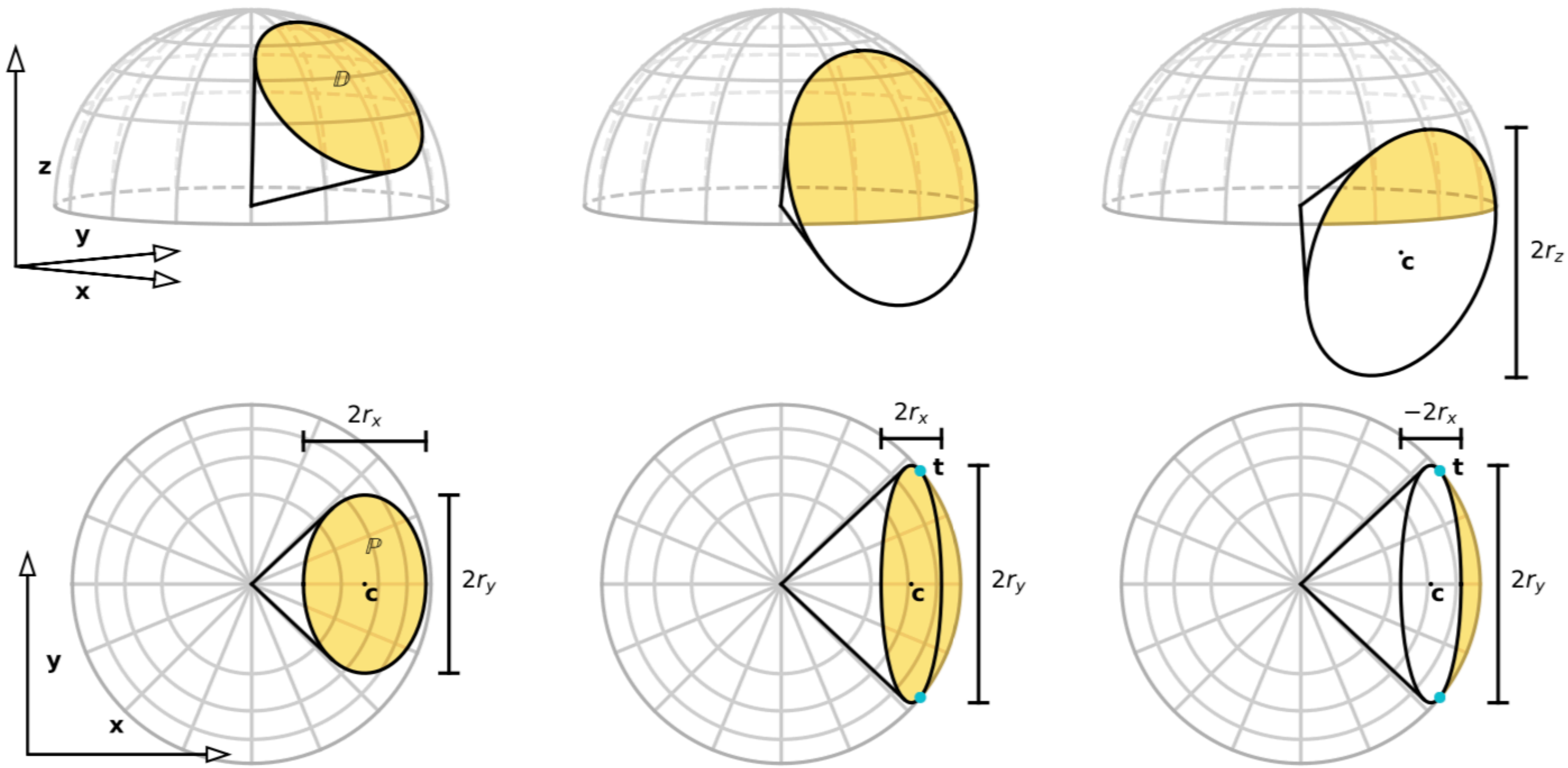


Sampling Spherical Lights: Directions

Problems with sampling uniformly by solid angle?



Sampling Spherical Caps



[Peters and Dachsbacher 2019]

Solid Angle vs. Projected Solid Angle

Solid angle estimator (diffuse, uniform direction):

$$L_{\mathrm{o}}(\mathrm{p}, \omega_{\mathrm{o}}) \approx \left[\frac{f_r\, L_i}{p}\right] V(\mathrm{p}, \omega_{\mathrm{i}})\, \cos\theta_{\mathrm{i}}$$

Projected solid angle estimator (diffuse, cosine-weighted direction):

$$L_{\mathrm{o}}(\mathrm{p}, \omega_{\mathrm{o}}) \approx \left[\frac{f_r\, L_i}{p}\right] V(\mathrm{p}, \omega_{\mathrm{i}})$$

Variance and Importance Sampling

The Monte Carlo Estimator

$$\int f(x)\,\mathrm{d}x \approx \frac{1}{n}\sum_{i=1}^{n}\frac{f(x_i)}{p(x_i)} \qquad\qquad x_i \sim p$$

Variance

Definition

$$V[Y] \equiv E[(Y - E[Y])^2]$$
$$= E[Y^2] - E[Y]^2$$

e.g.:

$$Y = f(X)$$
$$E[Y] = \int f(x)\,\mathrm{d}x$$

Importance Sampling

Sample according to f

$$\tilde{p}(x) = \frac{f(x)}{E[f]}$$

$$\tilde{f}(x) = \frac{f(x)}{\tilde{p}(x)}$$

Importance Sampling

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Variance

$$V[f] = E[f^2] - E^2[f]$$

Importance Sampling

Sample according to f

$$\tilde{p}(x) = \frac{f(x)}{E[f]}$$

$$\tilde{f}(x) = \frac{f(x)}{\tilde{p}(x)}$$

Variance

$$V[f] = E[f^2] - E^2[f]$$

$$E[\tilde{f}^2] = \int \left[\frac{f(x)}{\tilde{p}(x)} \right]^2 \tilde{p}(x)\, dx$$

$$= \int \left[\frac{f(x)}{f(x)/E[f]} \right]^2 \frac{f(x)}{E[f]} dx$$

$$= E[f] \int f(x)\, dx$$

$$= E^2[f]$$

Importance Sampling

Sample according to f

$$\tilde{p}(x) = \frac{f(x)}{E[f]}$$

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$$= E[f] \int f(x)\, dx$$

$$= E^2[f]$$

Zero variance!

$$V[\tilde{f}^2] = 0$$

Importance Sampling

Sample according to f

$$\tilde{p}(x) = \frac{f(x)}{E[f]}$$

$$\tilde{f}(x) = \frac{f(x)}{\tilde{p}(x)}$$

Variance

$$V[f] = E[f^2] - E^2[f]$$

$$E[\tilde{f}^2] = \int \left[\frac{f(x)}{\tilde{p}(x)} \right]^2 \tilde{p}(x)\, dx$$

$$= \int \left[\frac{f(x)}{f(x)/E[f]} \right]^2 \frac{f(x)}{E[f]} dx$$

$$= E[f] \int f(x)\, dx$$

$$= E^2[f]$$

Zero variance!

$$V[\tilde{f}^2] = 0$$

Gotcha?

Sample Variance

Recall $$V[Y] \equiv E[(Y - E[Y])^2] \\ = E[Y^2] - E[Y]^2$$

Compute $$\bar{y} = \frac{1}{n}\sum_i^n Y_i, \quad \bar{y}_2 = \frac{1}{n}\sum_i^n Y_i^2$$

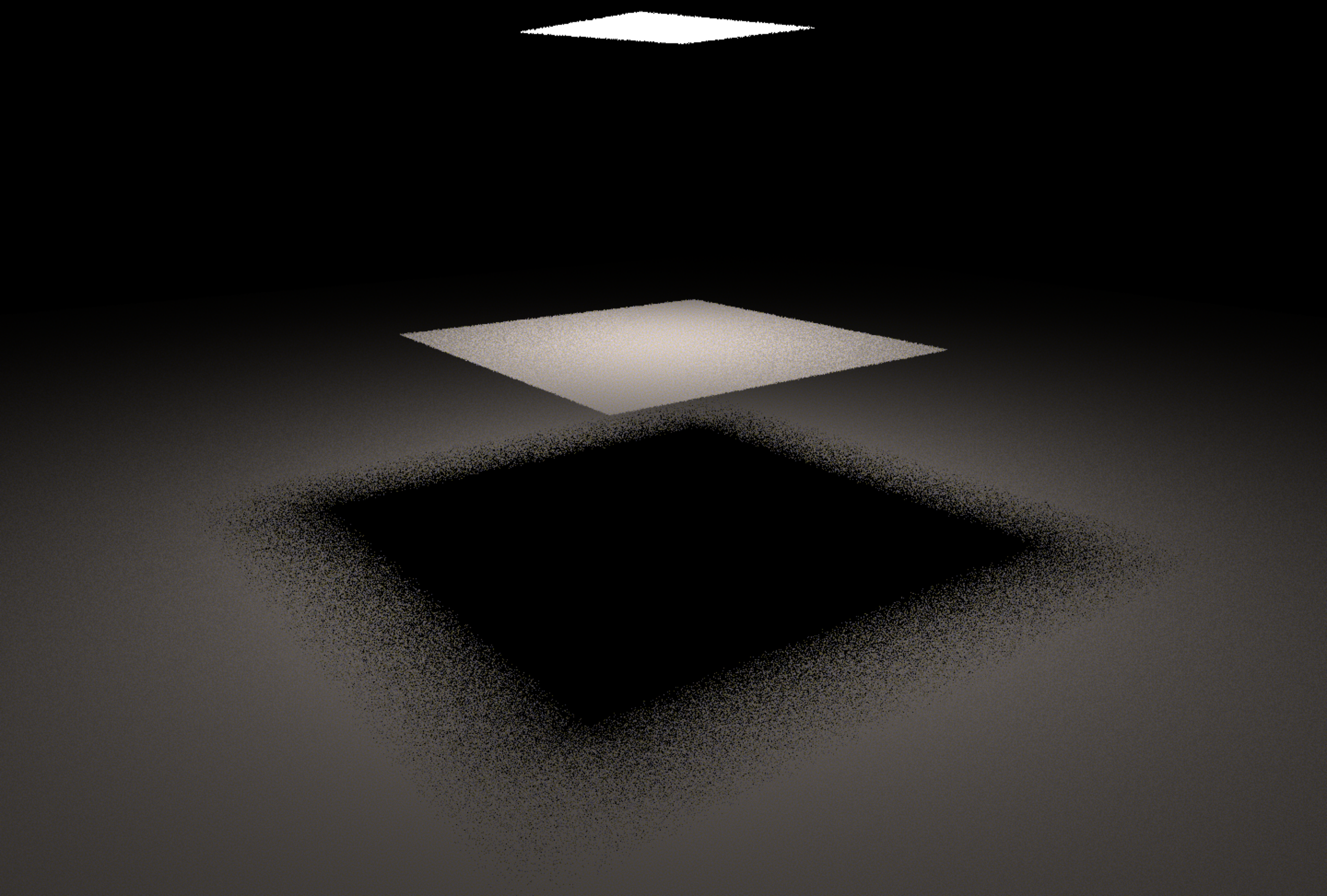
Estimate: $$\bar{y}_2 - (\bar{y})^2$$

Sample Variance: Welford's Algorithm

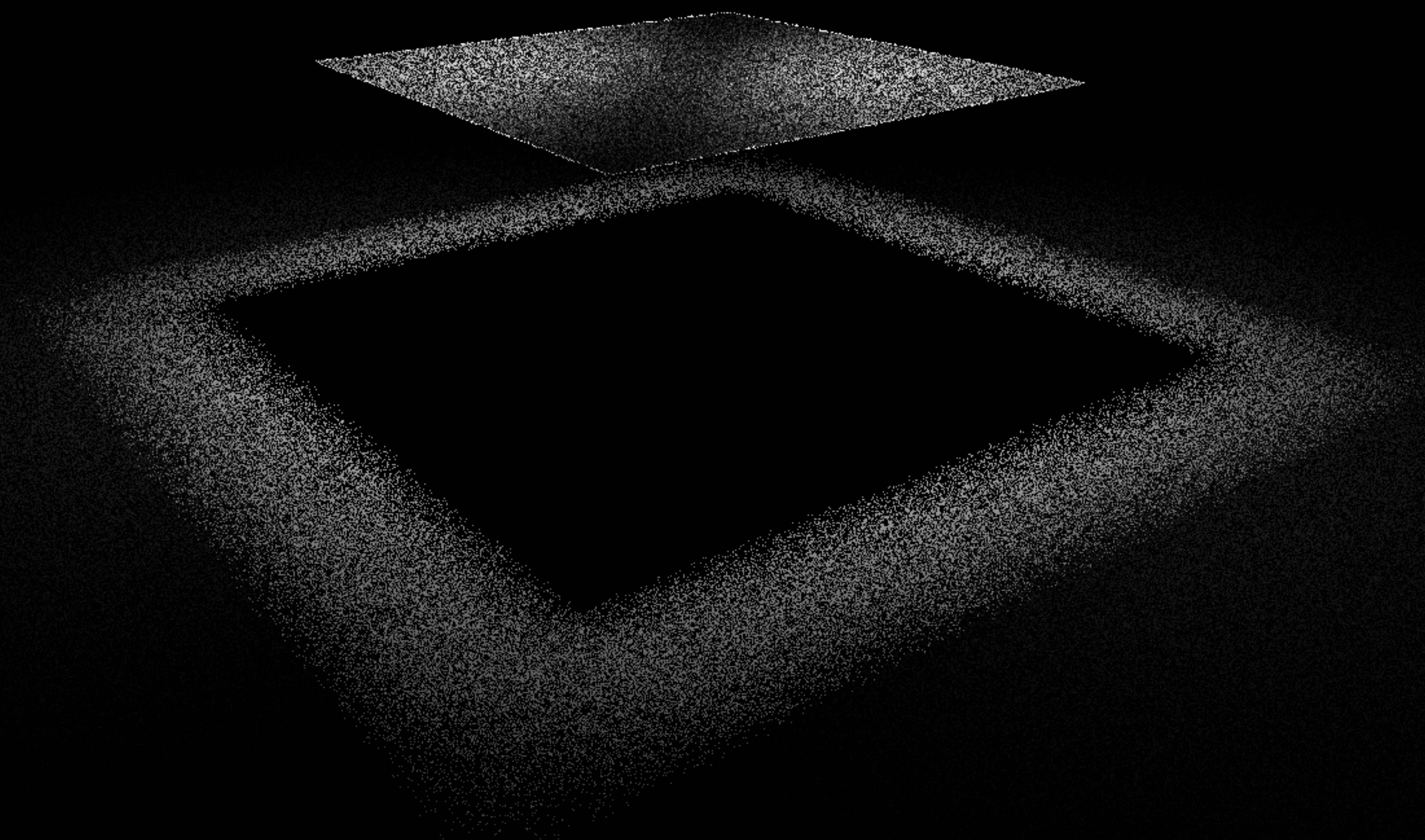
```
int n;
float mean, S;

void add(float v) {
  ++n;
  float delta = v - mean;
  mean += delta / n;
  float delta2 = v - mean;
  S += delta * delta2;
}

float variance() {
  return S / (n - 1);
}
```

$$S = \sum_i^n (x_i - \bar{x}_n)^2$$

Random Sampling (n=4)



Per-Pixel Sample Variance



Variance

Definition

$$V[Y] \equiv E[(Y - E[Y])^2]$$
$$= E[Y^2] - E[Y]^2$$

e.g.:

$$Y = f(X)$$
$$E[Y] = \int f(x)\,\mathrm{d}x$$

Variance decreases linearly with sample size

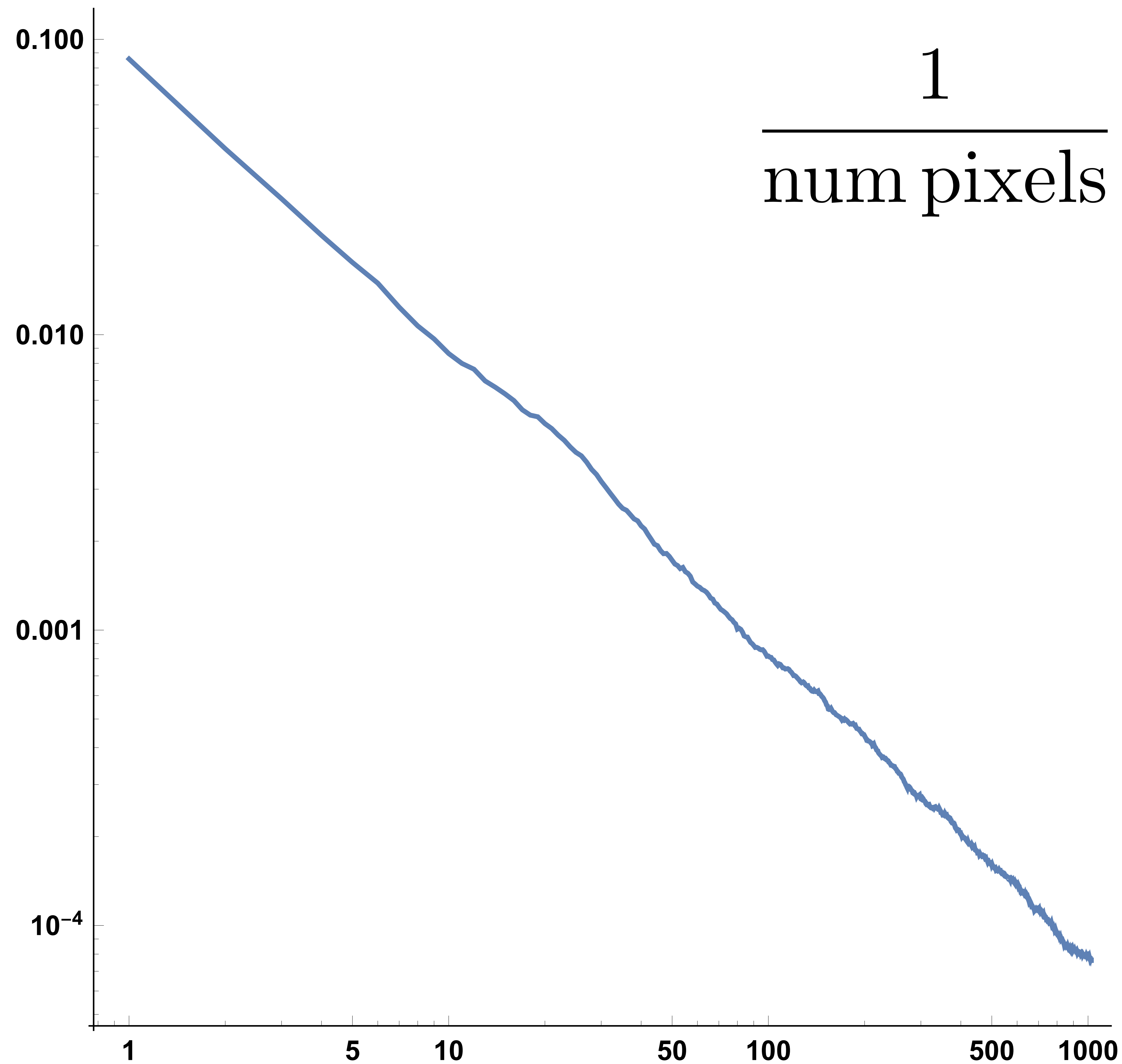
$$V\left[\frac{1}{N}\sum_{i=1}^{N} Y_i\right] = \frac{1}{N^2}\sum_{i=1}^{N} V[Y_i] = \frac{1}{N^2} N V[Y] = \frac{1}{N} V[Y]$$

$$E[aY] = aE[Y]$$
$$V[aY] = a^2 V[Y]$$

Mean Squared Error (Quads Scene)

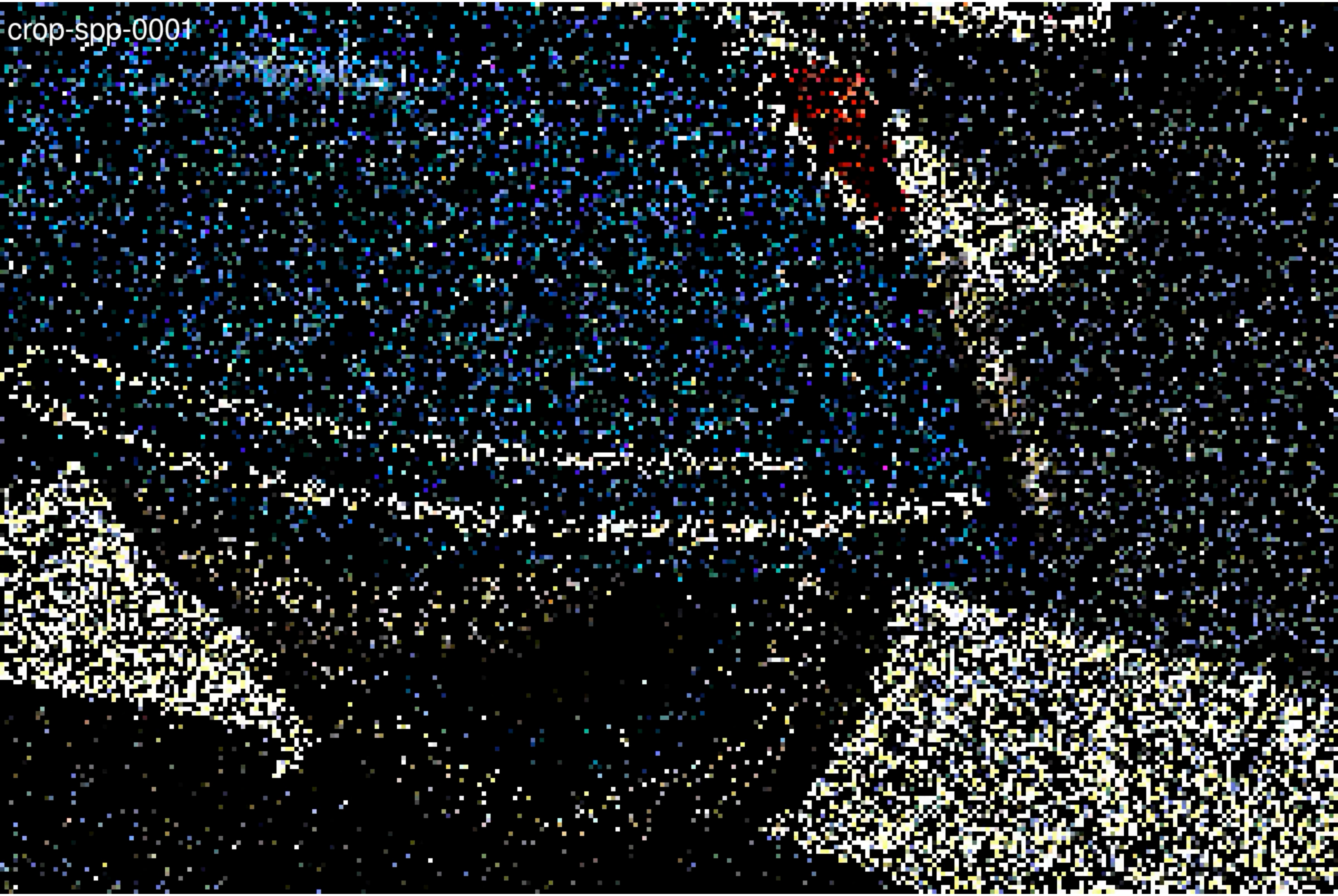
$$\frac{1}{\text{num pixels}} \sum_{\text{pixels}} (x_i - \bar{x}_i)^2$$

MSE



Number of Samples

Samples vs. Error



http://pharr.org/matt/assets/bistro-spp.mp4

Comparing Different Techniques

Monte Carlo Efficiency measure

$$\text{Efficiency} \propto \frac{1}{\text{Variance} \cdot \text{Cost}}$$

Comparing two sampling techniques A and B

If A has twice the variance as B, then it takes twice as many samples from A to achieve the same variance as B

If A has twice the cost of B, then it takes twice as much time reduce the variance using A compared to using B

The product of variance and cost is a constant independent of the number of samples

Recall: Variance goes as 1/N, time goes as C*N

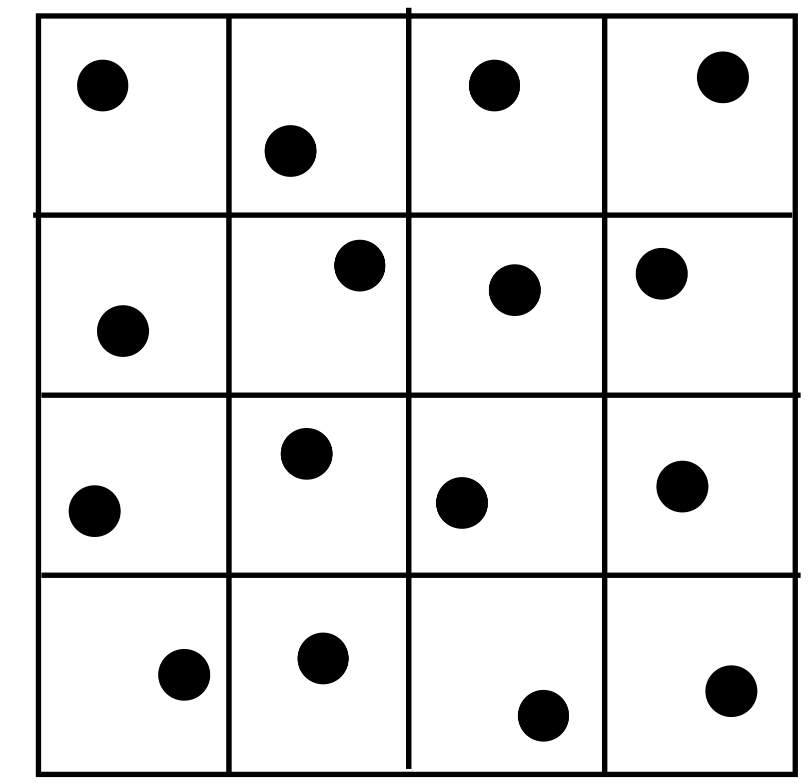
Stratified Sampling

Stratified Sampling

Allocate samples per region

Estimate each region separately

$$F_N = \frac{1}{N}\sum_{i=1}^{N} F_i$$



New variance

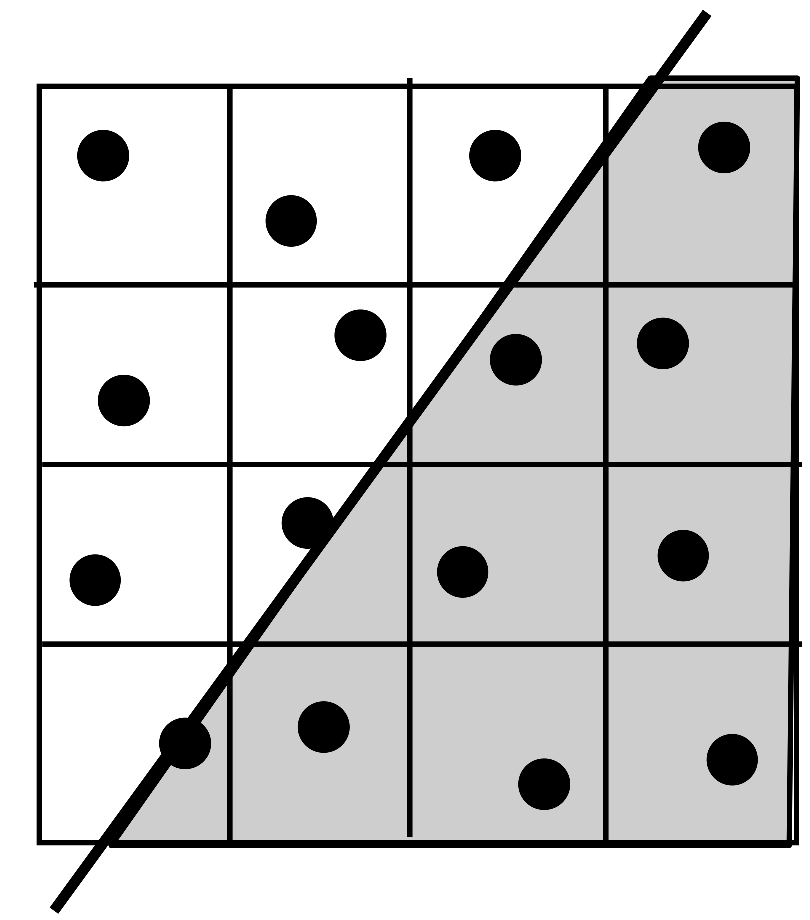
$$V[F_N] = \frac{1}{N^2}\sum_{i=1}^{N} V[F_i]$$

If the variance in each region is the same, then total variance goes as $1/N$

Stratified Sampling

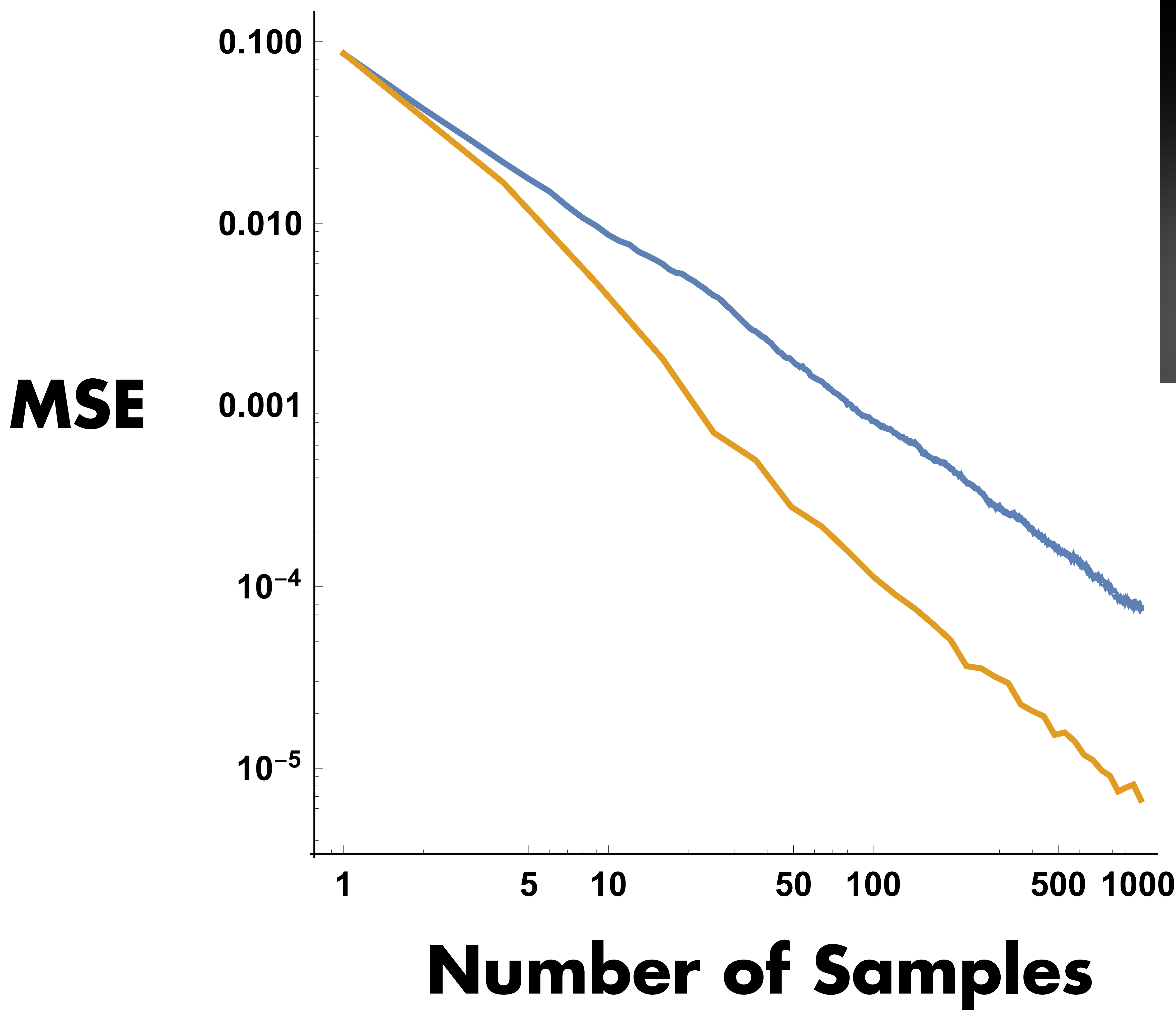
Sample a polygon

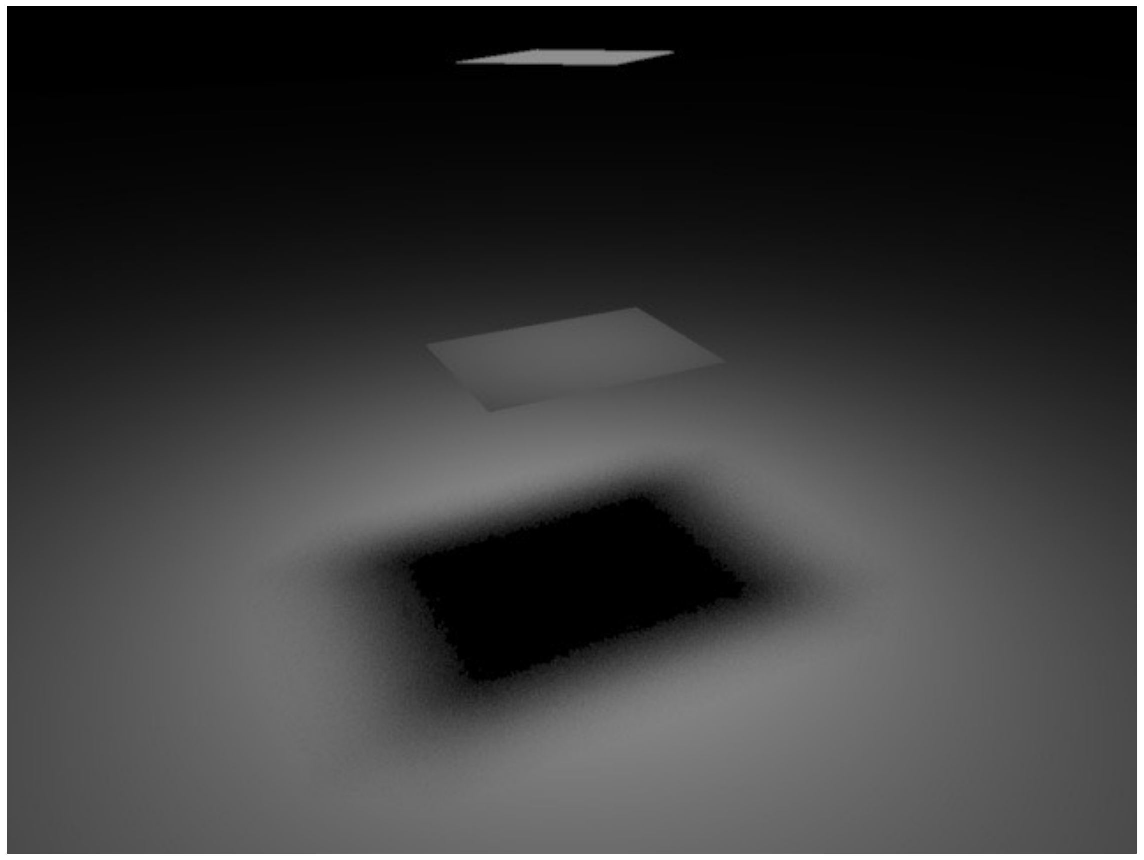
$$V[F_N] = \frac{1}{N^2} \sum_{i=1}^{\sqrt{N}} V[F_j] = \frac{V[F_E]}{N^{1.5}}$$



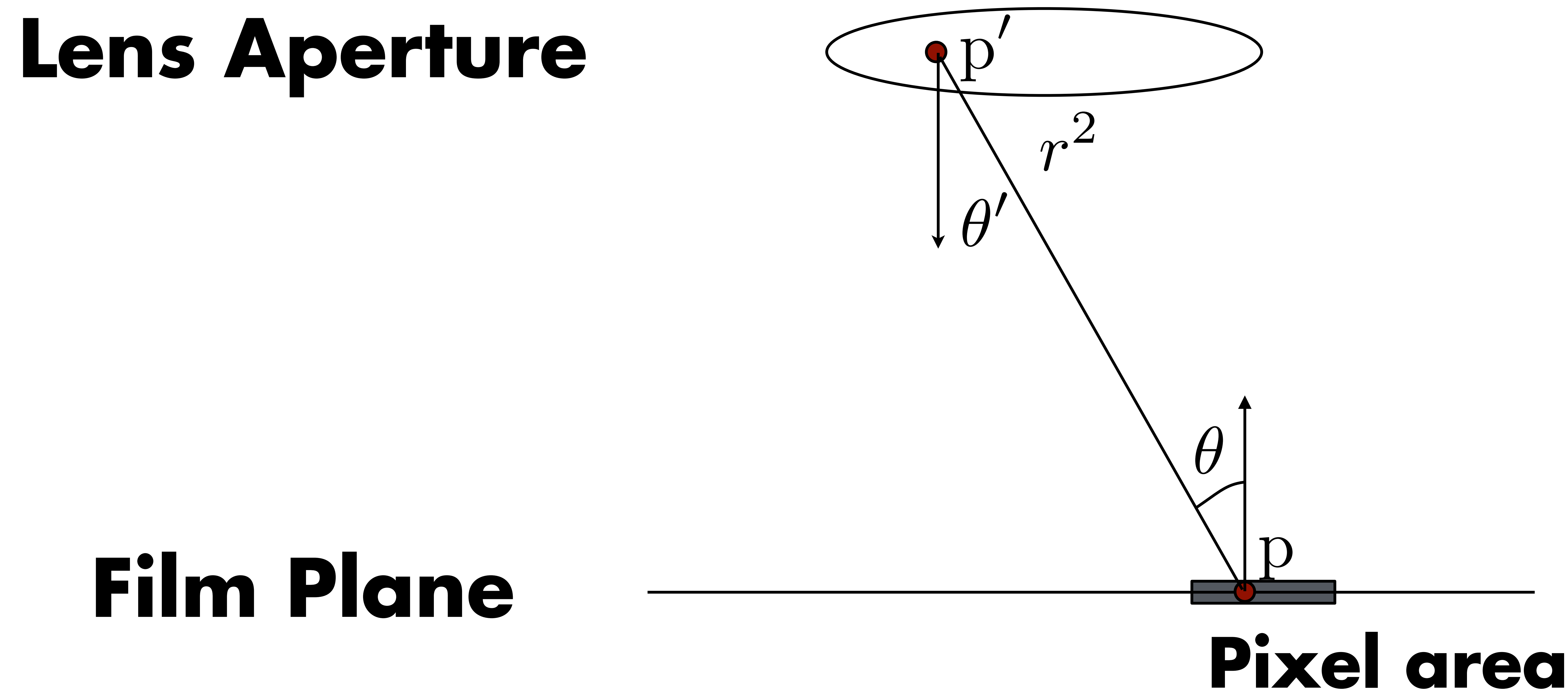
If the variance in some regions are smaller, then the overall variance will be reduced

Mean Squared Error (Quads Scene)





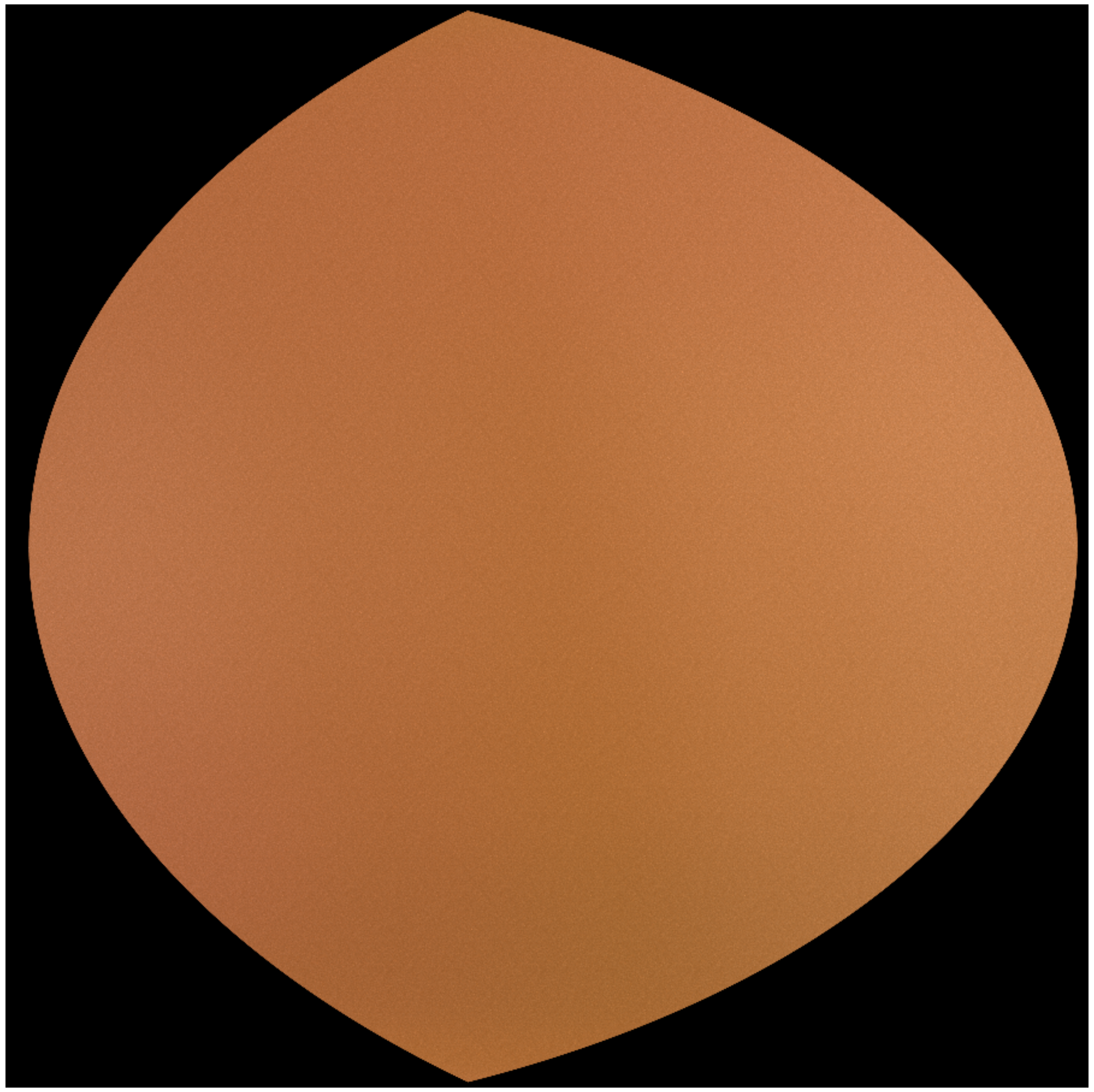
Computing Power at a Pixel



$$E(\mathrm{p}) = \int_{A_{\mathrm{lens}}} L(\mathrm{p}' \to \mathrm{p}) \frac{\cos\theta \cos\theta'}{||\mathrm{p}' - \mathrm{p}||^2} \, \mathrm{d}A'$$

$$W = \int_{A_{\mathrm{pixel}}} \int_{A_{\mathrm{lens}}} L(\mathrm{p}' \to \mathrm{p}) \frac{\cos\theta \cos\theta'}{||\mathrm{p}' - \mathrm{p}||^2} \, \mathrm{d}A' \, \mathrm{d}A$$

View of Lens from Pixel (Exit Pupil)



In Focus

Out of Focus

How to Stratify?

n strata in d dimensions: $O(n^d)$

- **8 strata in 4 dimensions: 4096 samples!**

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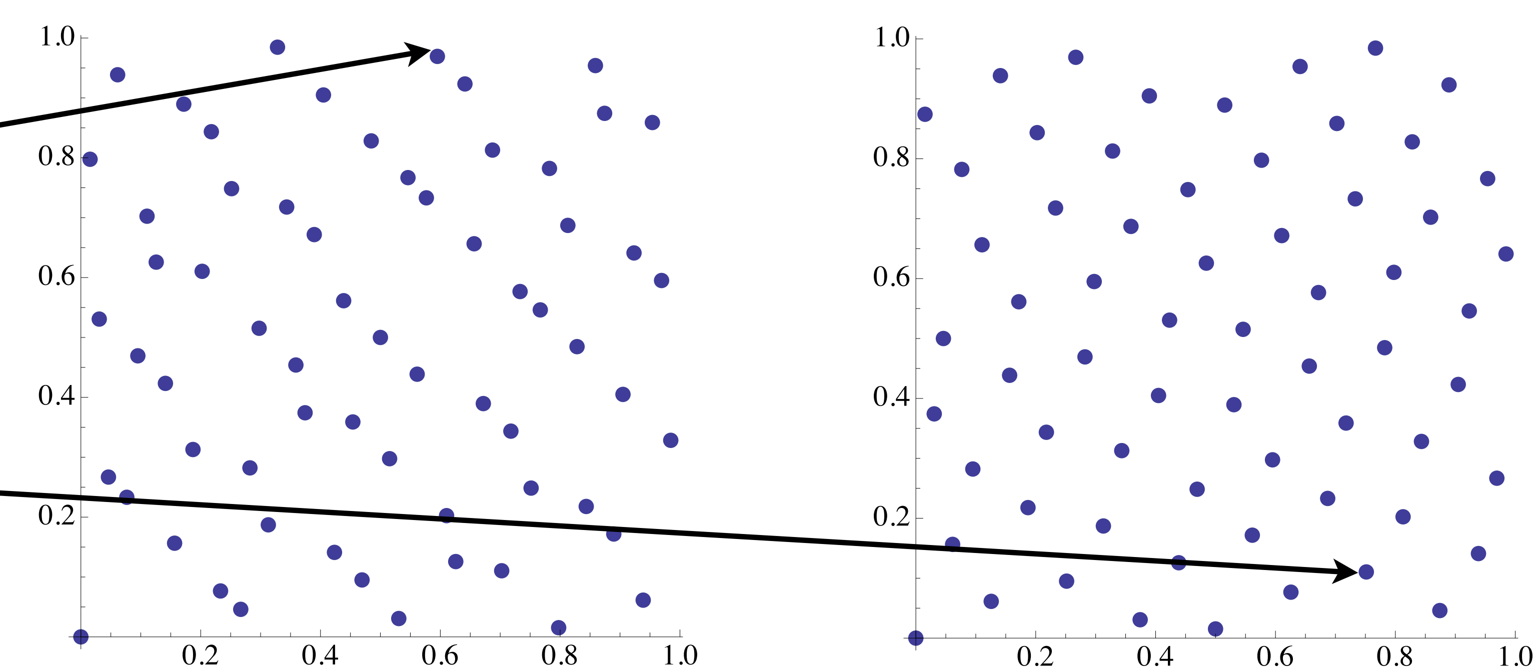
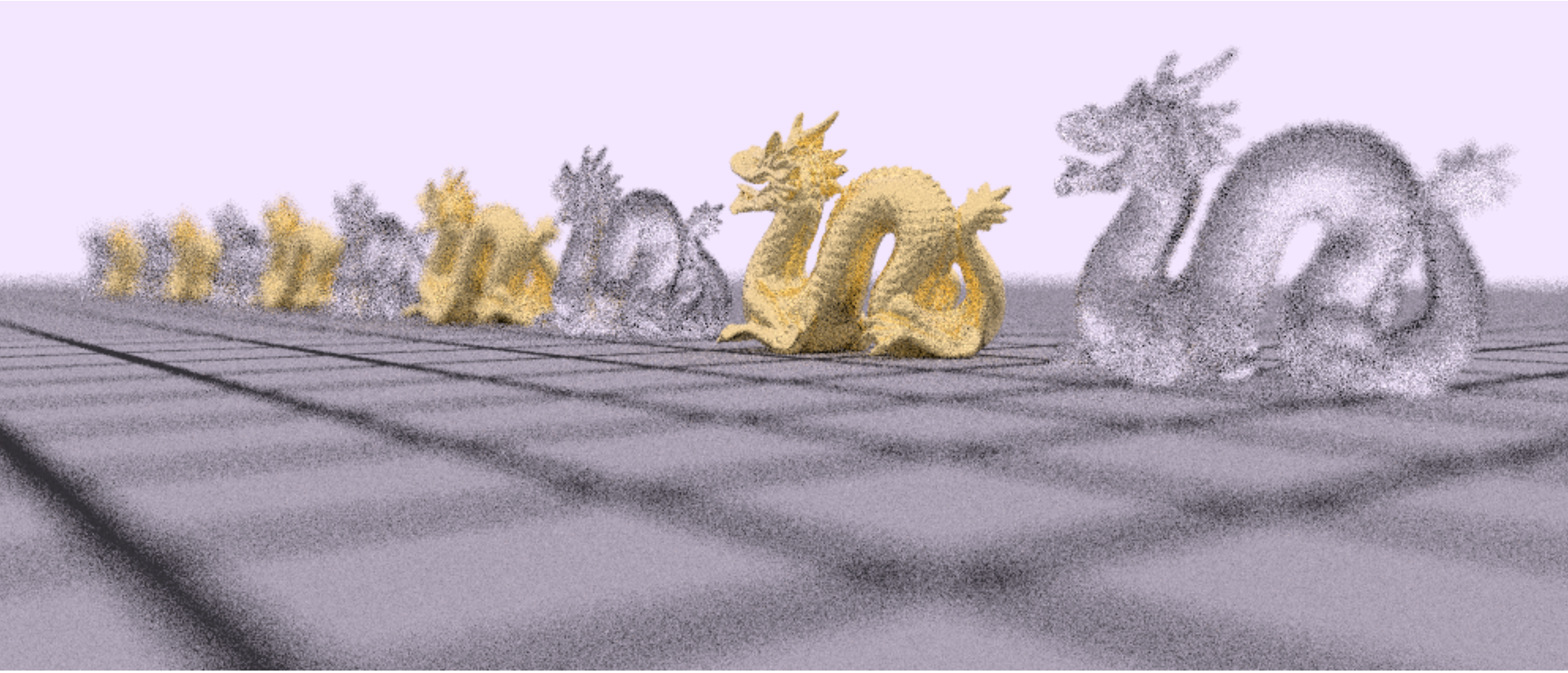
One solution: padding

- Generate stratified lower dimensional point sets, randomly associate pairs of samples

(p_0, p_1, p_2, p_3)



Independent Random Sampling



2x Padded 2D Stratified Sampling