Next up!!!

Monte Carlo

one really bad joke

Monte Carlo 1: Integration

Previous lecture:

- Radiometry & Photometry
- Light Fields
- Theme: *analytical* illumination formulae

This lecture: Monte Carlo Integration

- Review random variables and probability
- Sampling from distributions
- Sampling from shapes
- Numerical calculation of illumination

Further Reading: PBRT Book

Chapter 13: Monte Carlo Integration

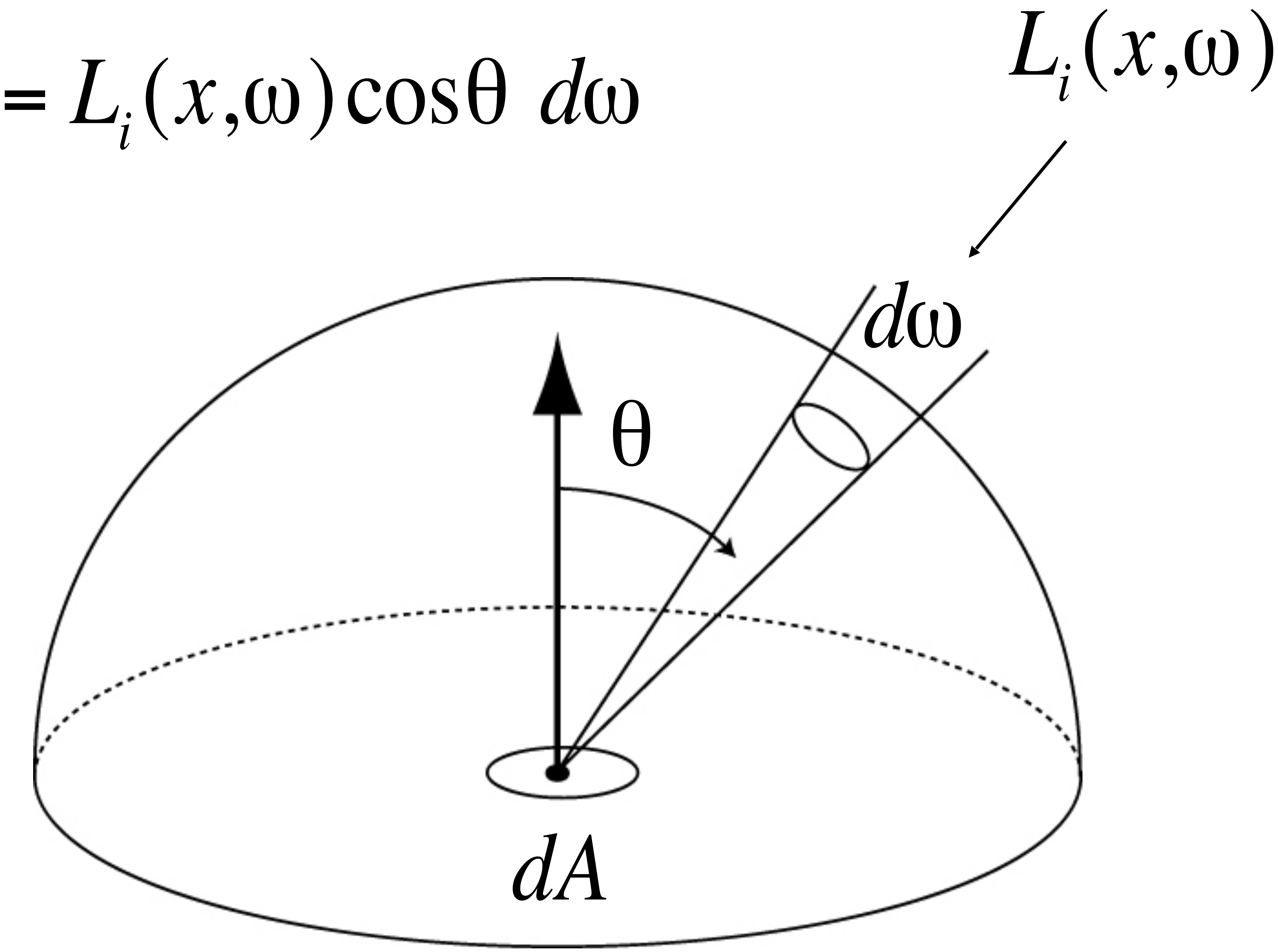
- Skip 13.4 Metropolis Sampling

Motivation:
Computing Irradiance

Irradiance from the Environment

$$d^2\Phi_i(x,\omega) = L_i(x,\omega)\cos\theta \, dA \, d\omega$$

$$dE(x,\omega) = L_i(x,\omega)\cos\theta \; d\omega$$



$$E(x) = \int_{H^2} L_i(x,\omega)\cos\theta \; d\omega$$

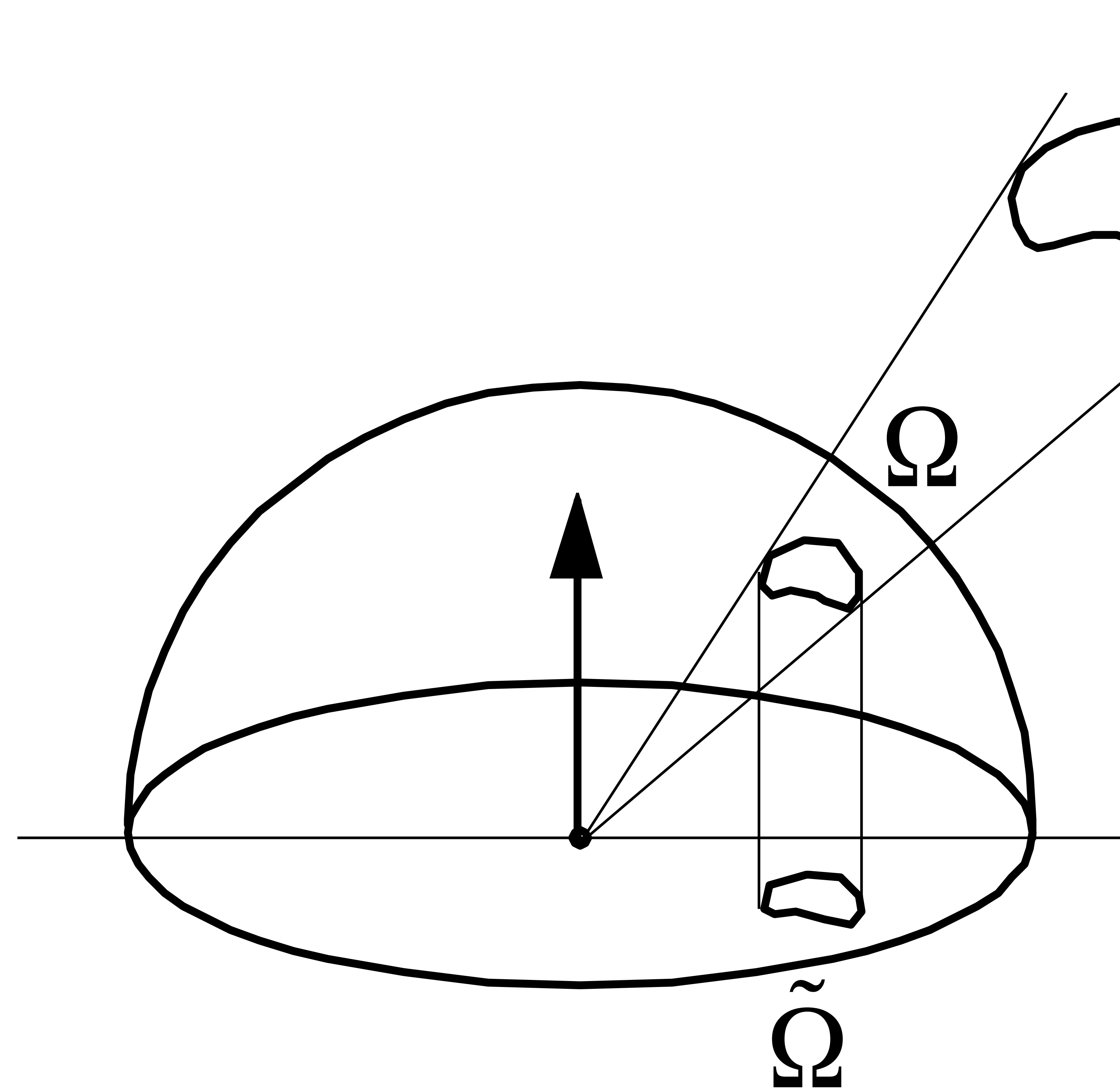
Irradiance from a Uniform Area Source

Constant

$$E(x) = \int_{H^2} L \cos\theta \, d\omega$$

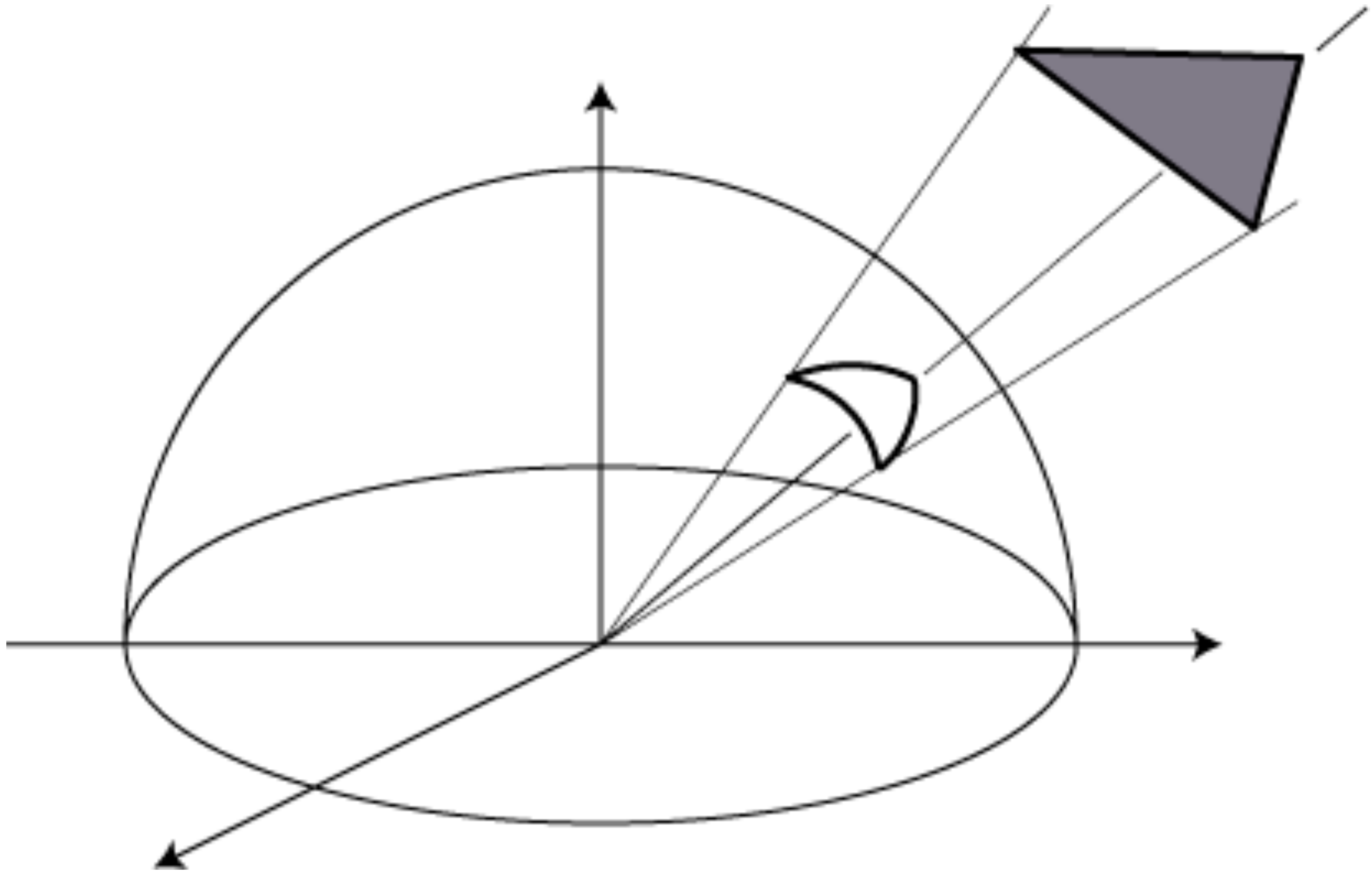
$$= L \int_{\Omega} \cos\theta \, d\omega$$

$$= L\tilde{\Omega}$$

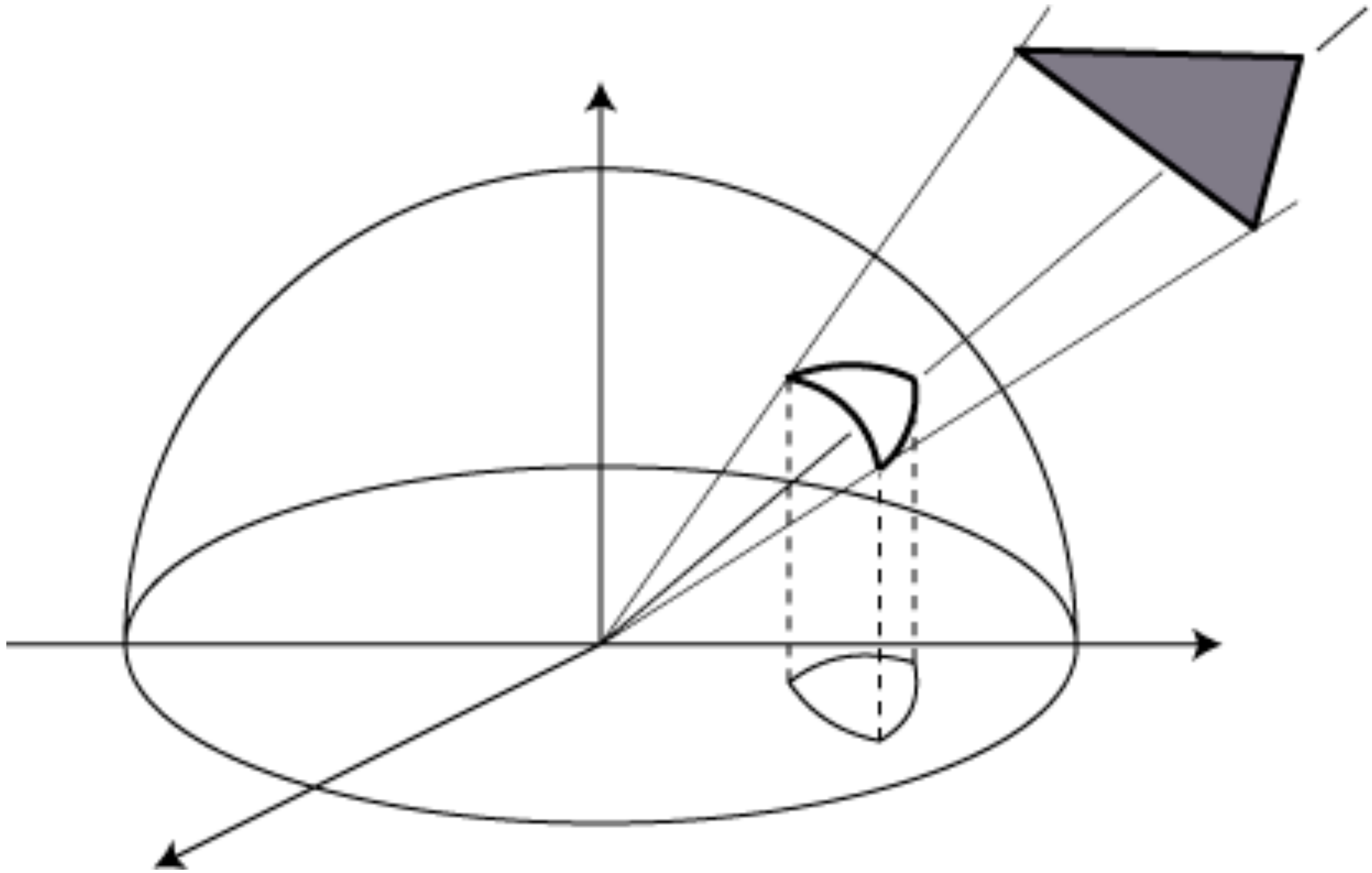


Direct Illumination

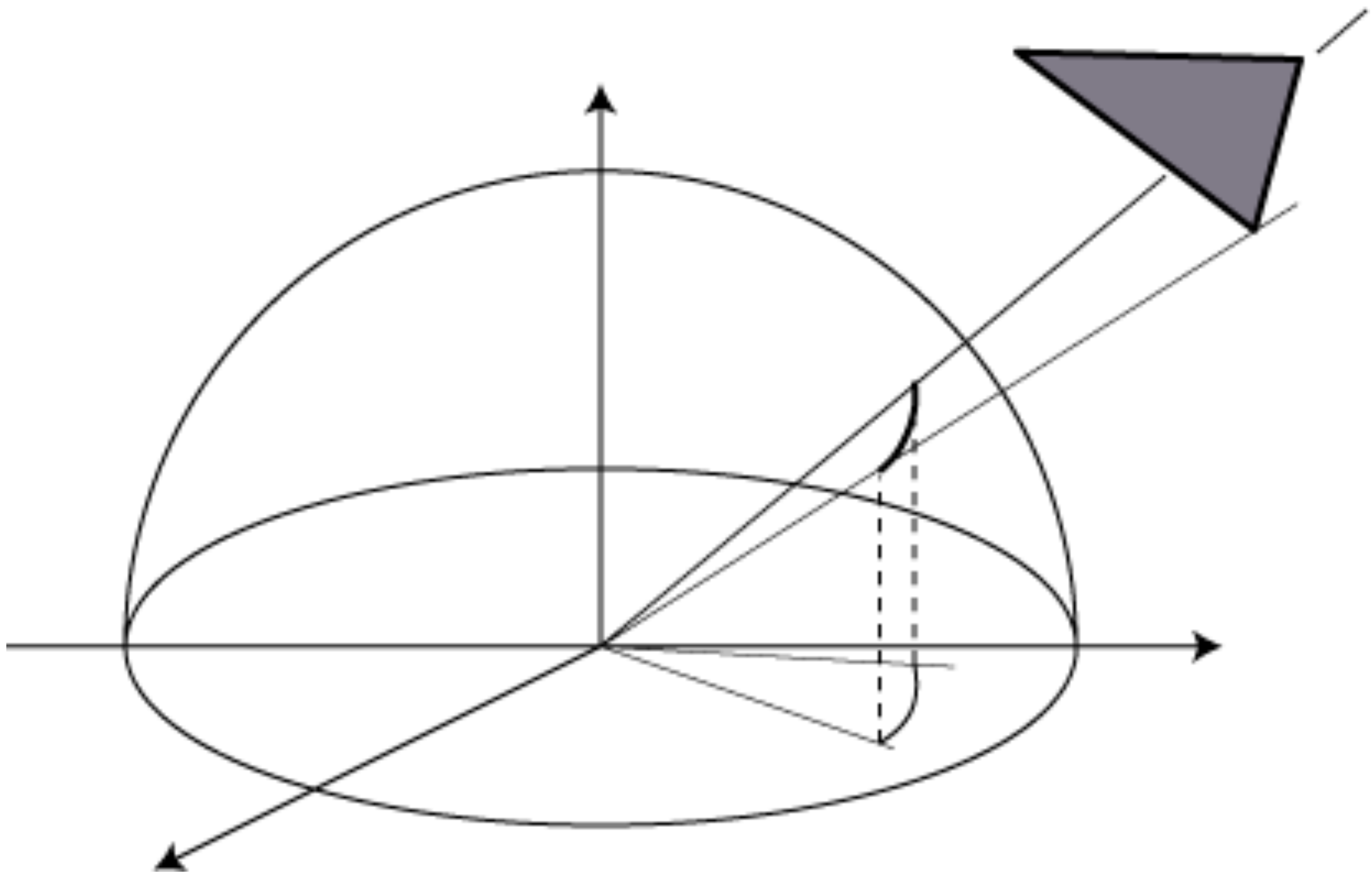
Uniform Triangle Source



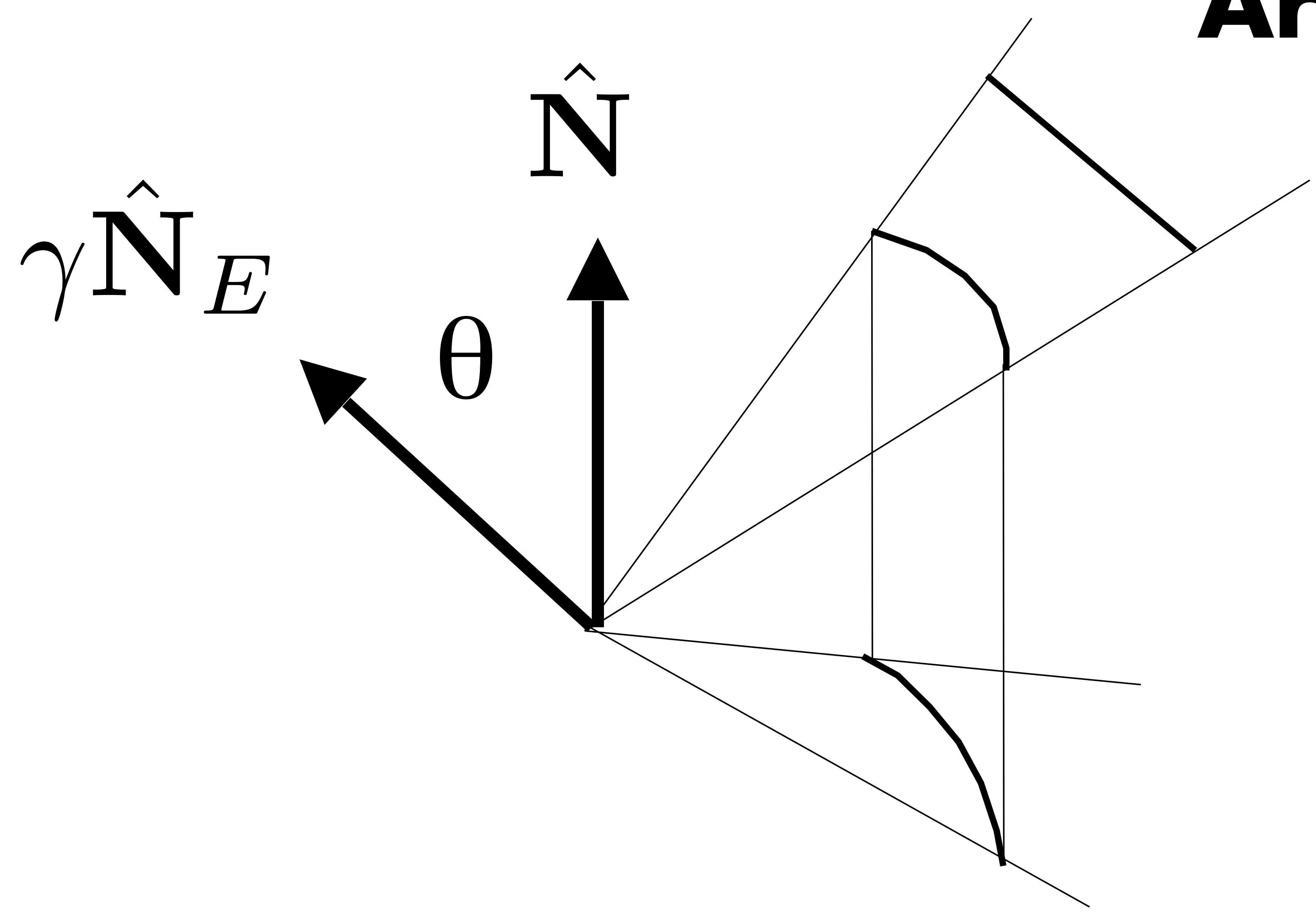
Uniform Triangle Source



Uniform Triangle Source



Consider 1 Edge

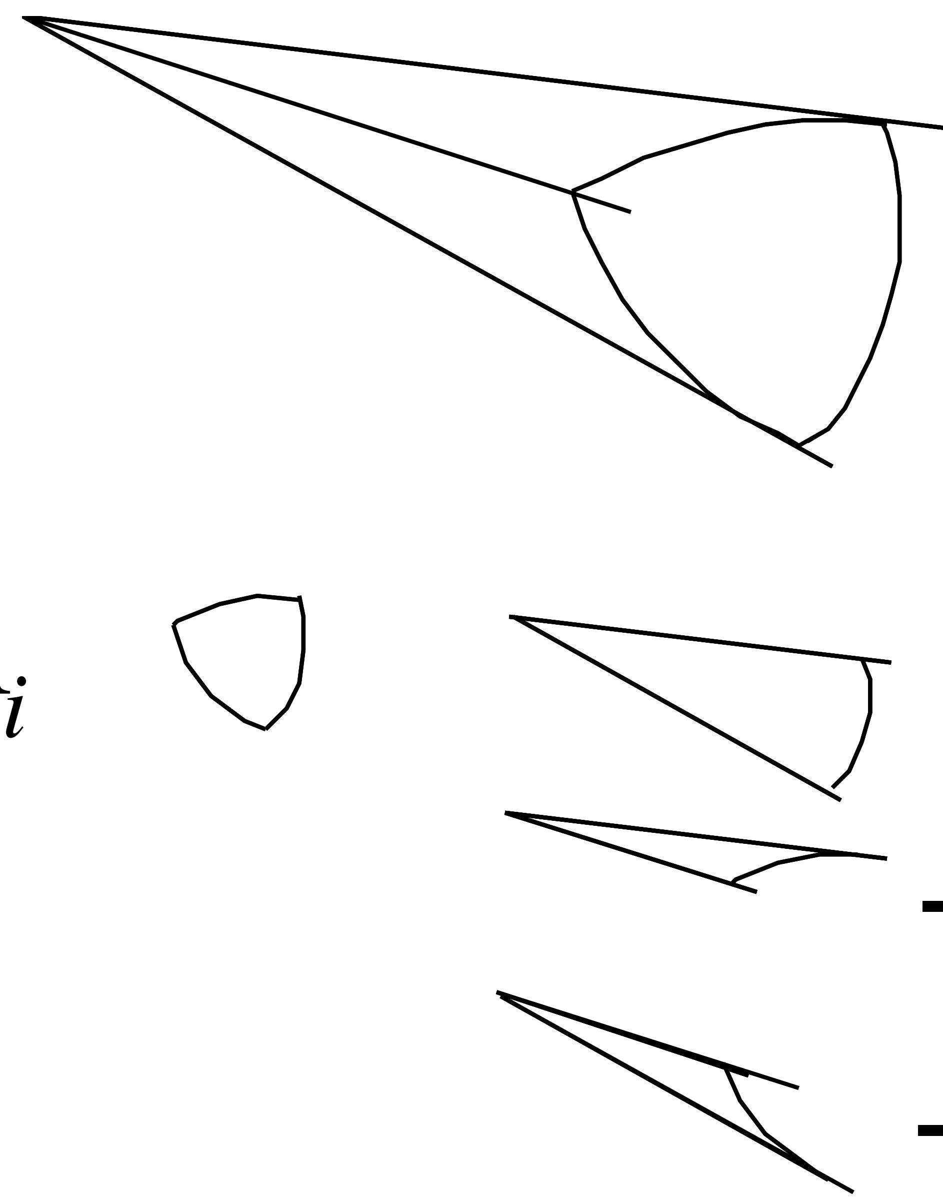


Area of sector γ

Projected area of sector

$$A = \gamma \cos\theta = \gamma \vec{\mathbf{N}}_E \bullet \vec{\mathbf{N}}$$

Lambert's Formula



$$\sum_{i=1}^{3} A_i$$

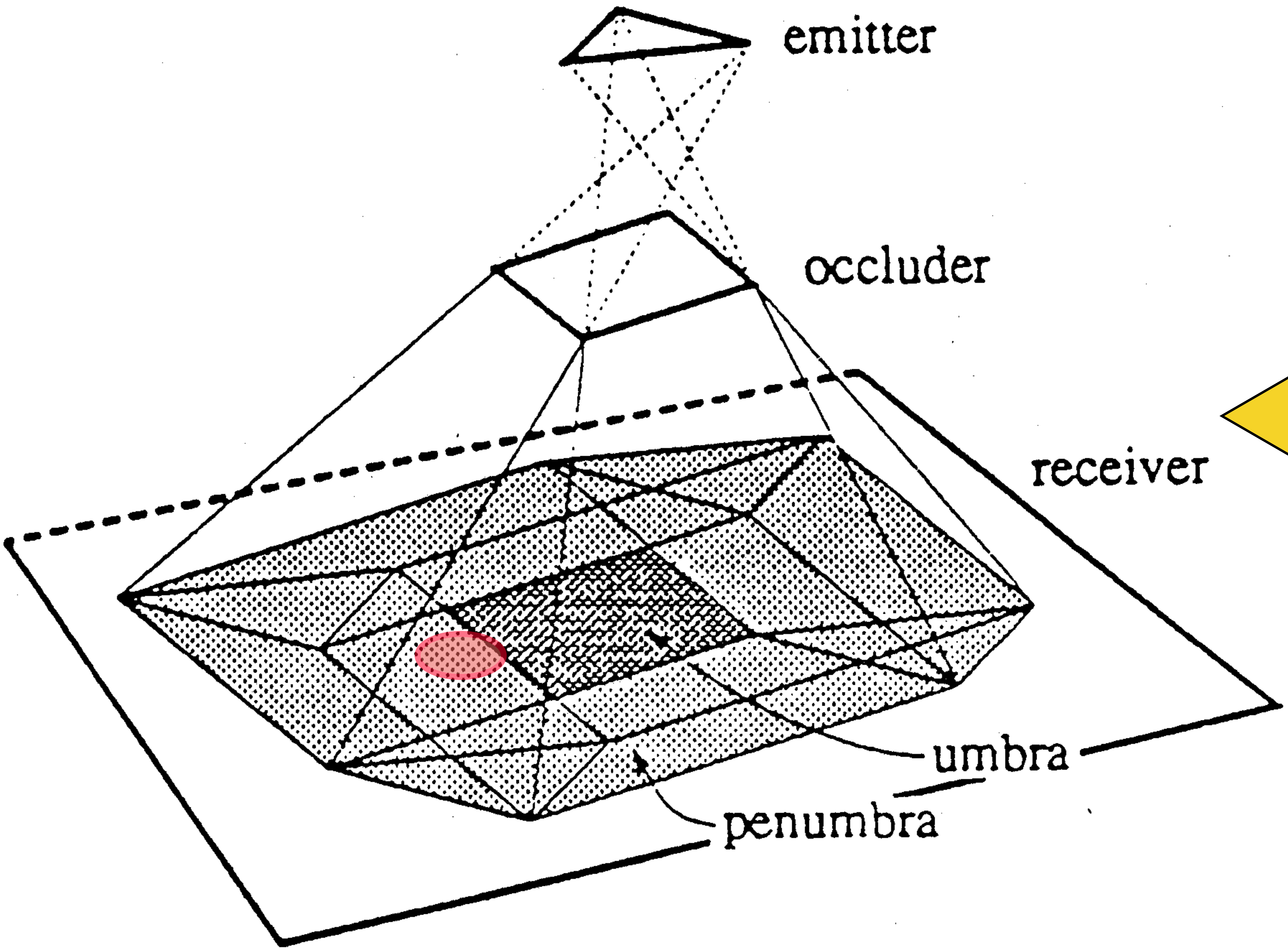
$$|A_1|$$

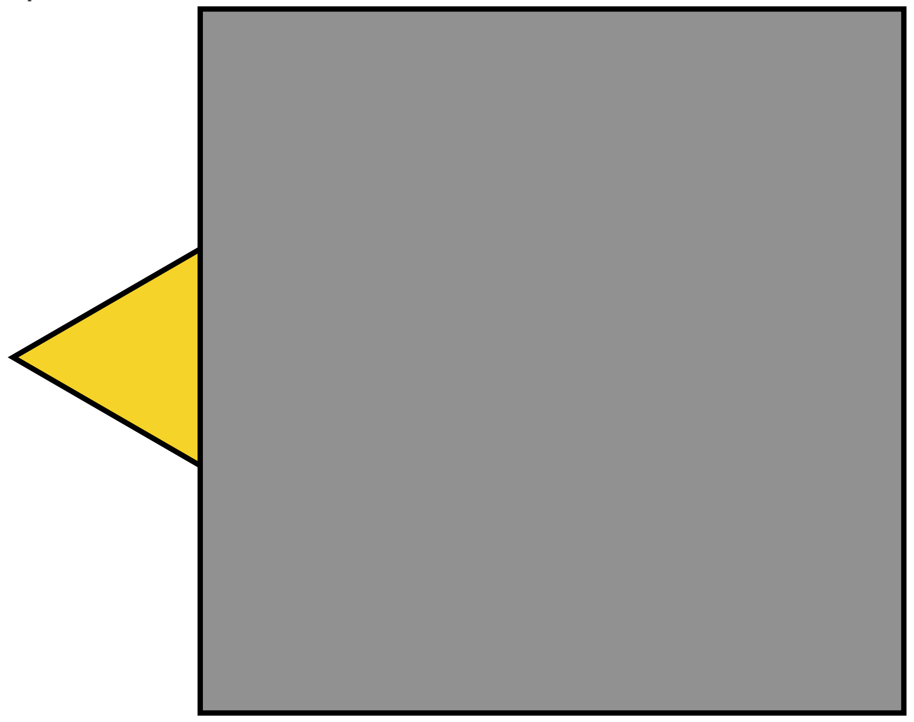
$$-|A_2|$$

$$-|A_3|$$

$$\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \gamma_i \, \vec{\mathbf{N}}_i \bullet \vec{\mathbf{N}}$$

Penumbras and Umbras



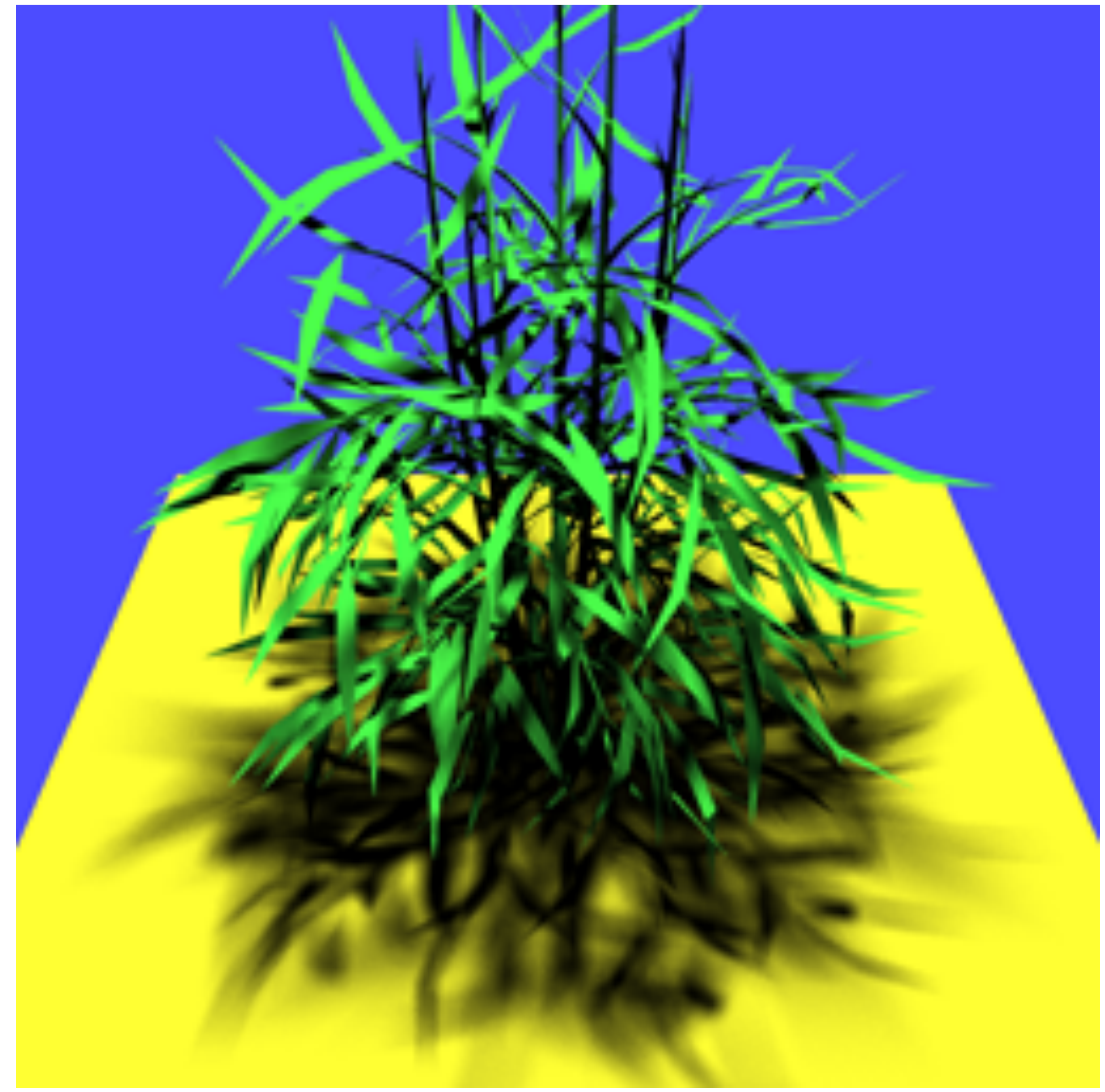


Lighting and Soft Shadows

$$E(x) = \int_{H^2} L_i(x,\omega)\cos\theta \, d\omega$$

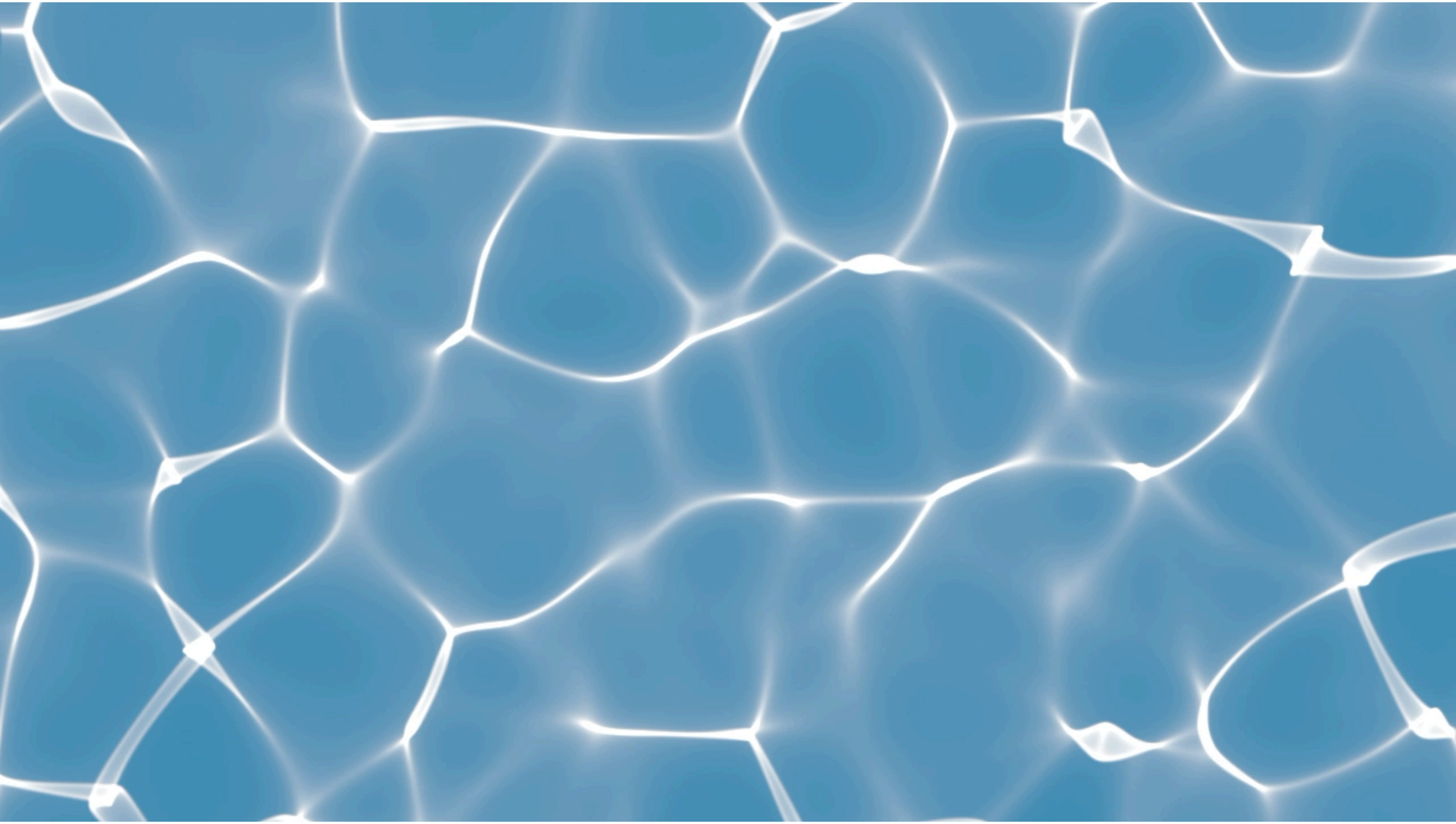
Challenges:

1. **Occluders**
   - **Complex geometry**
   - **Number of occluders**
2. **Non-uniform light sources**



Agrawala, Ramamoorthi, Heirich, Moll, 2000

Example: Non-uniform light sources



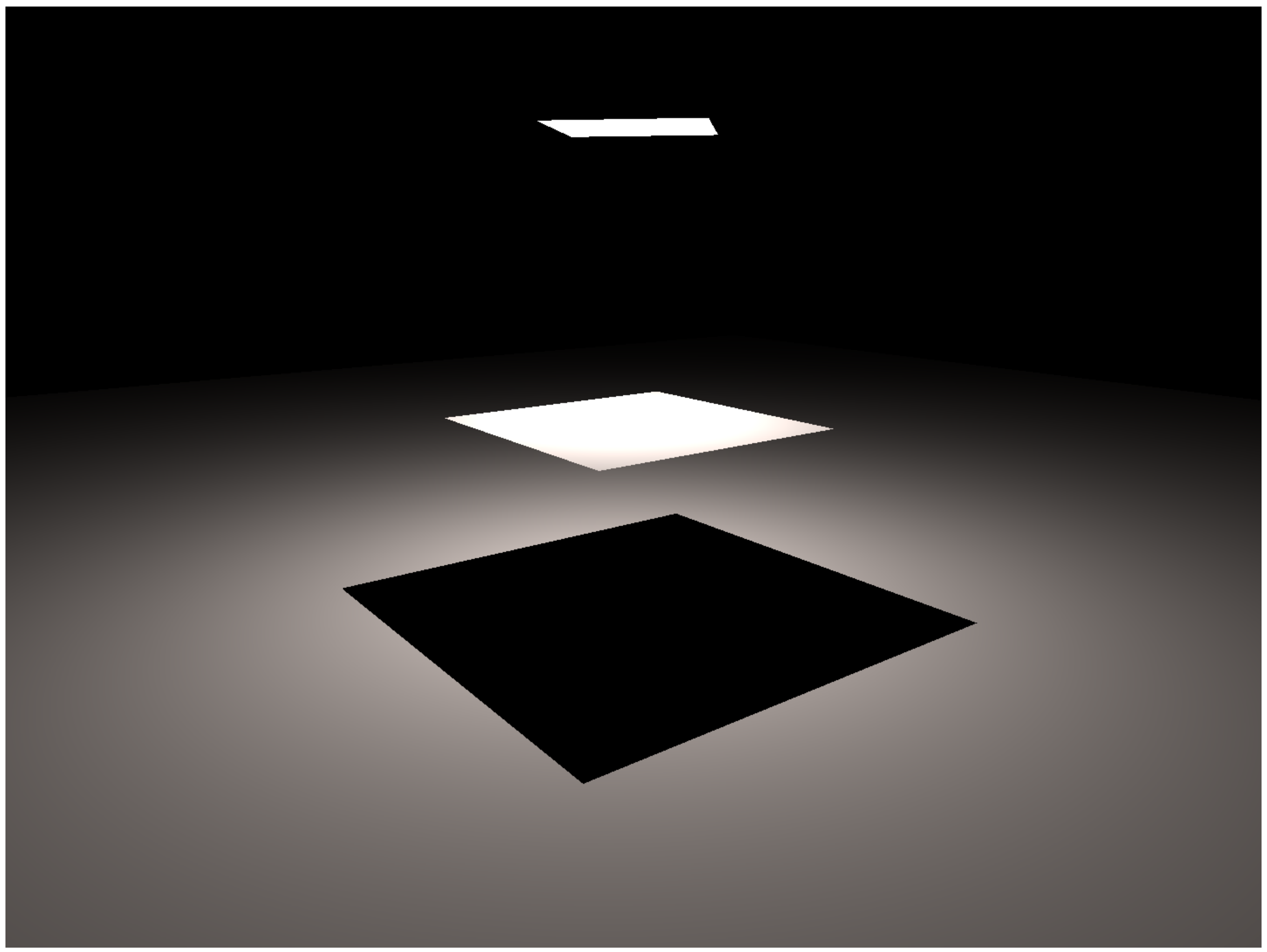
Example: Non-uniform light sources

https://digistatement.com/new-rtx-dlss-games-list-for-2022

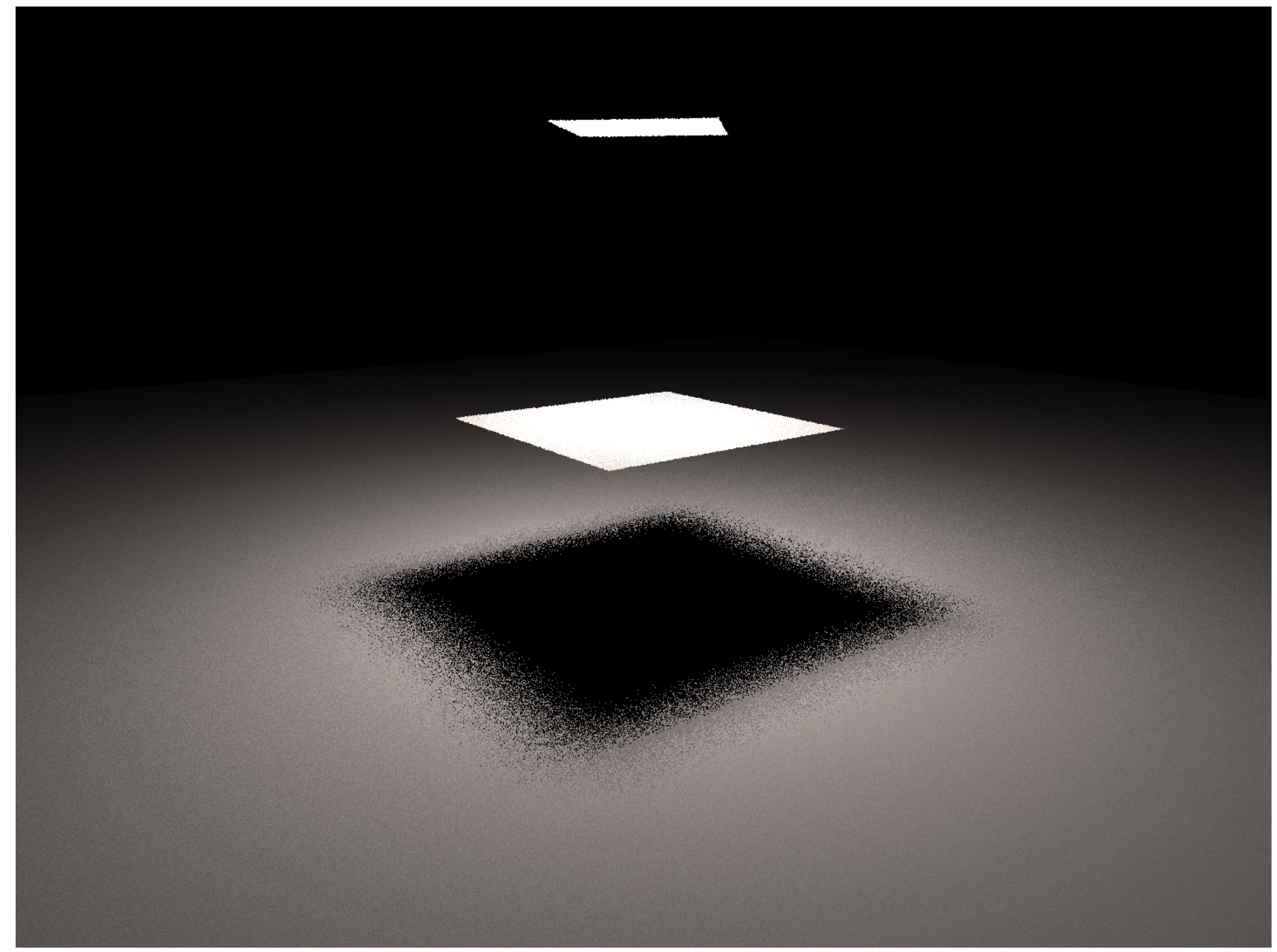
Monte Carlo Integration

Monte Carlo Illumination Calculation

$$E(x) = \int_{H^2} L_i(x,\omega)\cos\theta \, d\omega$$



Center



Random

1 shadow ray per eye ray

Monte Carlo Algorithms

Advantages

- Easy to implement
- Easy to think about (but be careful of subtleties)
- Robust when used with complex integrands (lights, BRDFs) and domains (shapes)
- Efficient for high-dimensional integrals
- Efficient when only need solution at a few points

Disadvantages

- Noisy
- Slow (many samples needed for convergence)

Random Variables

X **is a random variable**

A random variable takes on different values

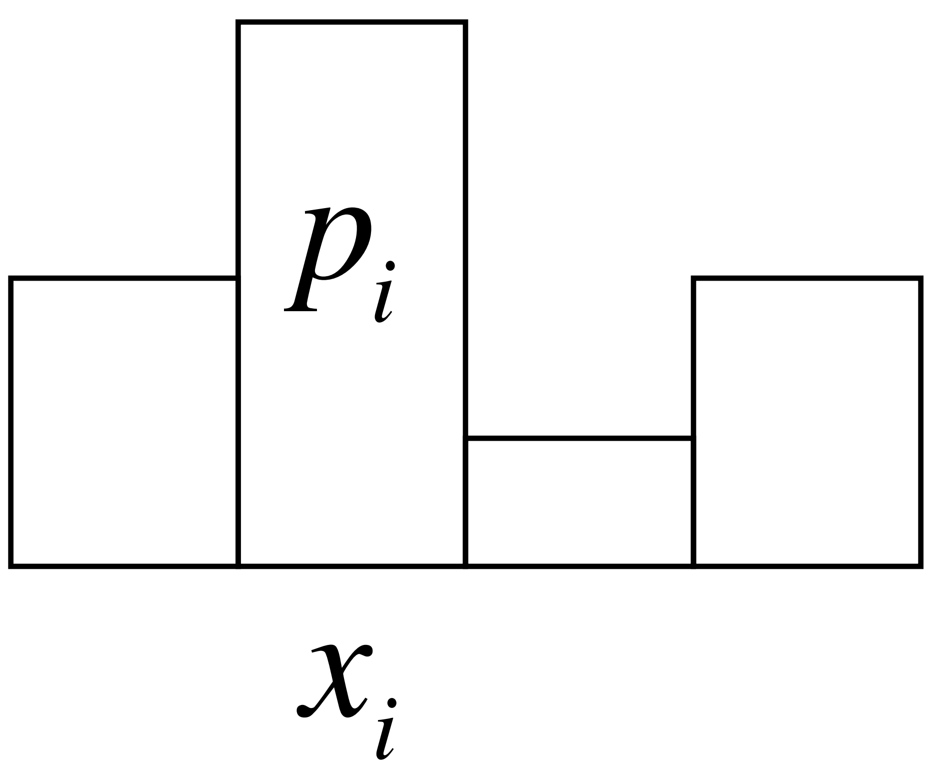
(represents a distribution of potential values)

$X \sim p(x)$ **probability distribution function (PDF)**

Discrete Probability Distributions

Discrete values x_i

with probability p_i

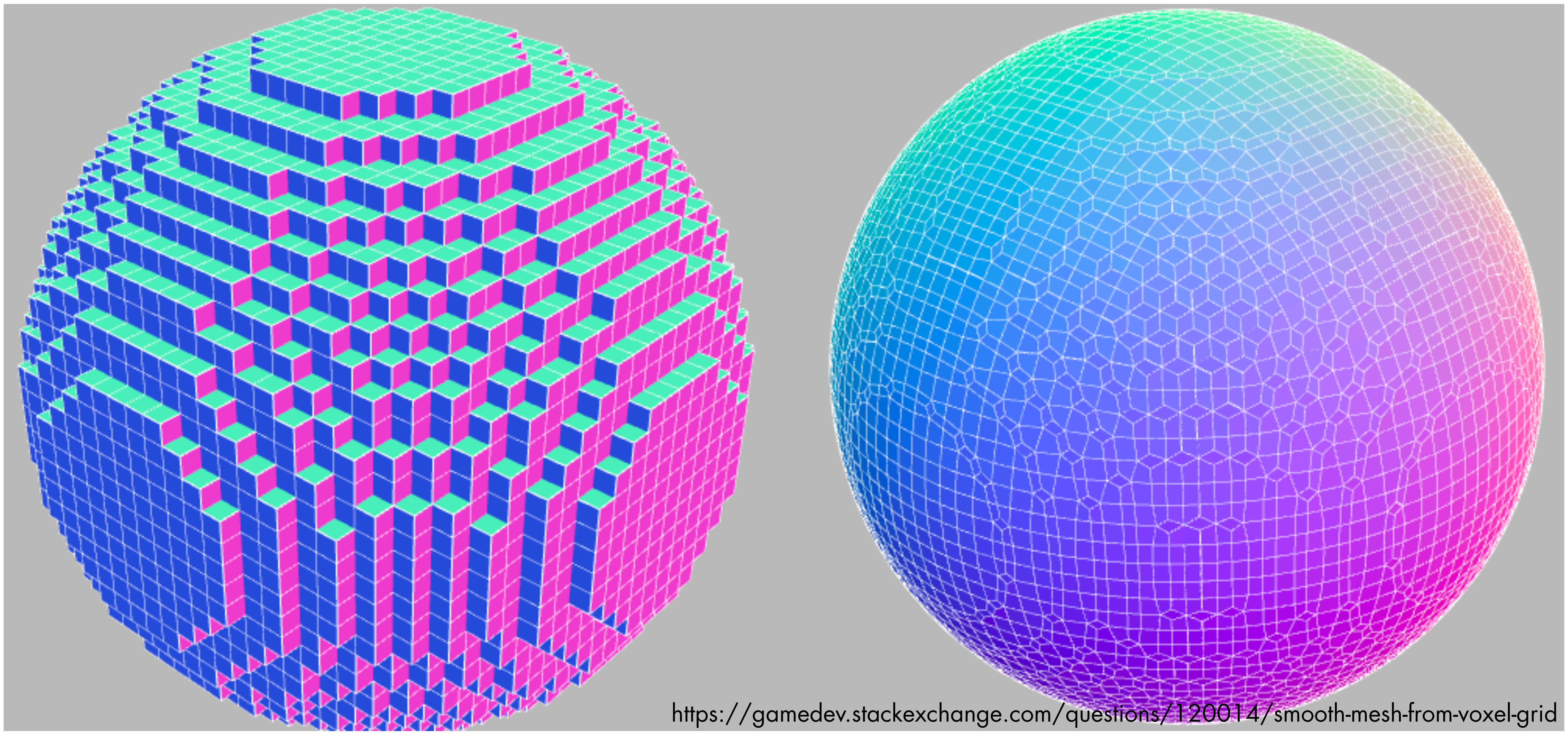


$$p_i \geq 0$$

$$\sum_{i=1}^{n} p_i = 1$$

Motivation:

Sampling polygons uniformly based on area



Equal probability

$$p_i = \frac{1}{n}$$

Nonequal probability

$$p_i = \frac{A_i}{A_{tot}}$$

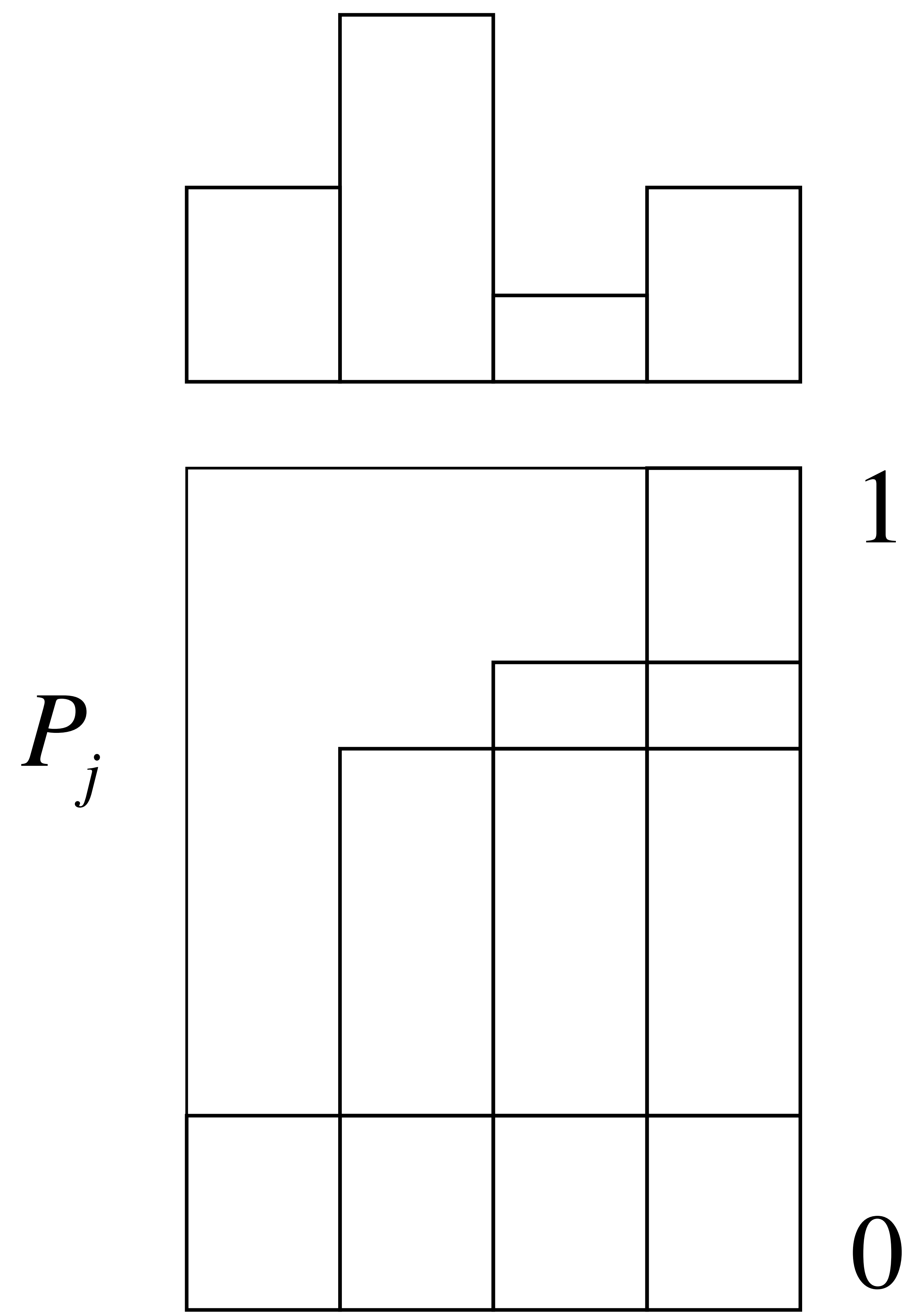
Discrete Probability Distributions

Cumulative PDF (CDF) P_j

$$P_j = \sum_{i=1}^{j} p_i$$

$$0 \leq P_i \leq 1$$

$$P_n = 1$$



Discrete Probability Distributions

Construction of samples

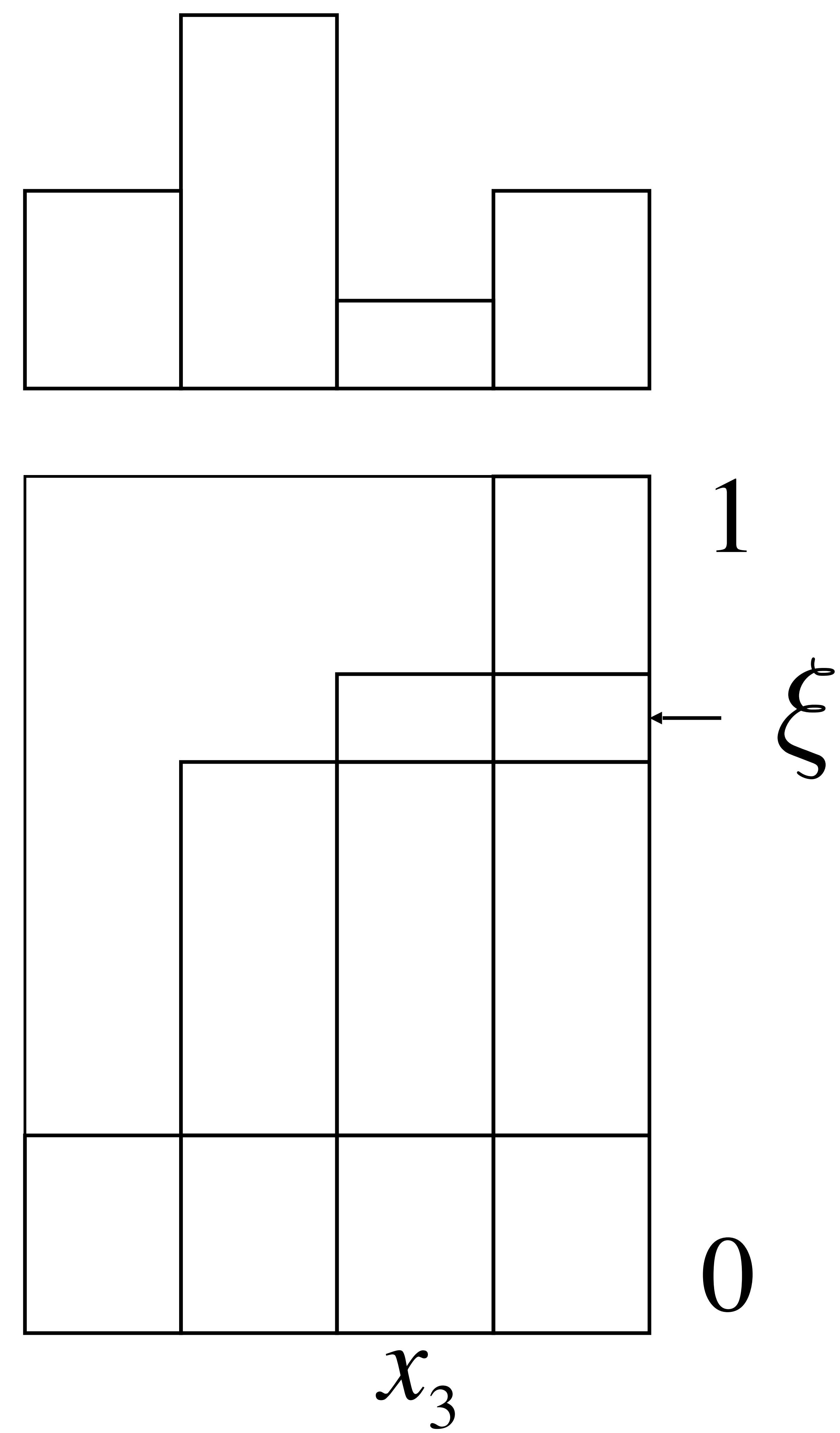
To randomly select an event,

Select x_i if

$$P_{i-1} < \xi \leq P_i$$

↑

Uniform random variable

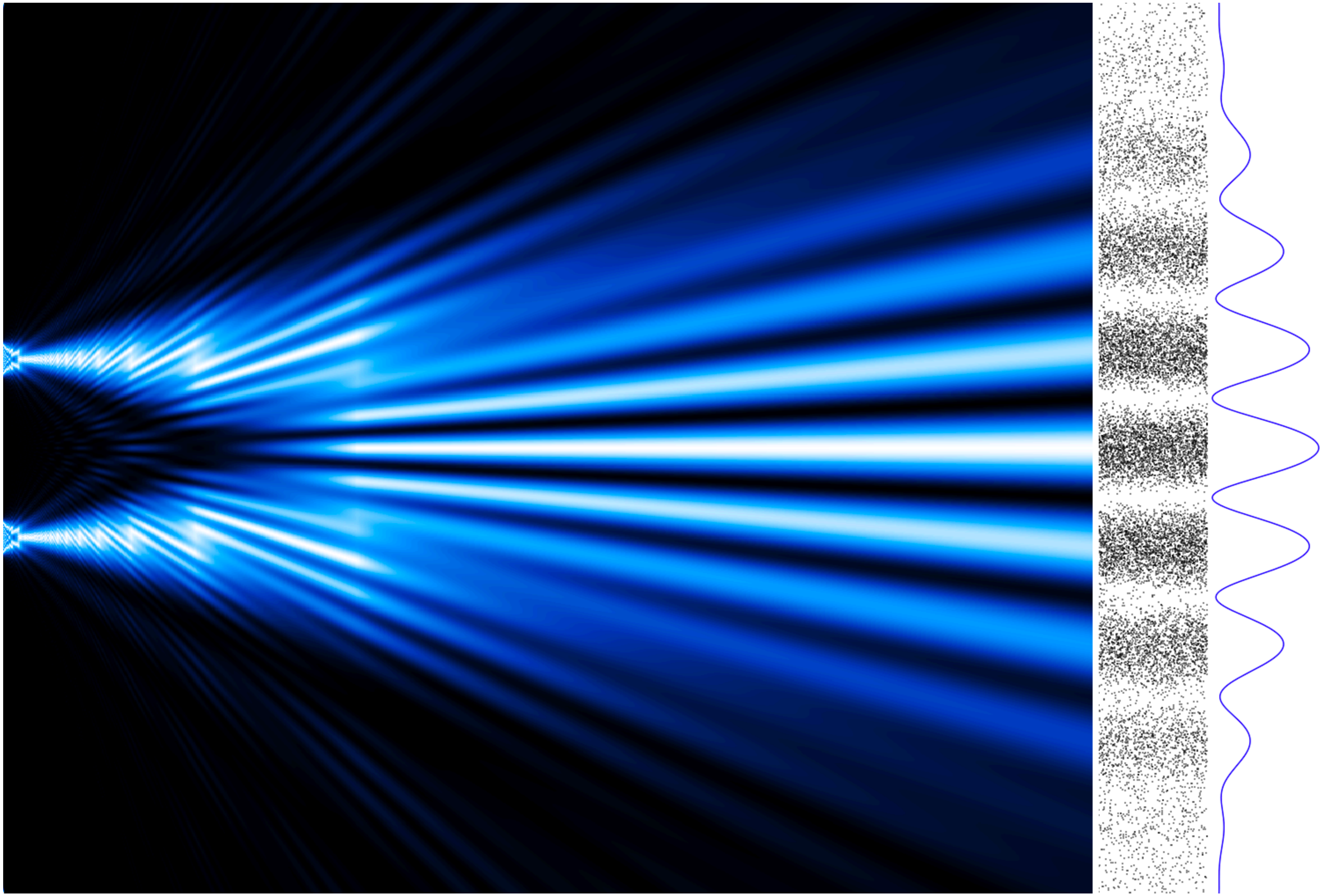


Example: Sampling a texture map

Monte Carlo

Example: Sampling a texture map

Motivation:
Continuous Probability Distributions



https://www.wikiwand.com/en/Double-slit_experiment

Continuous Probability Distributions

PDF $p(x)$

$$p(x) \geq 0$$

Uniform

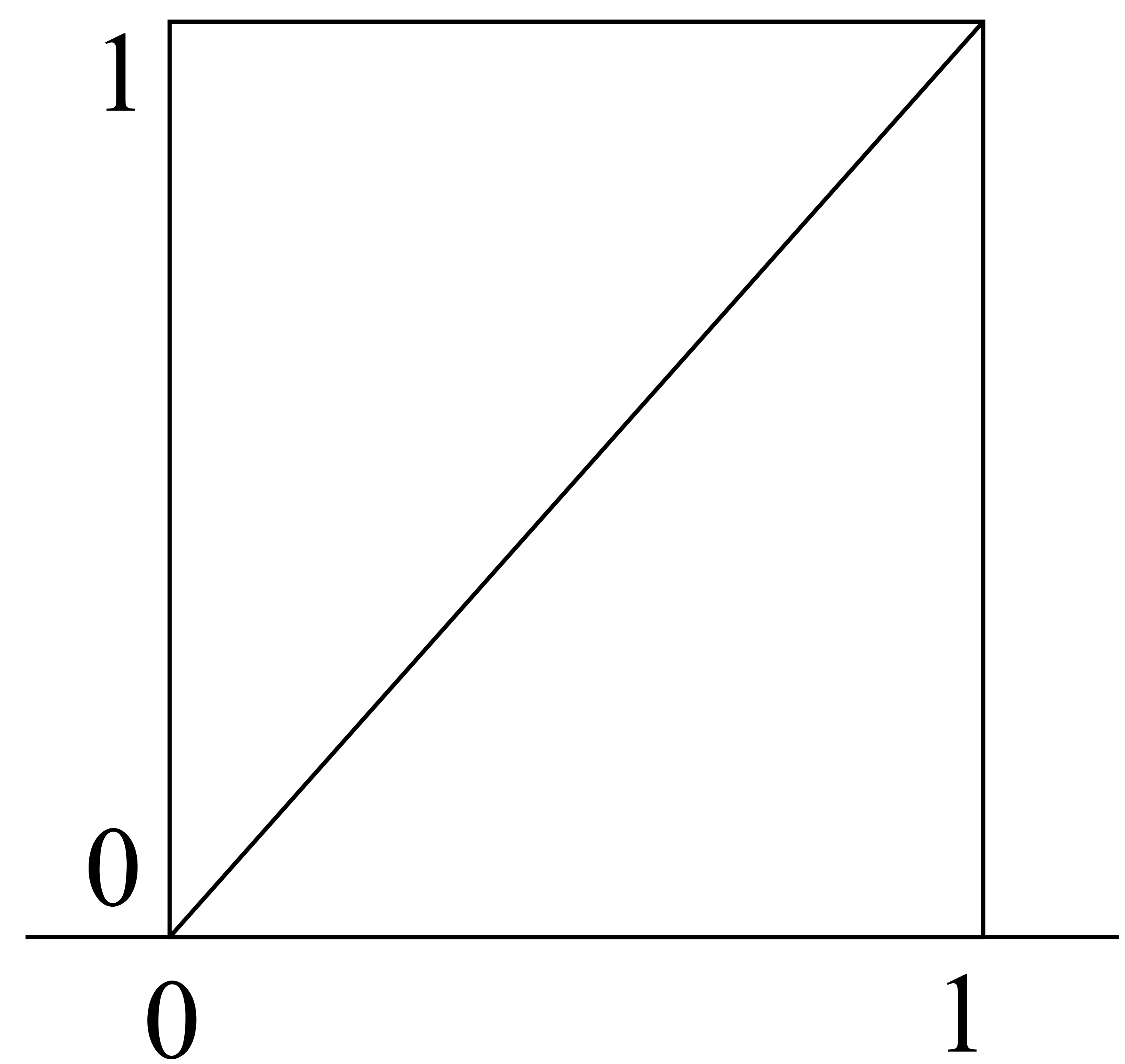
CDF $P(x)$

$$P(x) = \int_0^x p(x)\,dx$$

$$P(x) = \Pr(X < x) \qquad P(1) = 1$$

$$\Pr(\alpha \leq X \leq \beta) = \int_\alpha^\beta p(x)\,dx$$

$$= P(\beta) - P(\alpha)$$



Sampling Continuous Distributions using the *Inversion Method*

Cumulative probability distribution function

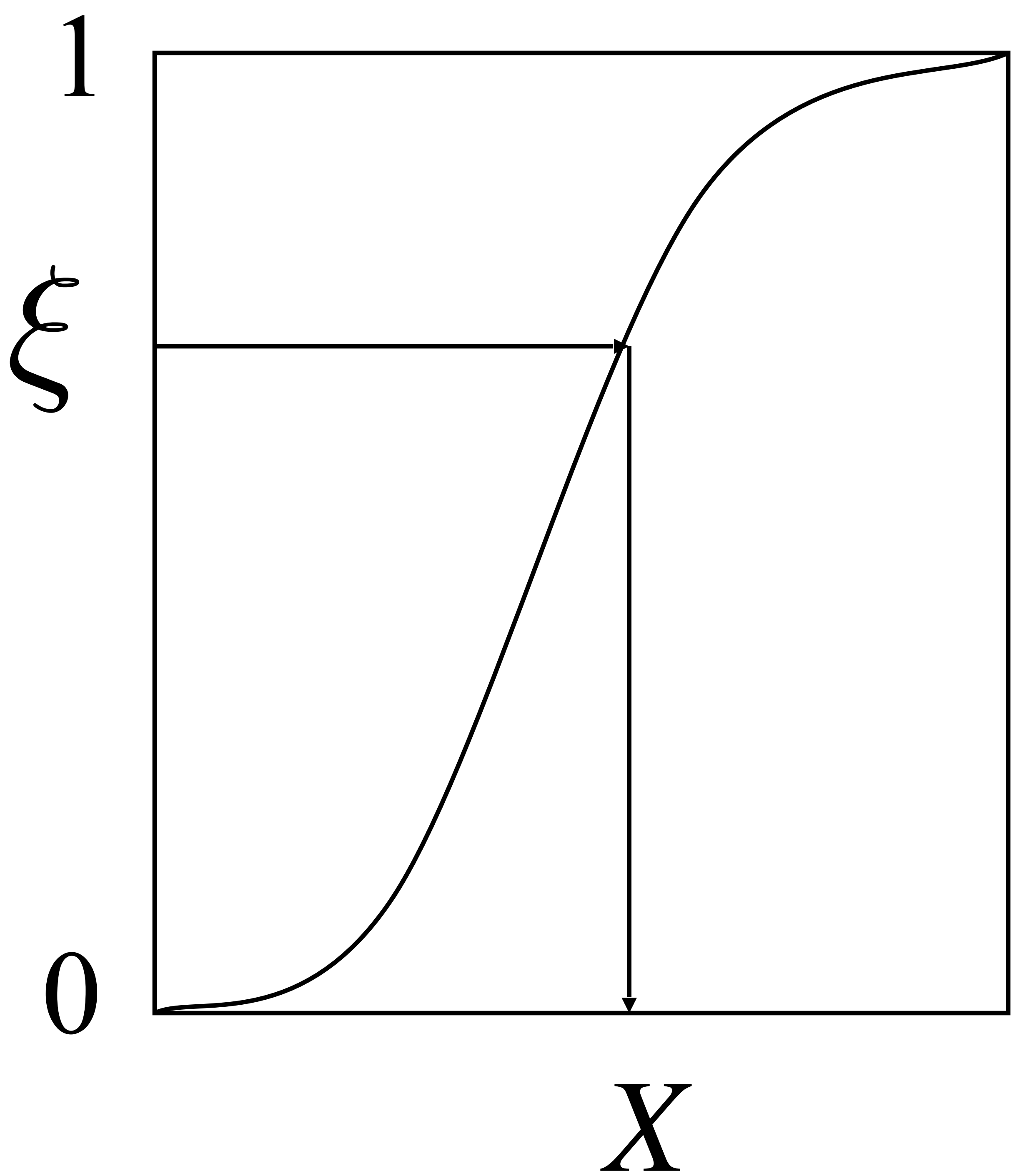
$$P(x) = \Pr(X < x)$$

Construction of samples

Solve for $X = P^{-1}(\xi)$

Must know the formula for:

1. **The integral of** $p(x)$
2. **The inverse function** $P^{-1}(x)$



Example: Power Distribution (p.755)

PDF: $p(x) = cx^n, \quad x \in [0,1)$

$$\int_0^1 cx^n \, \mathrm{d}x = 1$$

$$c \left. \frac{x^{n+1}}{n+1} \right|_0^1 = 1$$

$$\frac{c}{n+1} = 1$$

$$c = n+1.$$

Normalized: $p(x) = (n+1)x^n$

CDF: $P(x) = \int_0^x p(x') \, \mathrm{d}x' = x^{n+1}$

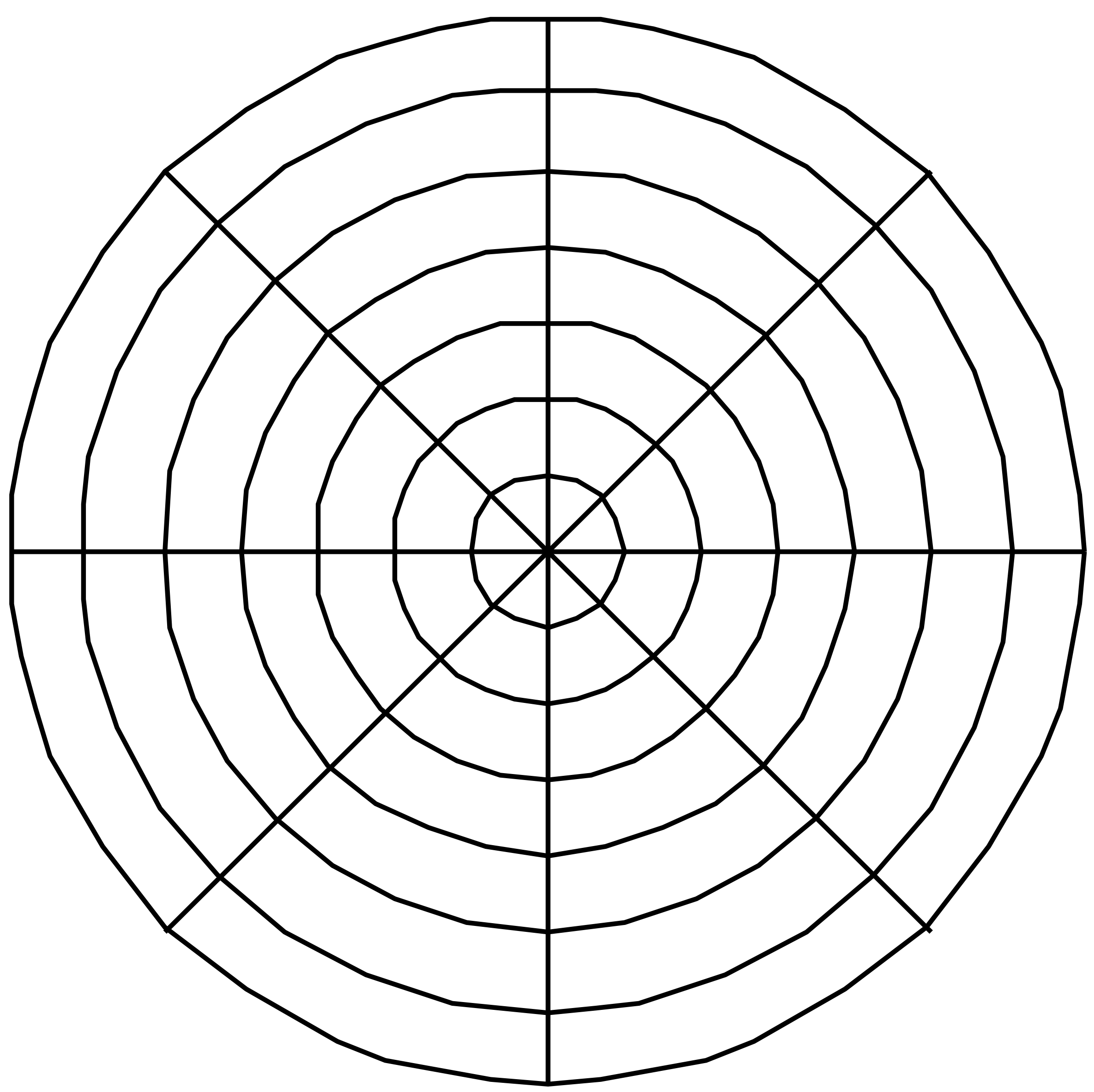
Inverse CDF: $P^{-1}(x) = \sqrt[n+1]{x}$

Sample PDF using: $X = \sqrt[n+1]{\xi}$

uniform random variable ξ

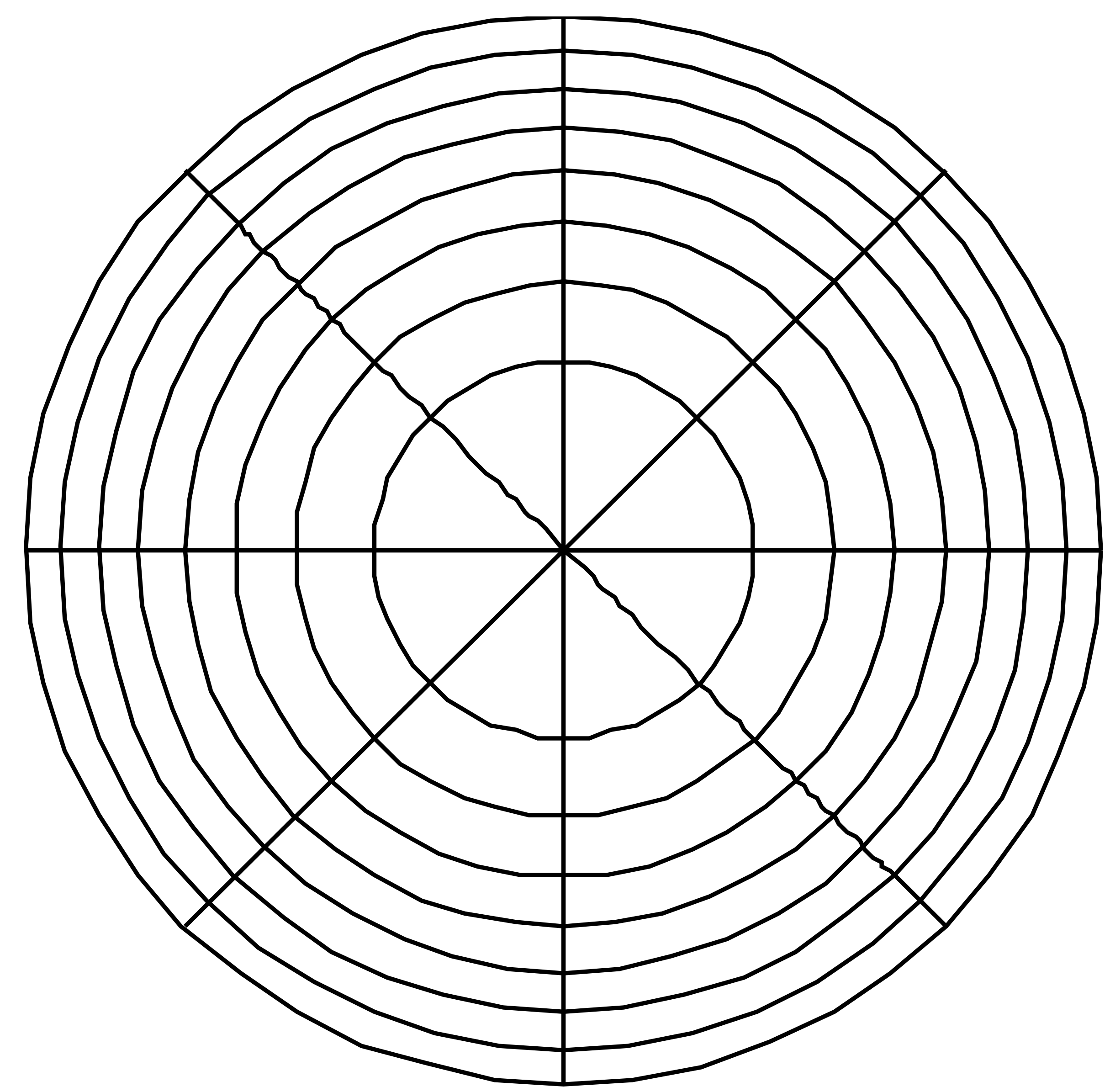
Sampling the Unit Circle

WRONG ≠ Equi-Areal



$$\theta = 2\pi\xi_1$$
$$r = \xi_2$$

RIGHT = Equi-Areal



$$\theta = 2\pi\xi_1$$
$$r = \sqrt{\xi_2}$$

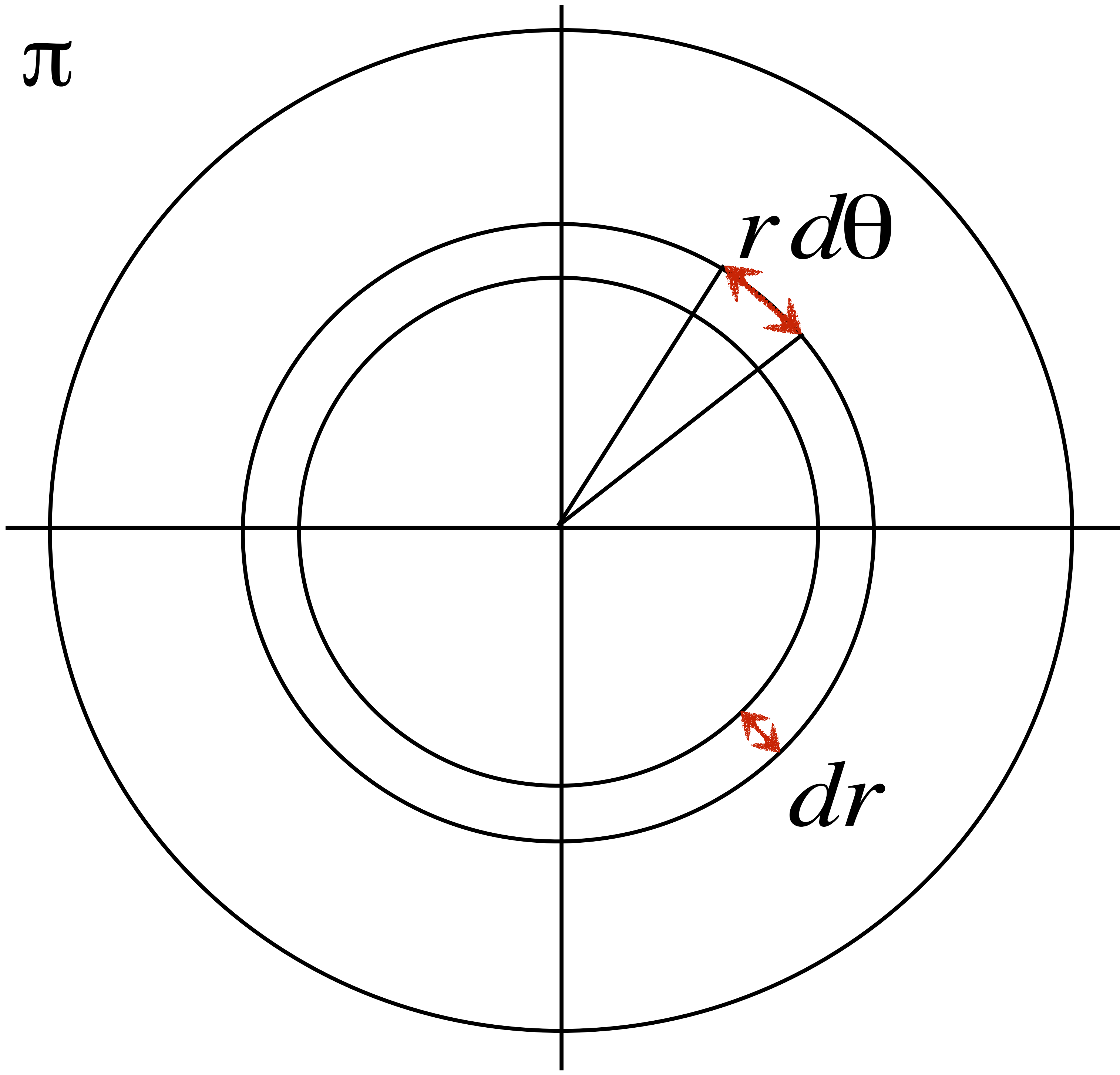
Sampling the Unit Circle (Inversion Method)

PDF:

$$A = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \int_0^1 r\,dr \int_0^{2\pi} d\theta = \left(\frac{r^2}{2}\right)\Bigg|_0^1 \theta\Big|_0^{2\pi} = \pi$$

$$p(r,\theta)\,dr\,d\theta = \frac{1}{\pi} r\,dr\,d\theta \Rightarrow p(r,\theta) = \frac{r}{\pi}$$

$$p(r,\theta) = p(r)\,p(\theta)$$ **(separable PDF)**



Apply inversion method to each factor:

$$p(\theta) = \frac{1}{2\pi}$$

$$P(\theta) = \frac{1}{2\pi}\theta$$

$$\theta = 2\pi\xi_1$$

$$p(r) = 2r$$

$$P(r) = r^2$$

$$r = \sqrt{\xi_2}$$

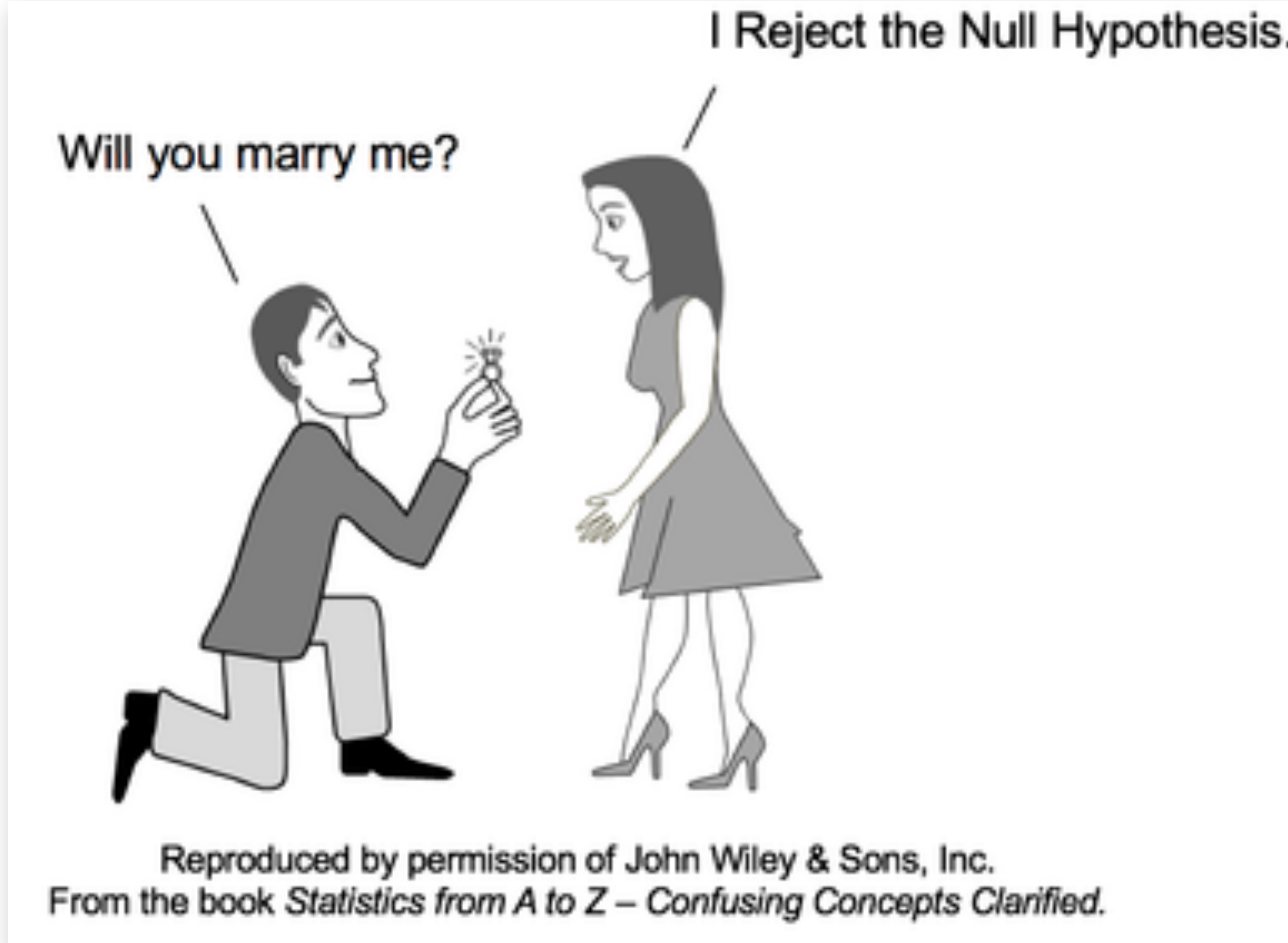
pbrt sampling (p. 837)

```
class Shape {
  public:
    // Shape Interface
   …
    virtual ~Shape();
    virtual Bounds3f ObjectBound() const = 0;
    virtual Bounds3f WorldBound() const;
    virtual bool Intersect(const Ray &ray, Float *tHit,
                           SurfaceInteraction *isect,
                           bool testAlphaTexture = true) const = 0;
    …
    virtual Float Area() const = 0;
    …
    // Sample a point on the surface of the shape
    // and return the PDF with respect to area on the surface.
    virtual Interaction Sample(const Point2f &u, Float *pdf) const = 0;
    virtual Float Pdf(const Interaction &) const { return 1 / Area(); }
    ..
};
```

two random numbers between 0 and 1

includes the point and the normal on the surface

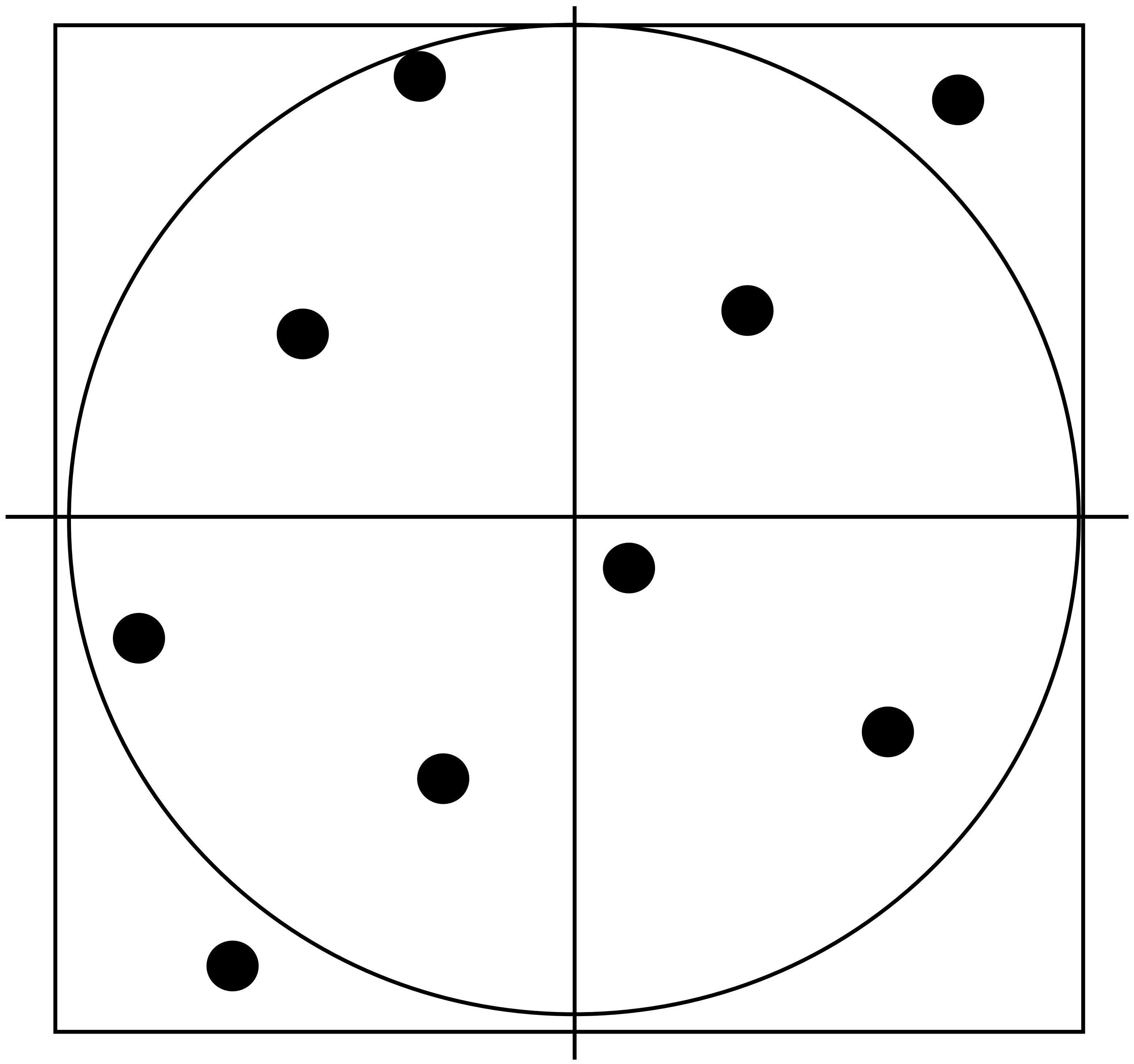
Default implementation of Pdf assumes that the shape is uniformly sampled by area



Rejection Sampling

https://www.statisticsfromatoz.com/blog/statistics-tip-of-the-week-understanding-reject-the-null-hypothesis

Rejection Sampling

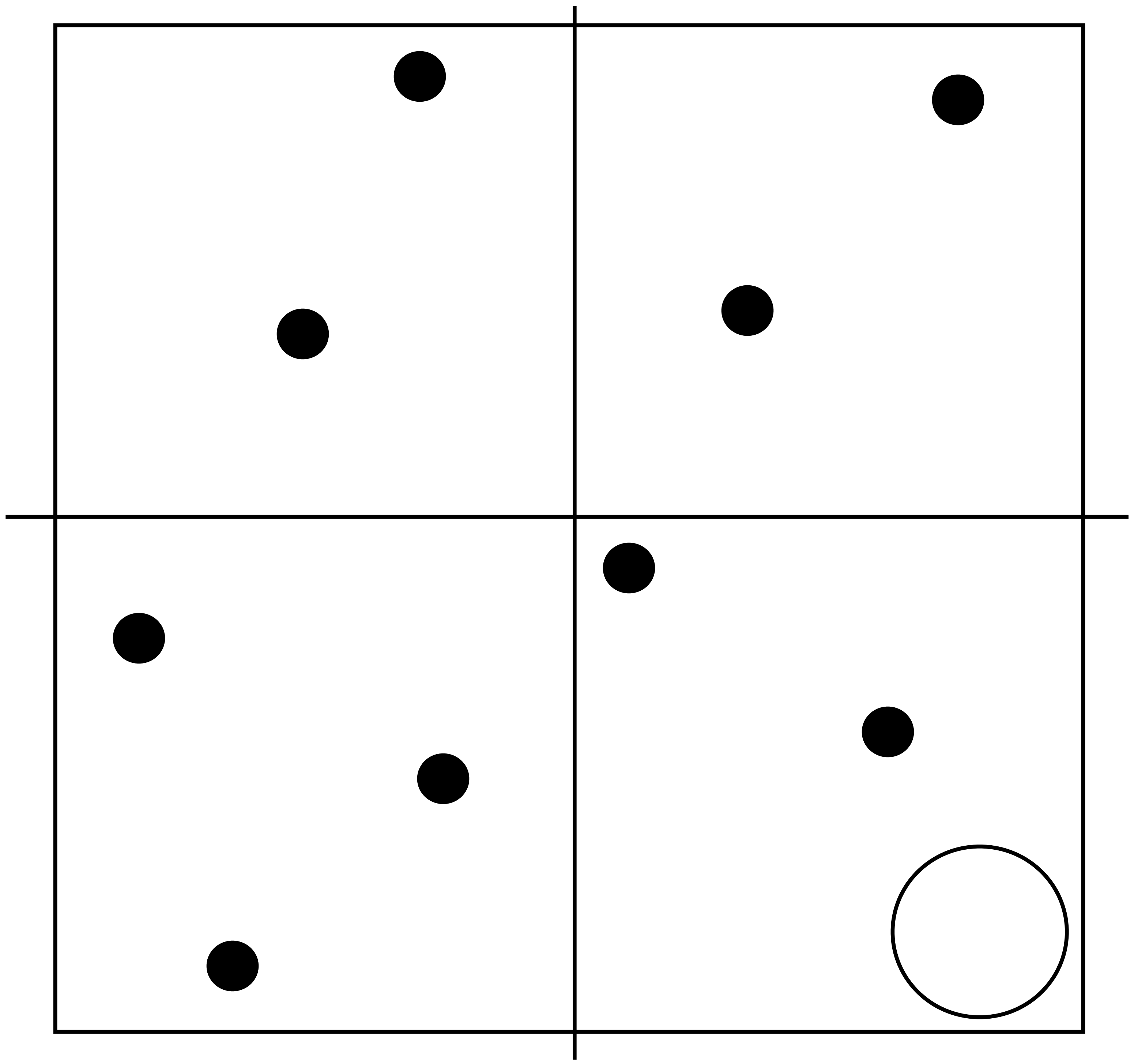


```
do {
   X=Uniform(-1,1)
   Y=Uniform(-1,1)
} while (X*X+Y*Y>1);
```

Efficiency?

Area of circle / Area of square

Rejection Sampling

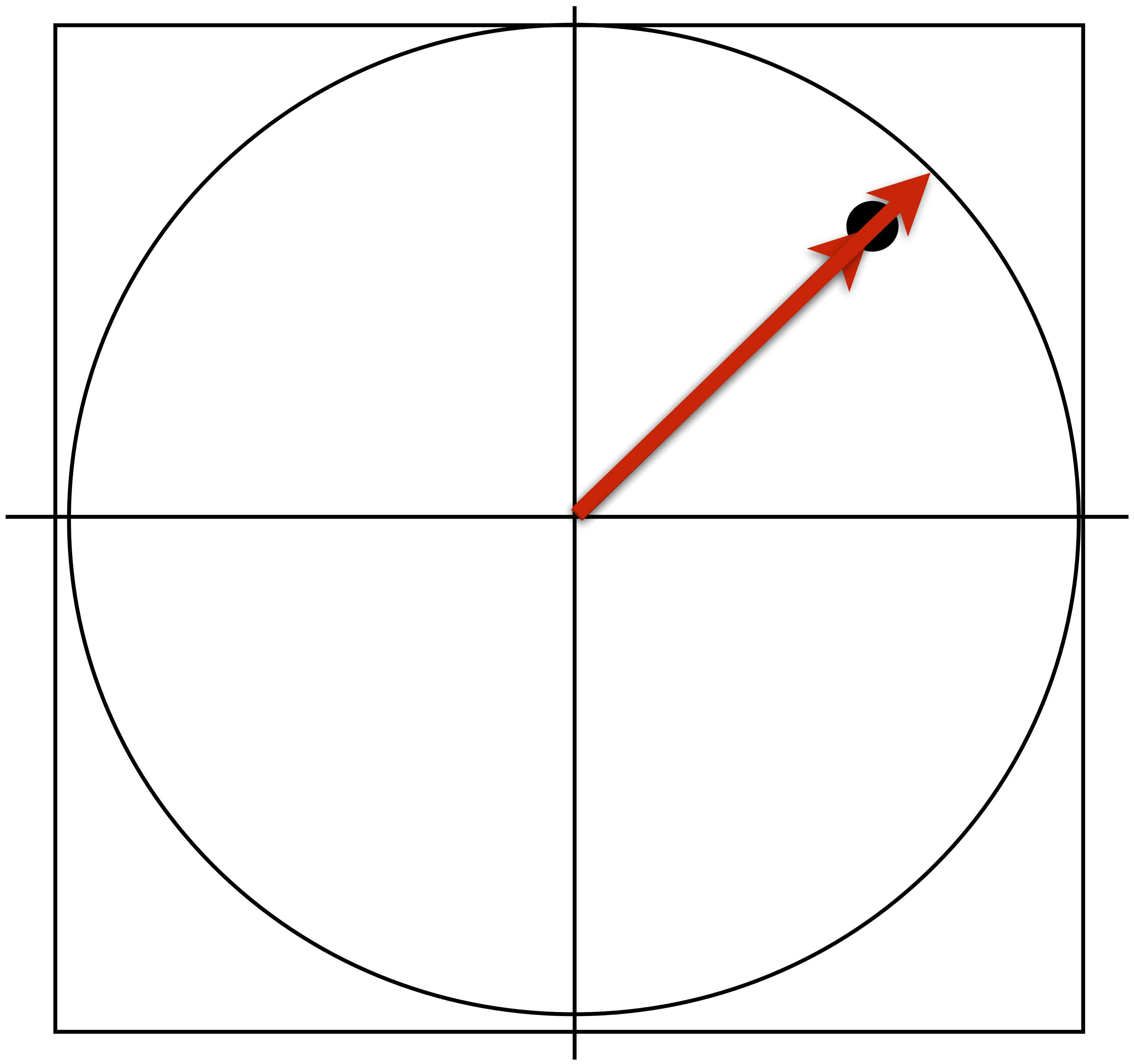


```
do {
   X=Uniform(-1,1)
   Y=Uniform(-1,1)
} while (X*X+Y*Y>1);
```

Efficiency?

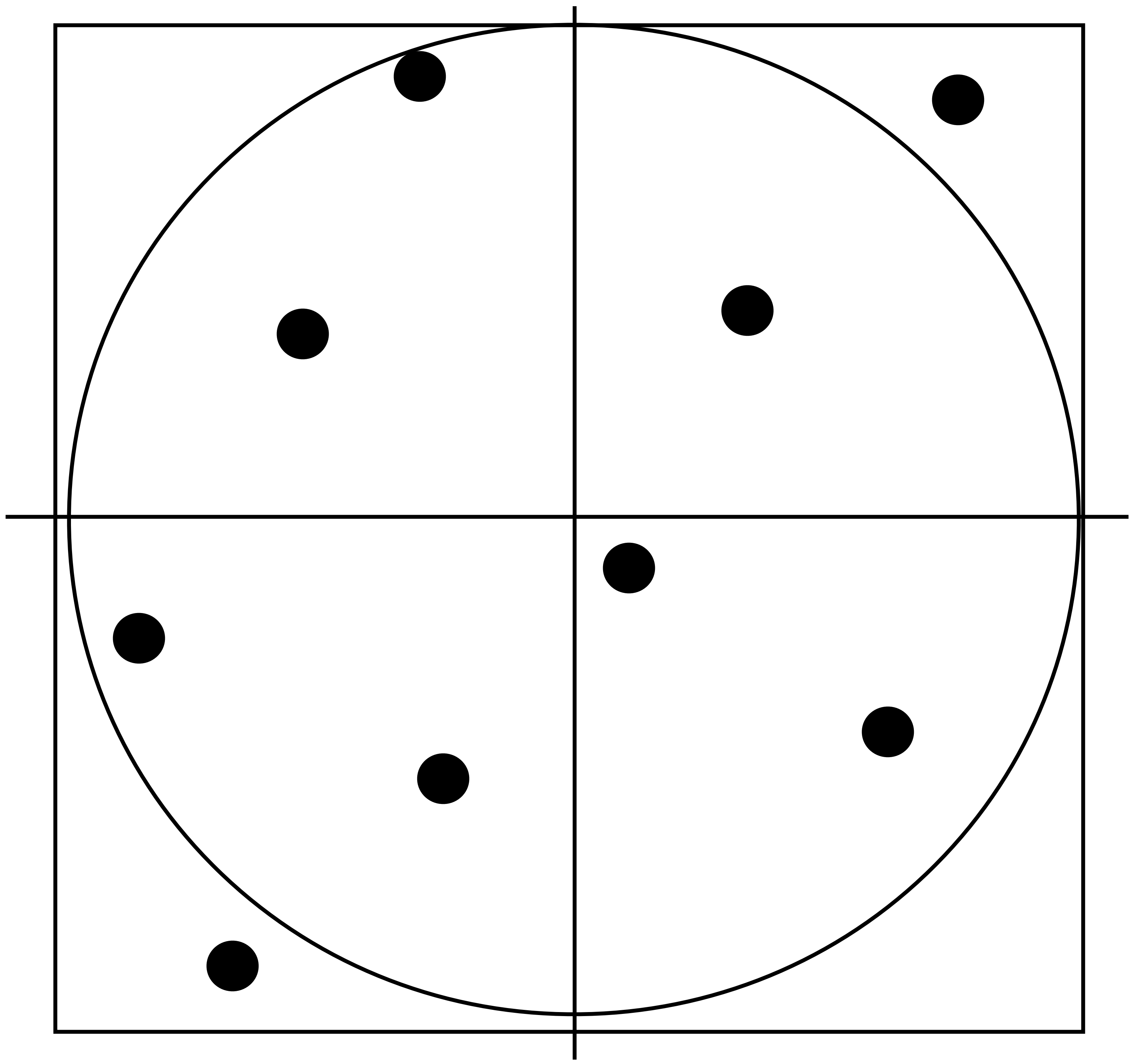
Area of circle / Area of square

Sampling 2D Directions



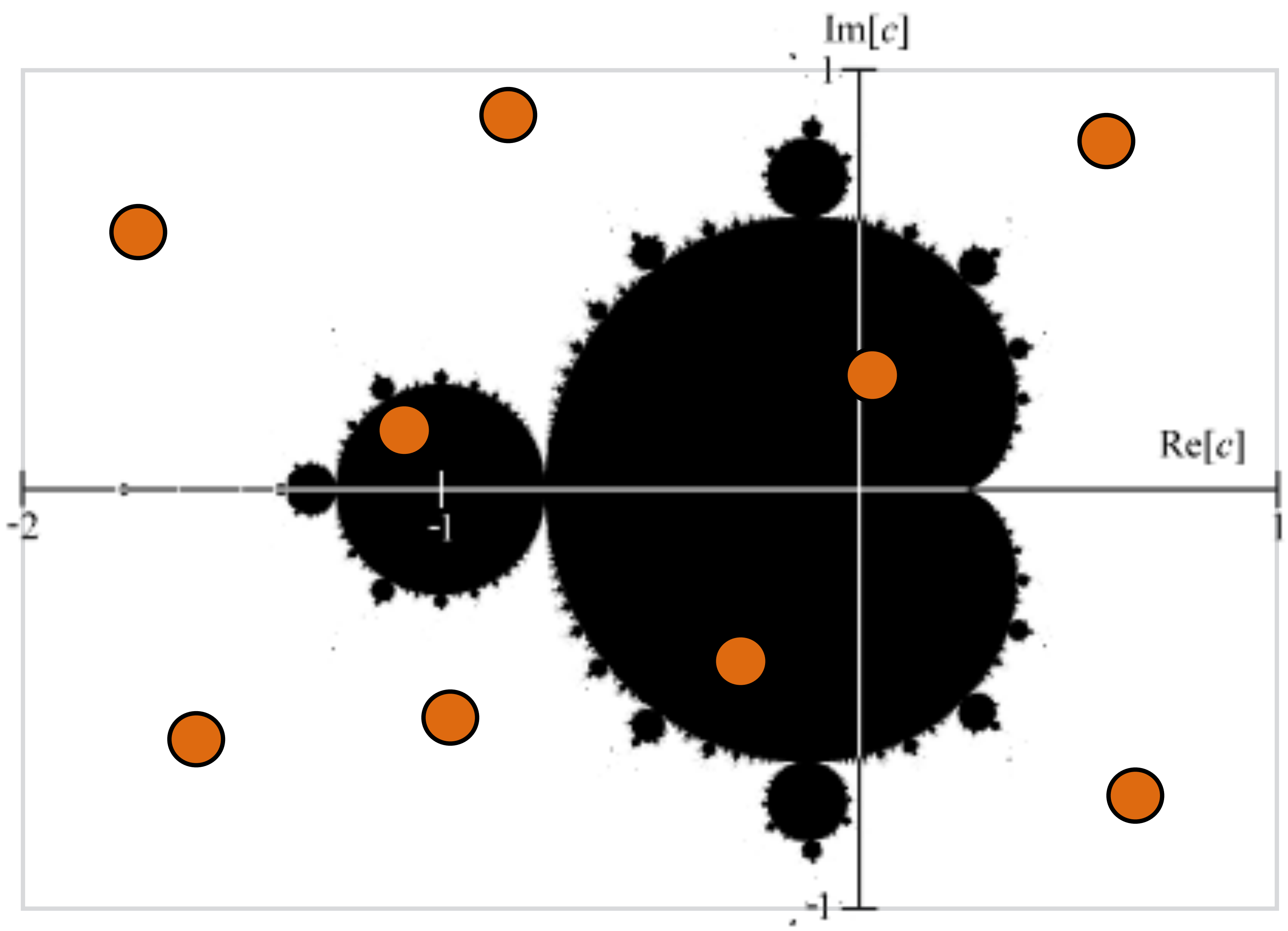
```
do {
   X=Uniform(-1,1);
   Y=Uniform(-1,1);
} while (X*X+Y*Y>1);

R = sqrt(X*X+Y*Y)
dx = X/R
dy = Y/R
```

Computing Area of a Circle



```
A = 0
for( i=0; i<N; i++ ) {
   X=Uniform(-1,1);
   Y=Uniform(-1,1);
   if(X*X+Y*Y < 1)
      A += 1 ;
}
A = 4*A/N
```

Computing Area of the Mandelbrot Set




```
A = 0
for( i=0; i<N; i++ ) {
   X=Uniform(-2,1);
   Y=Uniform(-1,1);
   if(Mdlbrt(X,Y))
      A += 1 ;
}
A = 6*A/N
```

$A \approx 1.506484$

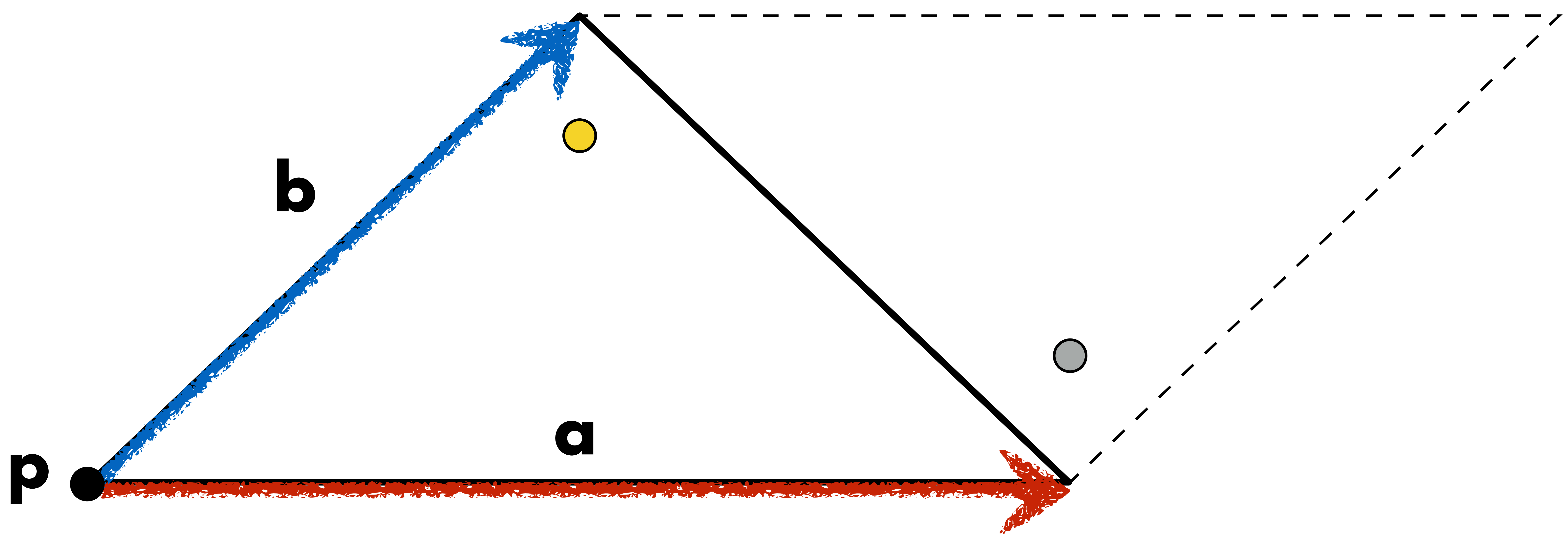
https://www.fractalus.com/kerry/articles/area/mandelbrot-area.html

https://ncatlab.org/nlab/show/Mandelbrot+set

2D Sampling with Multidimensional Transformations

(Section 13.6)

Sampling a Triangle



Possible approaches:

- **Simple: Sample parallelogram & reflect**
  - $X = \mathbf{p} + \xi_1 \mathbf{a} + \xi_2 \mathbf{b}$ **reflect if** $1 - \xi_1 - \xi_2 < 0$
- **Better: Use marginal density functions** (see 13.6, p.773)

Marginal Density Functions (p.773)

Motivation: What if 2D PDF is nonseparable?

$$p(x, y) \neq p_x(x) p_y(y)$$

Marginal density function: $p(x) = \int p(x, y)\, \mathrm{d}y$

Condition density function: $p(y|x) = \dfrac{p(x, y)}{p(x)}$

The basic idea for 2D sampling from joint distributions is to first compute the marginal density to isolate one particular variable and draw a sample from that density using standard 1D techniques. Once that sample is drawn, one can then compute the conditional density function given that value and draw a sample from that distribution, again using standard 1D sampling techniques.

Sampling a Triangle (S13.6.5)

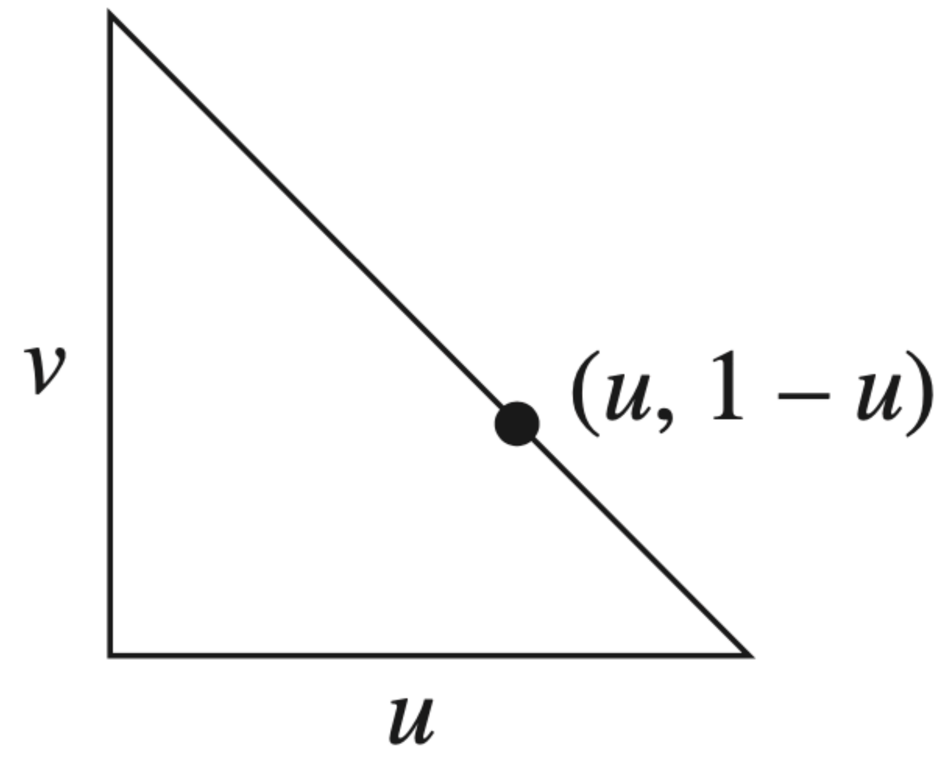
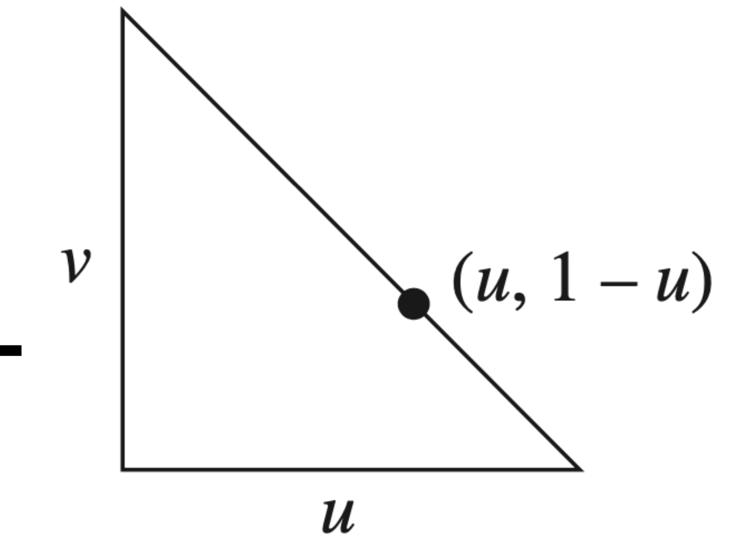


Figure 13.15: Sampling an Isosceles Right Triangle. Note that the equation of the hypotenuse is $v = 1 - u$.

$$p(u, v) = 2$$

Sampling a Triangle (S13.6.5)



Marginal density:

$$p(u) = \int_0^{1-u} p(u,v)\,\mathrm{d}v = 2\int_0^{1-u} \mathrm{d}v = 2(1-u)$$

Condition density:

$$p(v|u) = \frac{p(u,v)}{p(u)} = \frac{2}{2(1-u)} = \frac{1}{1-u}$$

CDFs:

$$P(u) = \int_0^u p(u')\,\mathrm{d}u' = 2u - u^2$$

$$P(v) = \int_0^v p(v'|u)\,\mathrm{d}v' = \frac{v}{1-u}$$

Apply Inversion Method (2X):

$$u = 1 - \sqrt{\xi_1}$$

$$v = \xi_2\sqrt{\xi_1}.$$

Back to ...

Monte Carlo Integration

Monte Carlo Integration

Definite integral $$I(f) \equiv \int_0^1 f(x)\,dx$$

Expectation of f $$E[f] \equiv \int_0^1 f(x)\,p(x)\,dx$$

Random variables $$X_i \sim p(x)$$ ← **Assume uniform probability distribution for now (p=1)**

$$Y_i = f(X_i)$$

Estimator $$F_N = \frac{1}{N}\sum_{i=1}^{N} Y_i$$

Unbiased Estimator

$$E[F_N] = I(f)$$

Properties

$$E[\sum_i Y_i] = \sum_i E[Y_i]$$

$$E[aY] = aE[Y]$$

$$E[F_N] = E[\frac{1}{N}\sum_{i=1}^{N} Y_i]$$

$$= \frac{1}{N}\sum_{i=1}^{N} E[Y_i] = \frac{1}{N}\sum_{i=1}^{N} E[f(X_i)]$$

$$= \frac{1}{N}\sum_{i=1}^{N}\int_0^1 f(x)\,p(x)\,dx$$

$$= \frac{1}{N}\sum_{i=1}^{N}\int_0^1 f(x)dx$$

$$= \int_0^1 f(x)\,dx$$

Assume uniform probability distribution for now (p=1)

General case: Arbitrary PDF (p. 752)

Estimator

$$F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)}$$

"Unbiased proof"

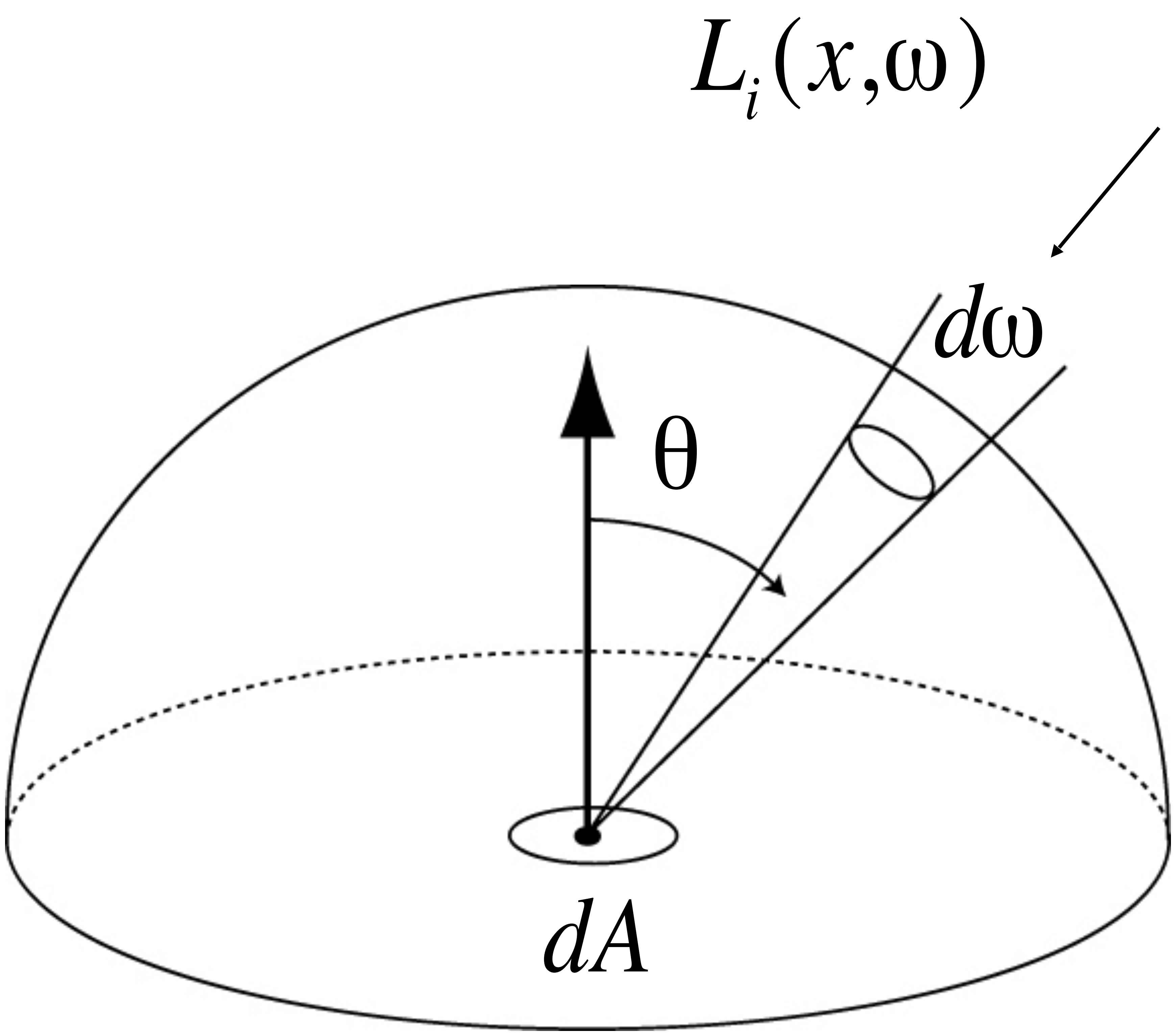
$$E[F_N] = E\left[\frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)}\right]$$

$$= \frac{1}{N} \sum_{i=1}^{N} \int_a^b \frac{f(x)}{p(x)} p(x)\, \mathrm{d}x$$

$$= \frac{1}{N} \sum_{i=1}^{N} \int_a^b f(x)\, \mathrm{d}x = \int_a^b f(x)\, \mathrm{d}x.$$

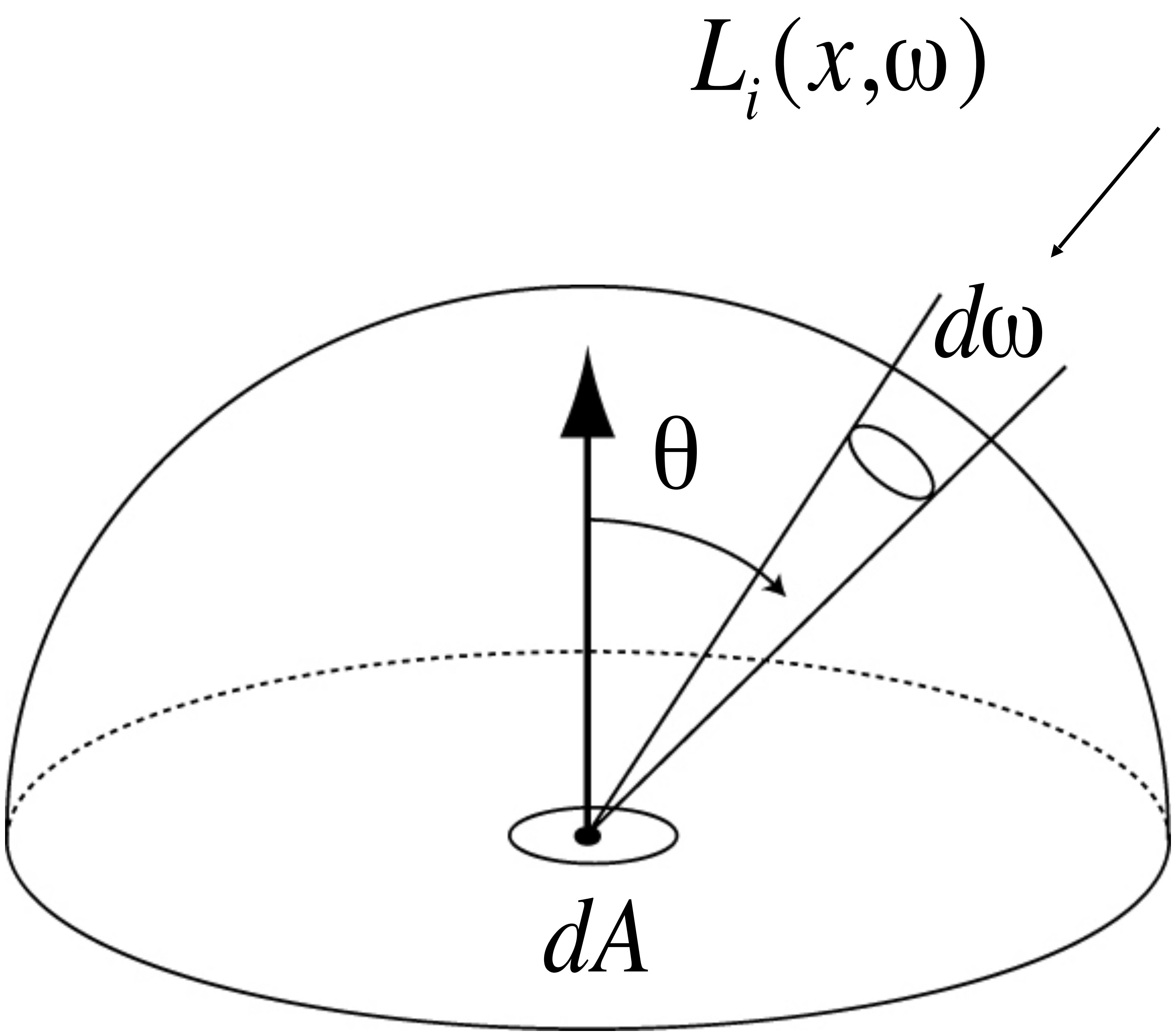
Direct Lighting: Hemispherical Integral

$$E(x) = \int_{H^2} L(x,\omega)\cos\theta \, d\omega$$



Direct Lighting: Solid Angle Sampling

Sample hemisphere uniformly

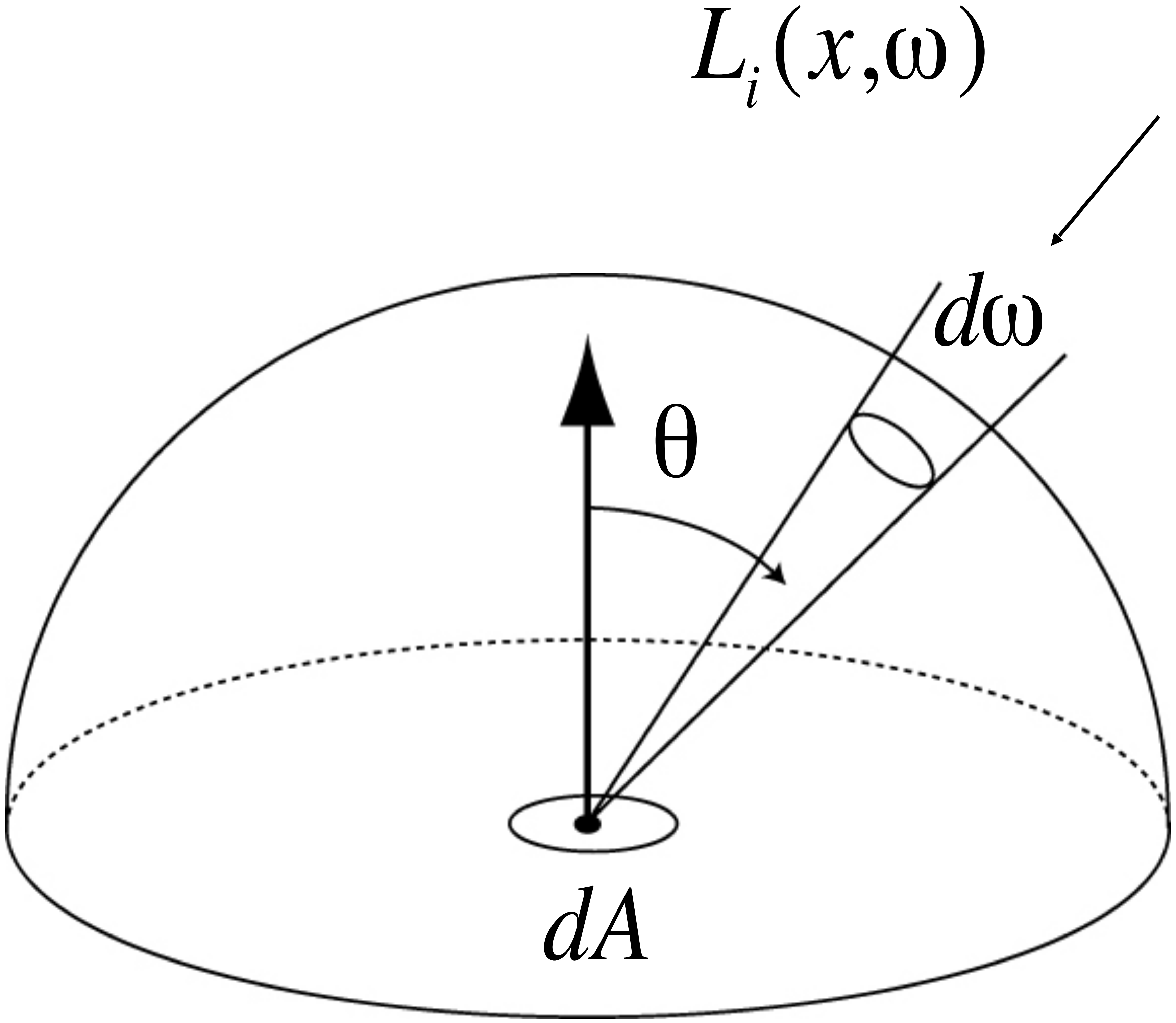


$$\int_{H^2} p(\omega)\, d\omega = 1$$

$$p(\omega) = \frac{1}{2\pi}$$

Direct Lighting: Solid Angle Sampling

$$E(x) = 2\pi \int L(x,\omega) \cos\theta \frac{1}{2\pi}\, d\omega$$
$$= 2\pi \int L(x,\omega) \cos\theta \, p(\omega)\, d\omega$$

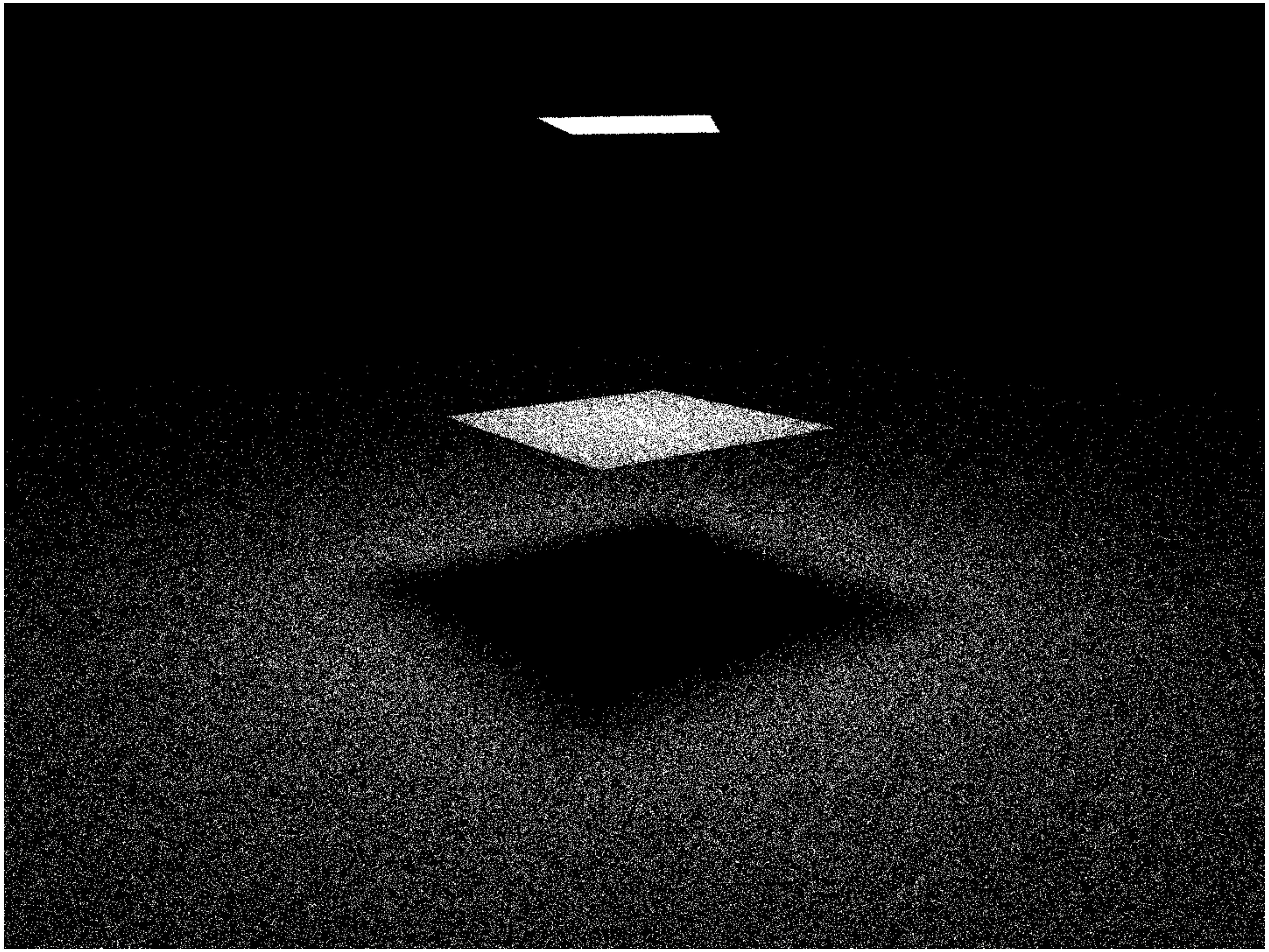


Estimator

$$Y_i = 2\pi L(x,\omega_i) \cos\theta_i$$

$$F_N = \frac{1}{N}\sum_{i=1}^{N} Y_i$$

Direct Lighting: Hemisphere Sampling

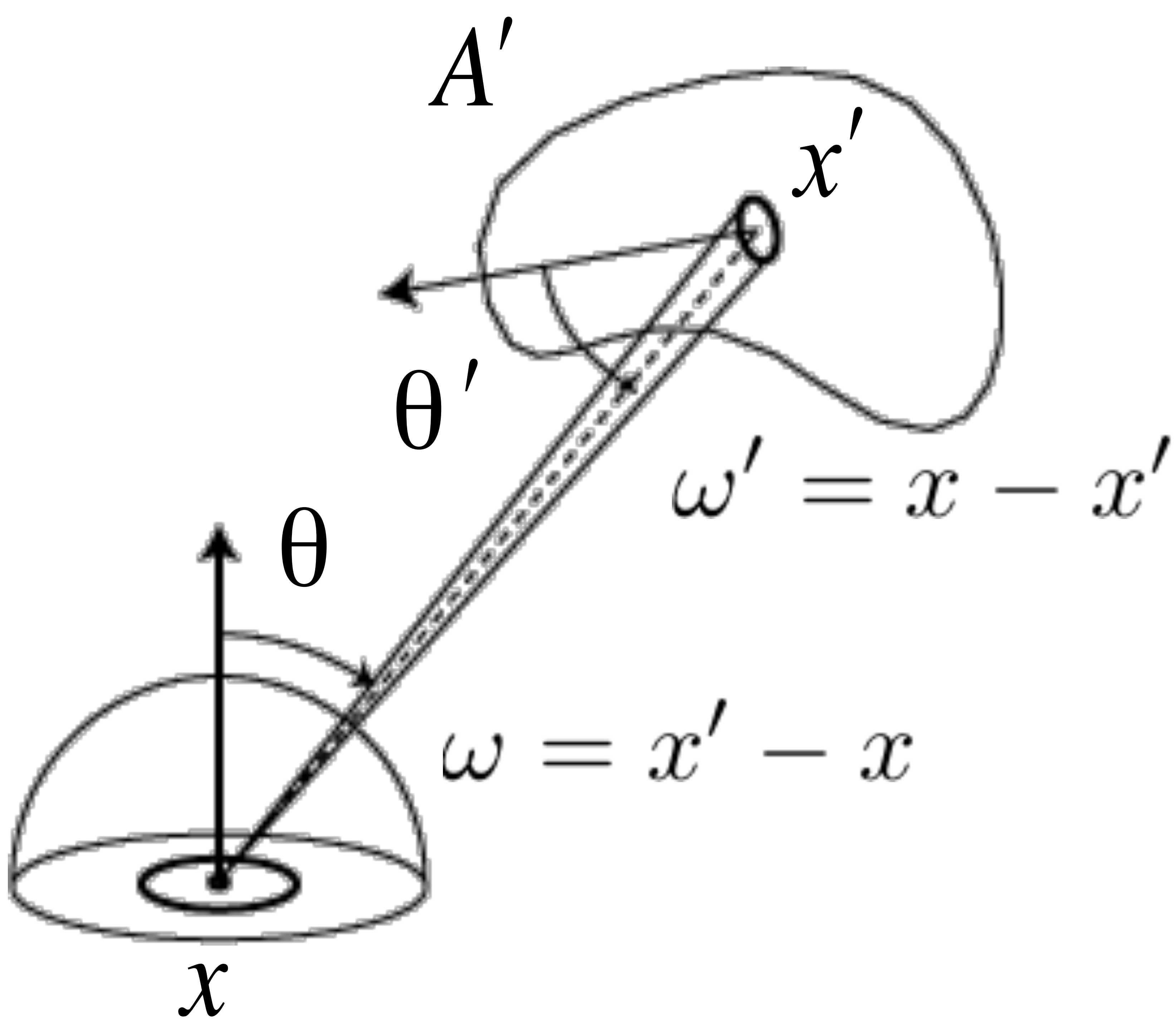


Hemisphere

16 shadow rays

Direct Lighting: Area Integral

$$E(x) = \int_{H^2} L_i(x,\omega)\, \cos\theta \, d\omega = \int_{A'} L_o(x',\omega') V(x,x') \frac{\cos\theta \cos\theta'}{|x-x'|^2} dA'$$



Integral

$$d\omega = \frac{\cos\theta'}{|x-x'|^2} dA'$$

Visibility

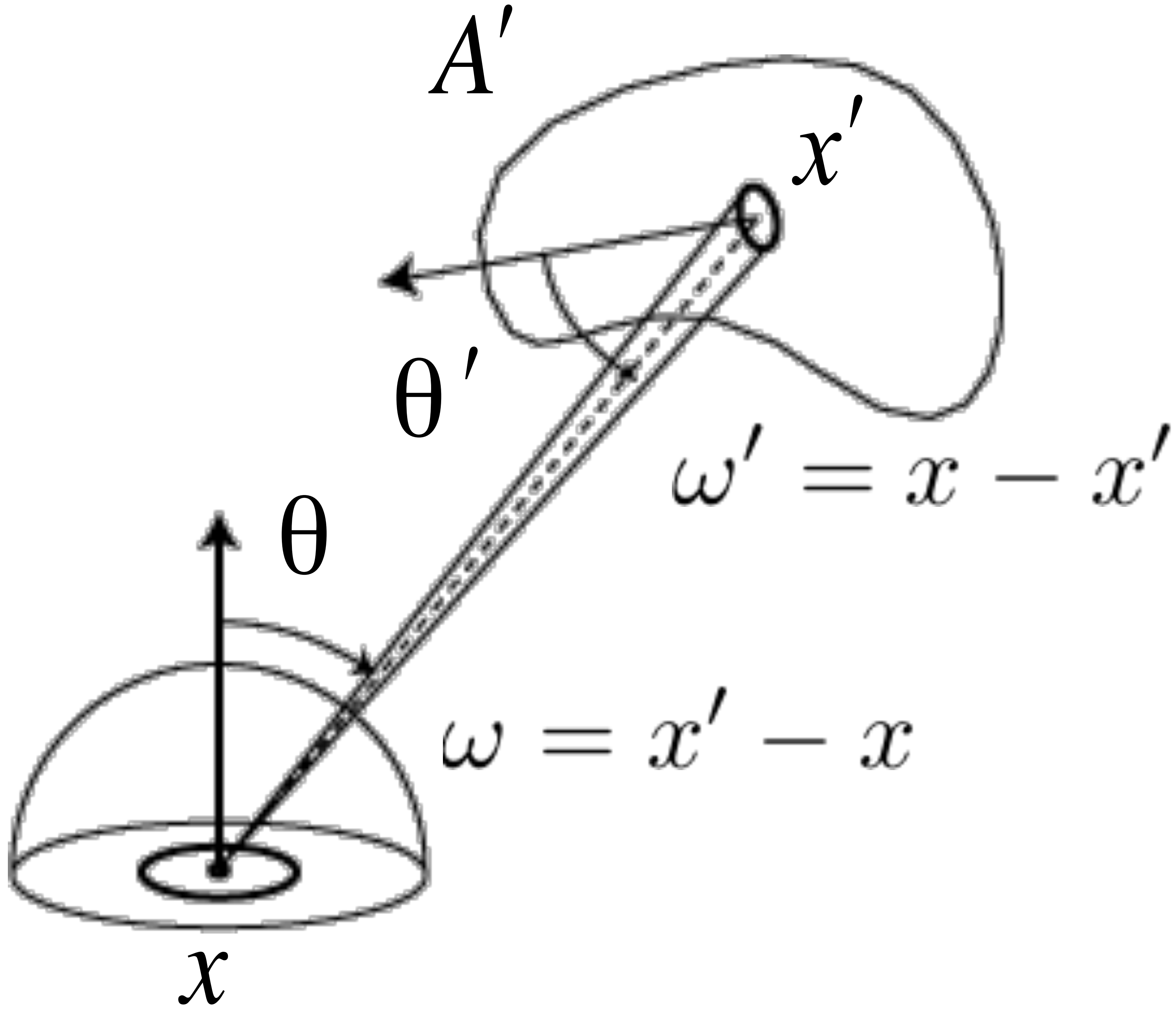
$$V(x,x') = \begin{cases} 0 & \text{blocked} \\ 1 & \text{visible} \end{cases}$$

Radiance

$$L_i(x,\omega) = L_o(x',\omega')$$

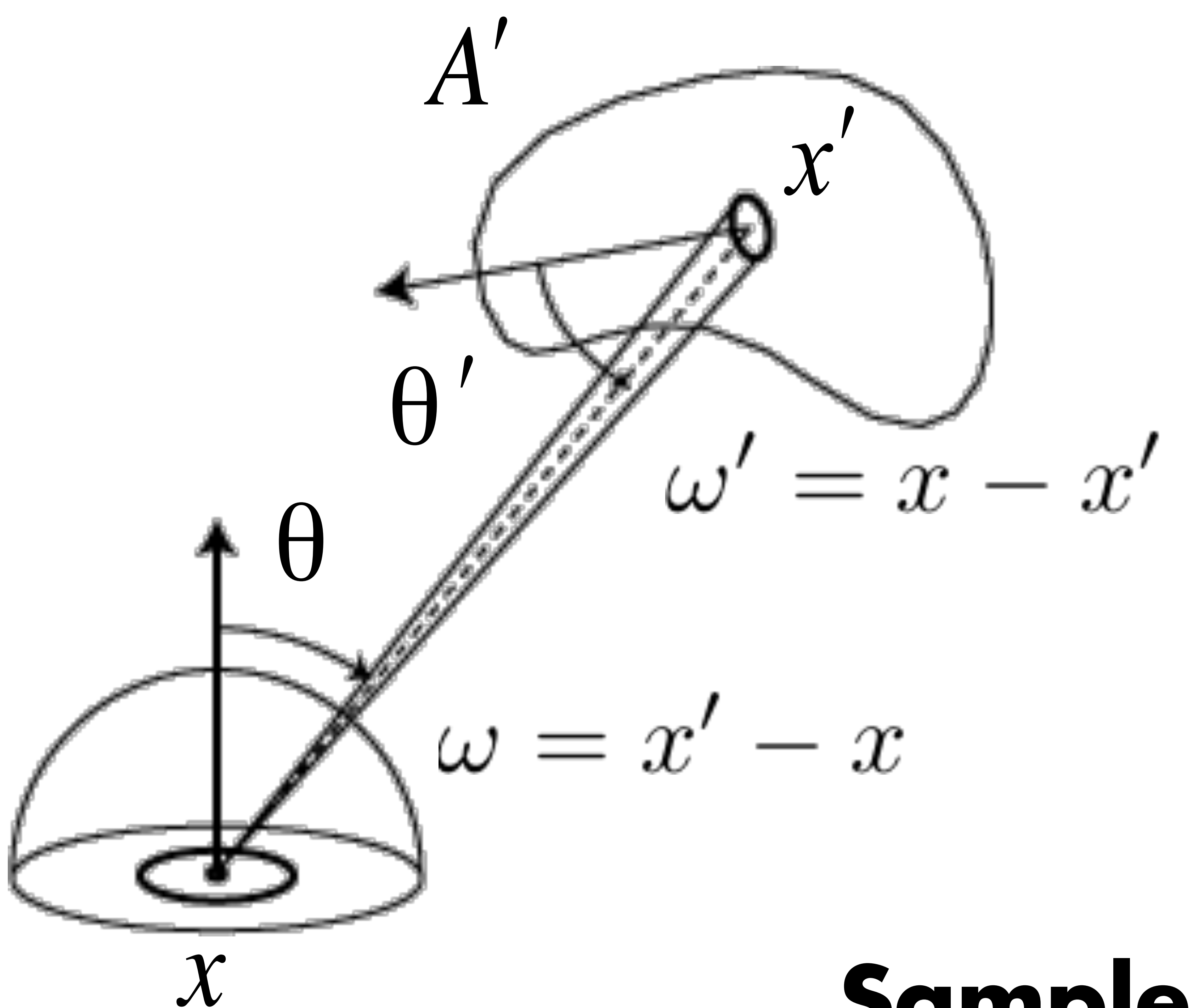
Direct Lighting: Area Integral

$$E(x) = \int_{A'} L_o(x', \omega') V(x, x') \frac{\cos\theta \cos\theta'}{|x - x'|^2} dA'$$



Direct Lighting: Area Sampling

$$E(x) = \int_{A'} L_o(x', \omega') V(x, x') \frac{\cos\theta \cos\theta'}{|x - x'|^2} dA'$$



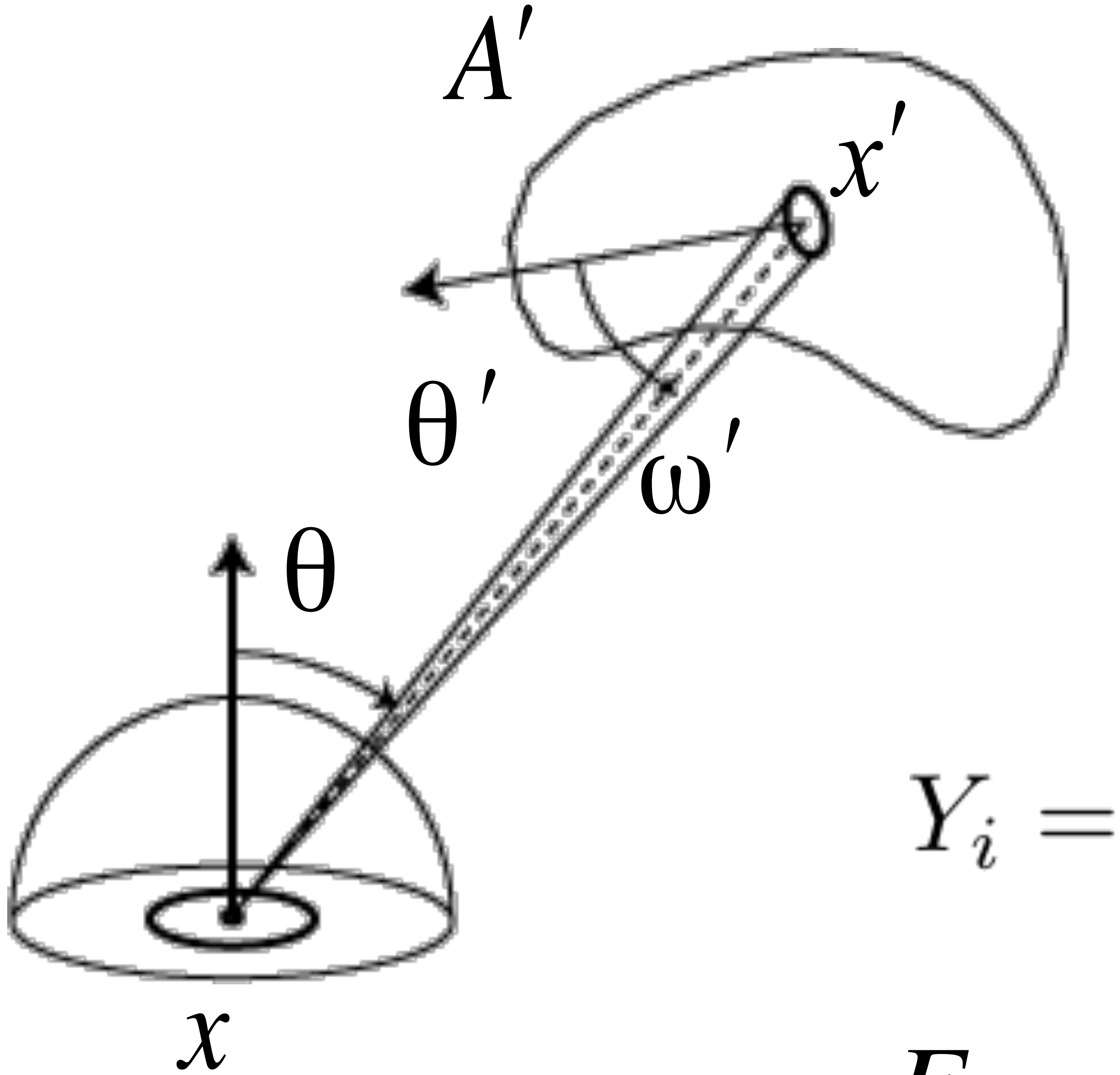
$$\int_{A'} p(x')\, dA' = 1$$

$$p(x') = \frac{1}{A'}$$

Sample shape uniformly by area

Direct Lighting: Area Sampling

$$E(x) = \int_{A'} L_o(x', \omega') V(x, x') \, \frac{\cos\theta \cos\theta'}{|x - x'|^2} dA'$$

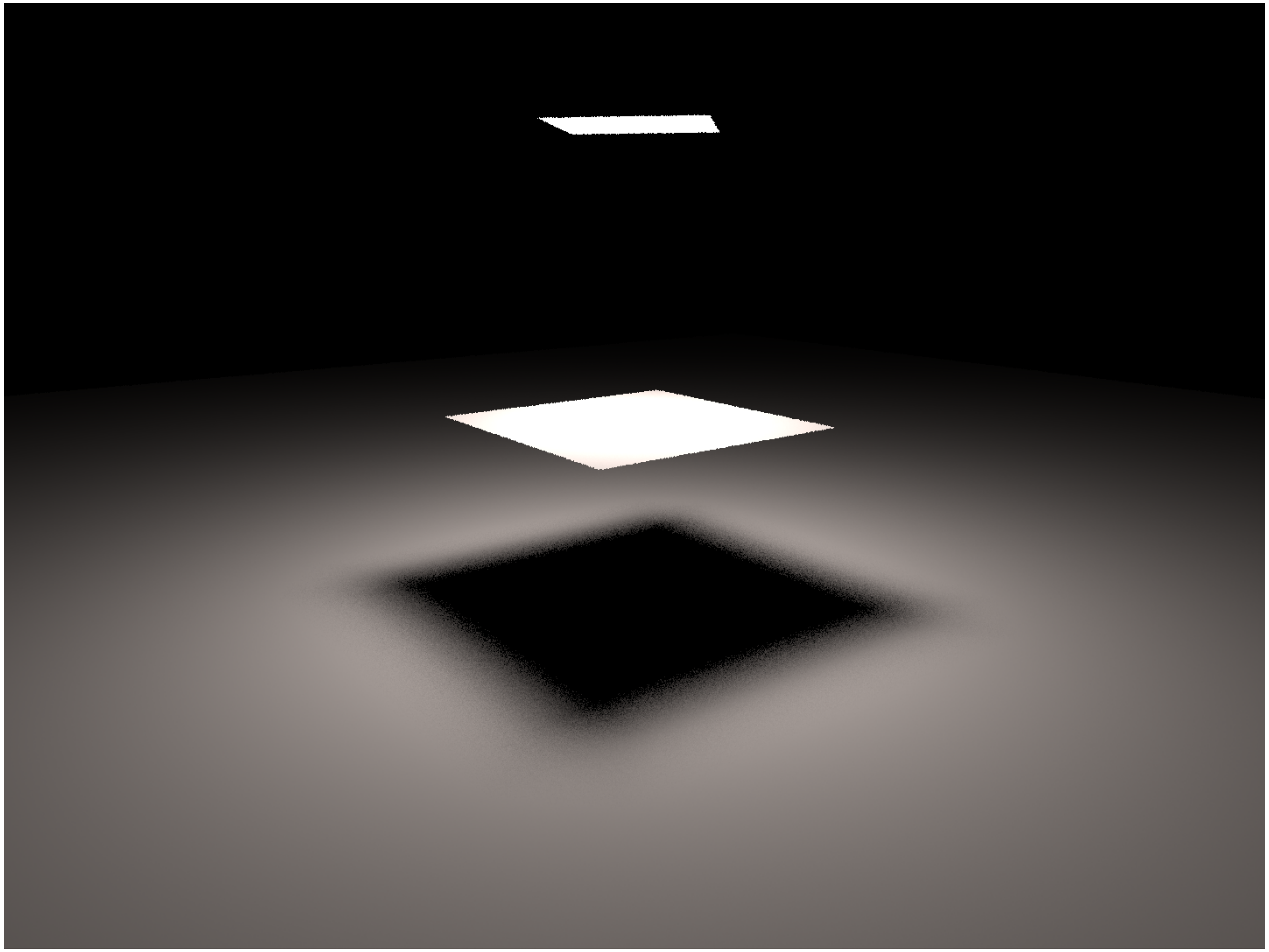


Estimator

$$Y_i = L_o(x_i', \omega_i') V(x, x_i') \, \frac{\cos\theta_i \cos\theta_i'}{|x - x_i'|^2} A'$$

$$F_N = \frac{1}{N} \sum_{i=1}^{N} Y_i$$

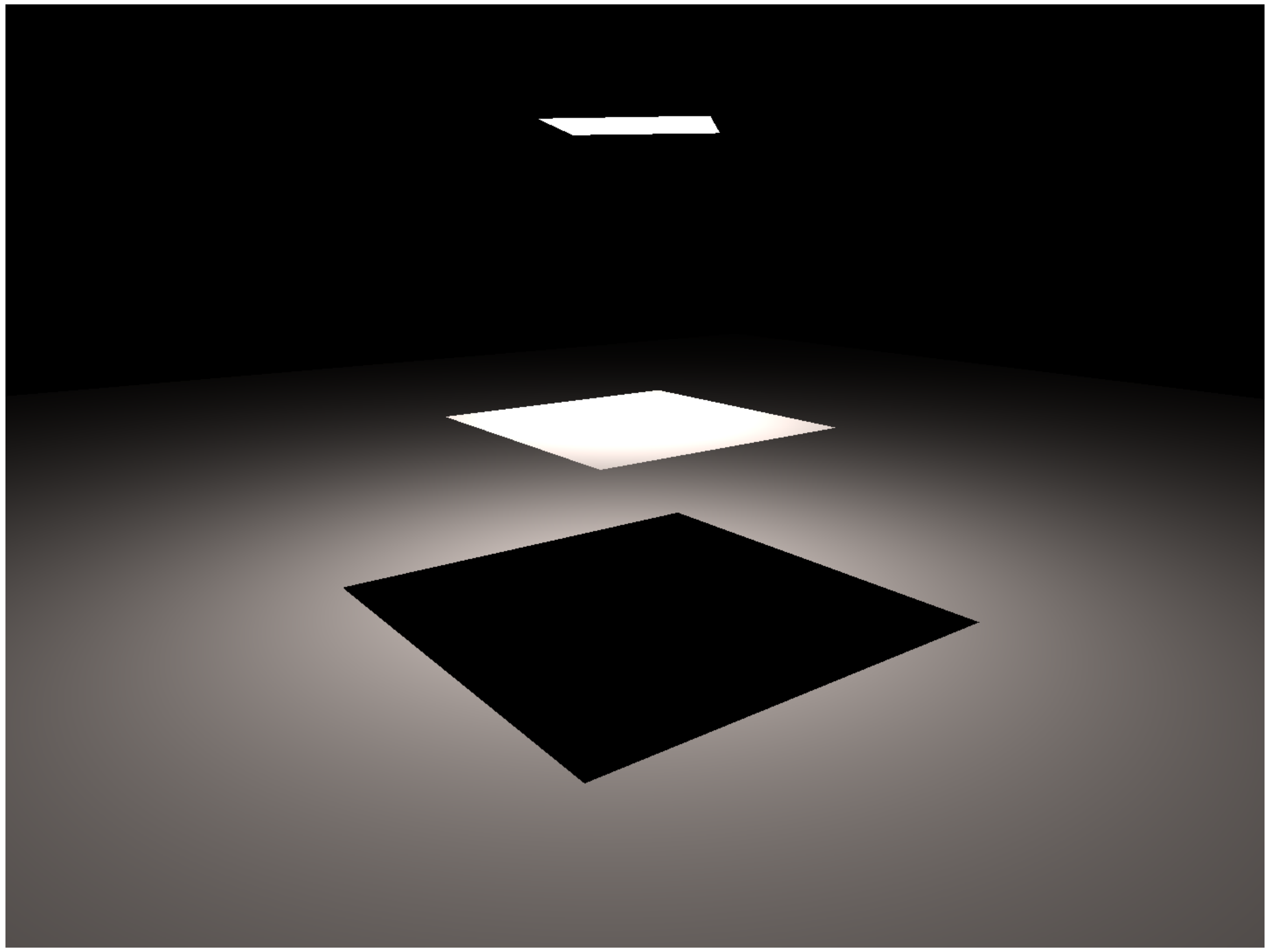
Direct Lighting: Area Sampling



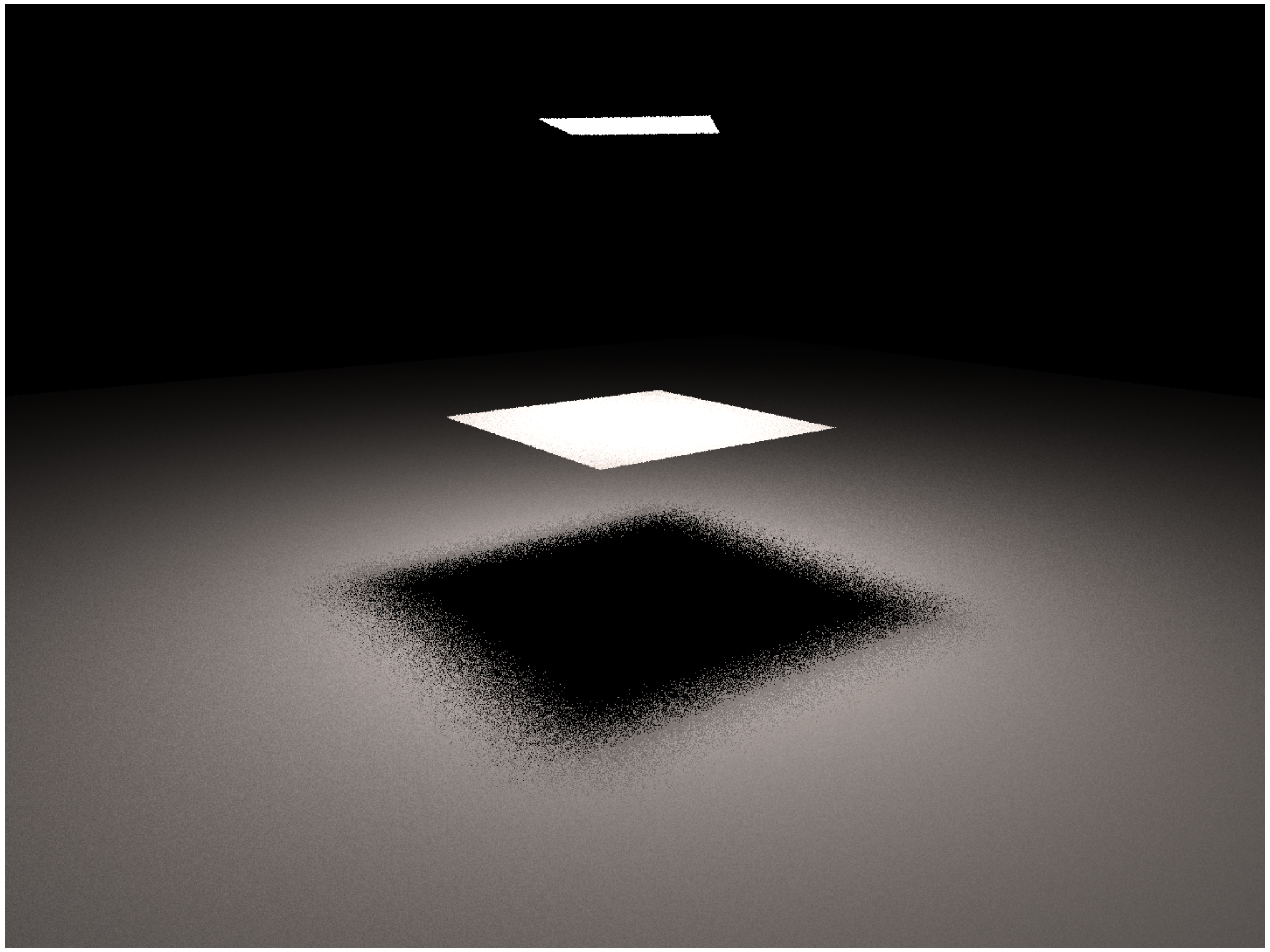
Area

16 shadow rays

Random Sampling Introduces Noise



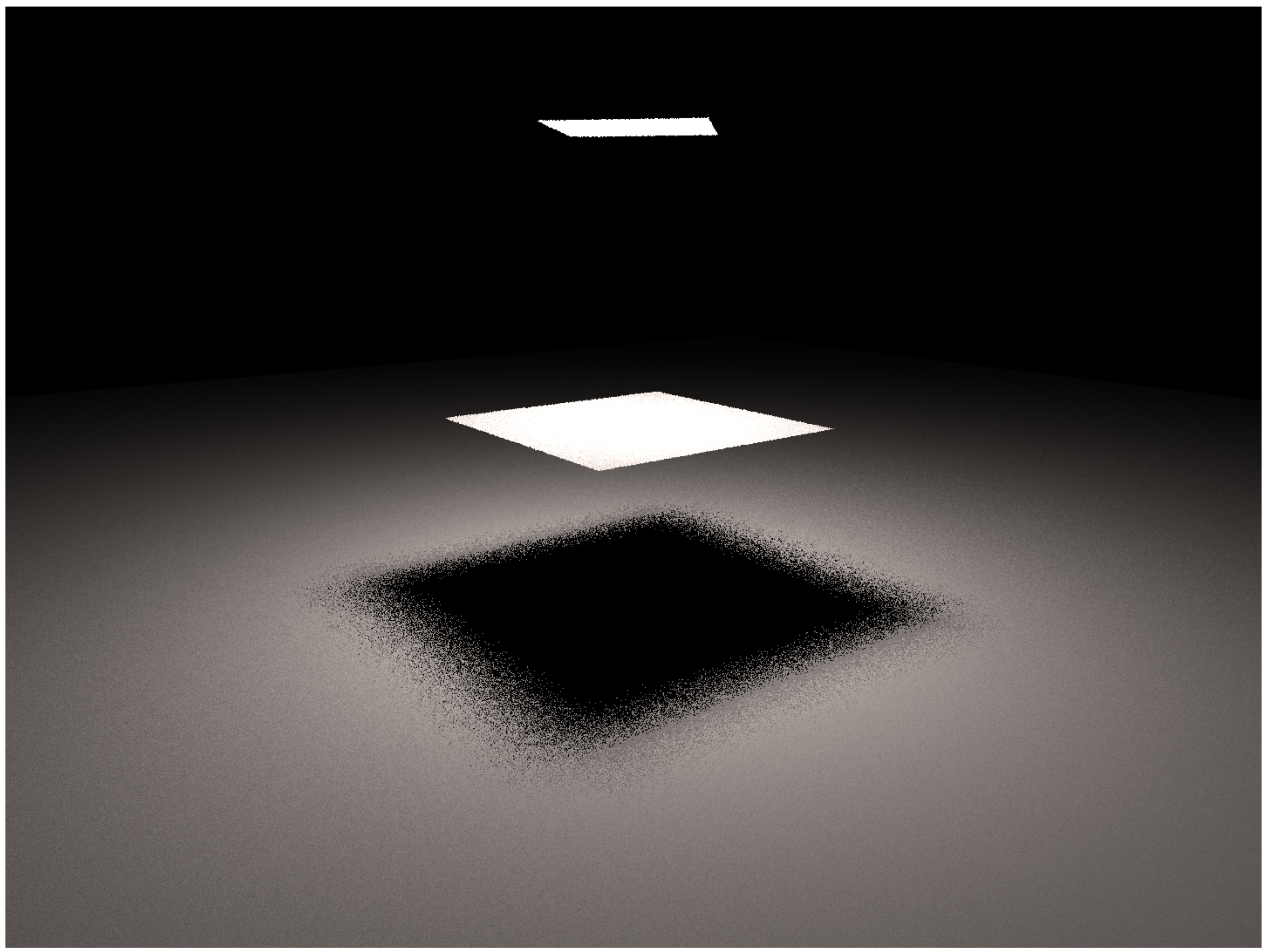
Center



Random

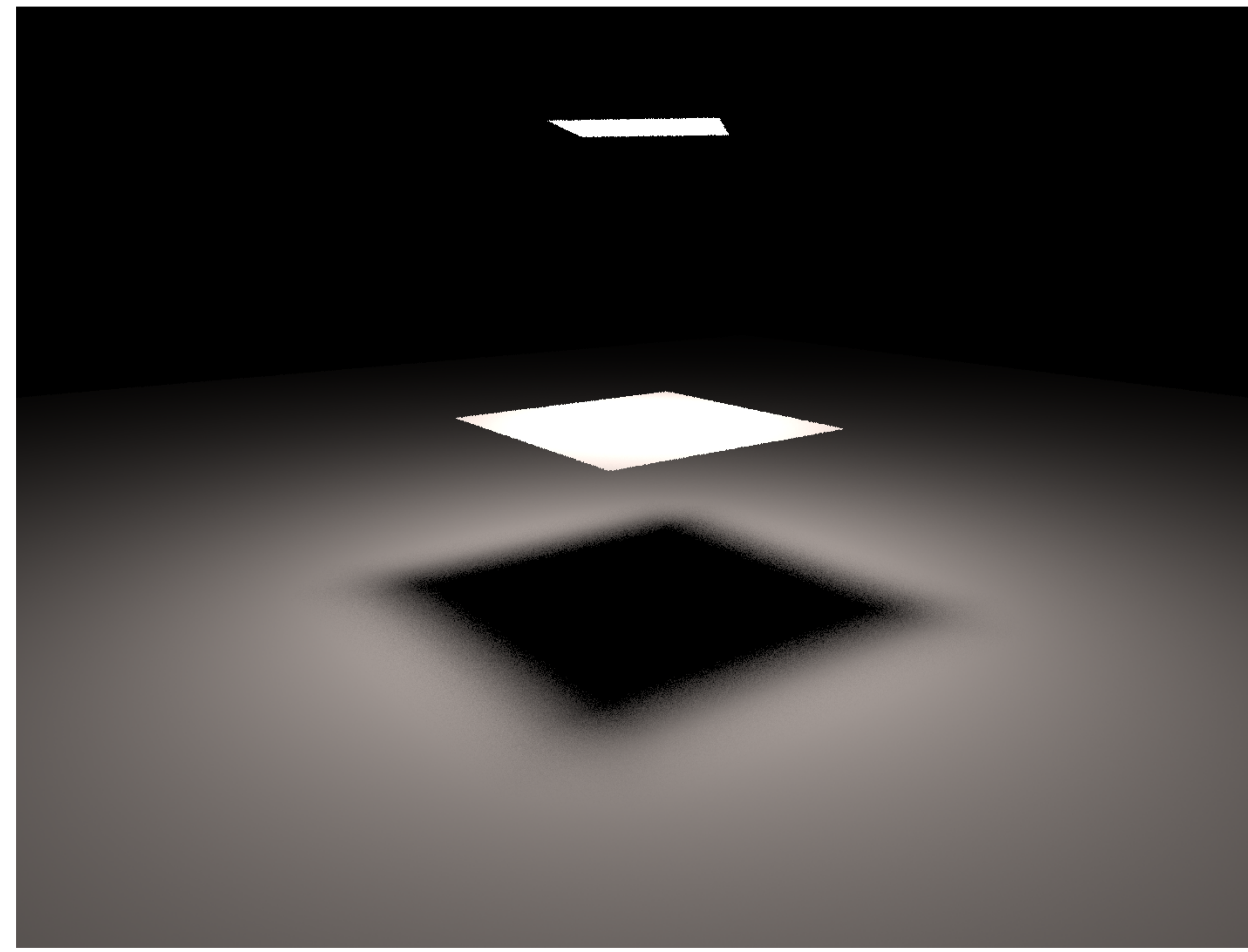
1 shadow ray per eye ray

Quality Improves with More Rays



Area

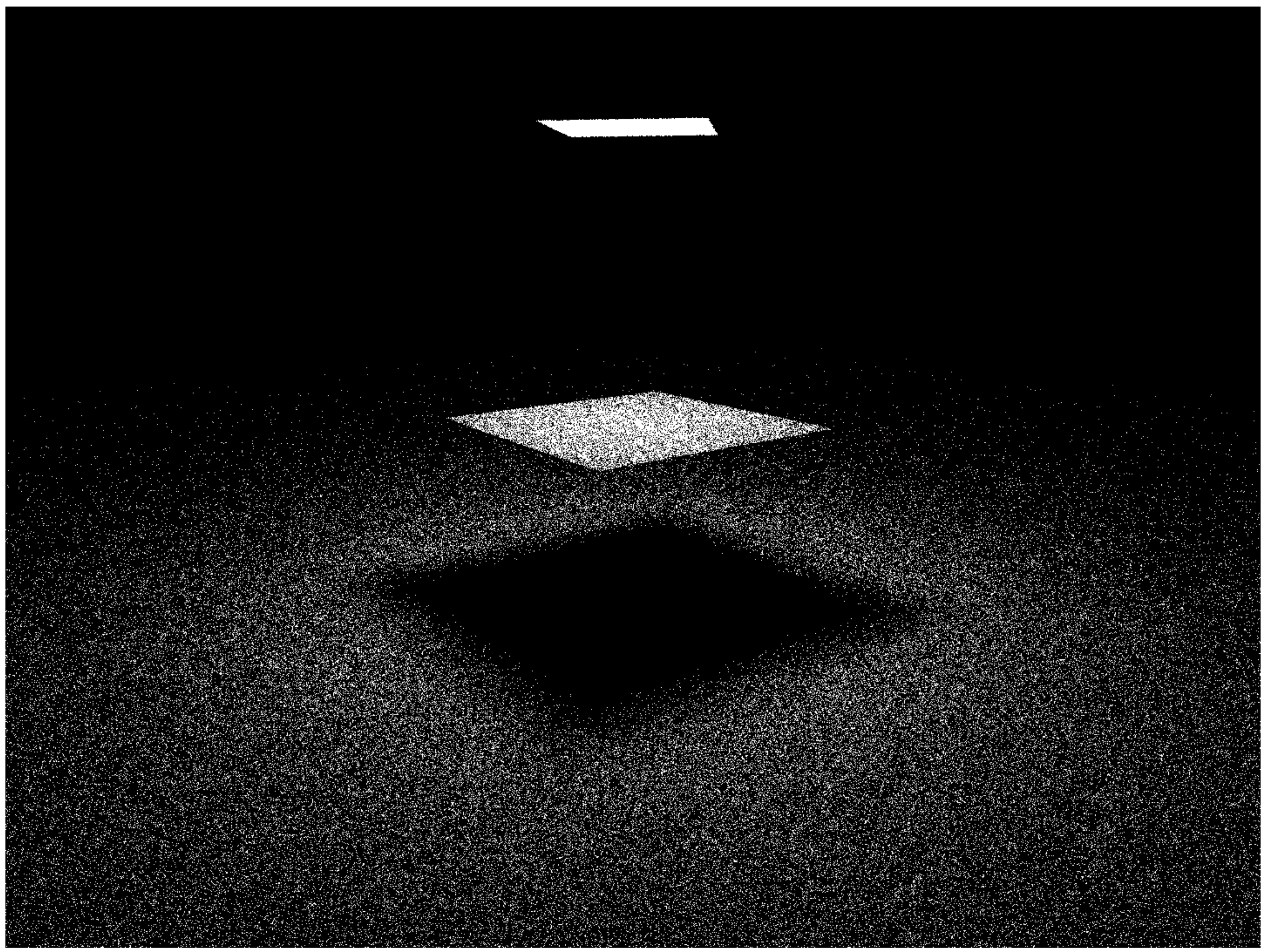
1 shadow ray

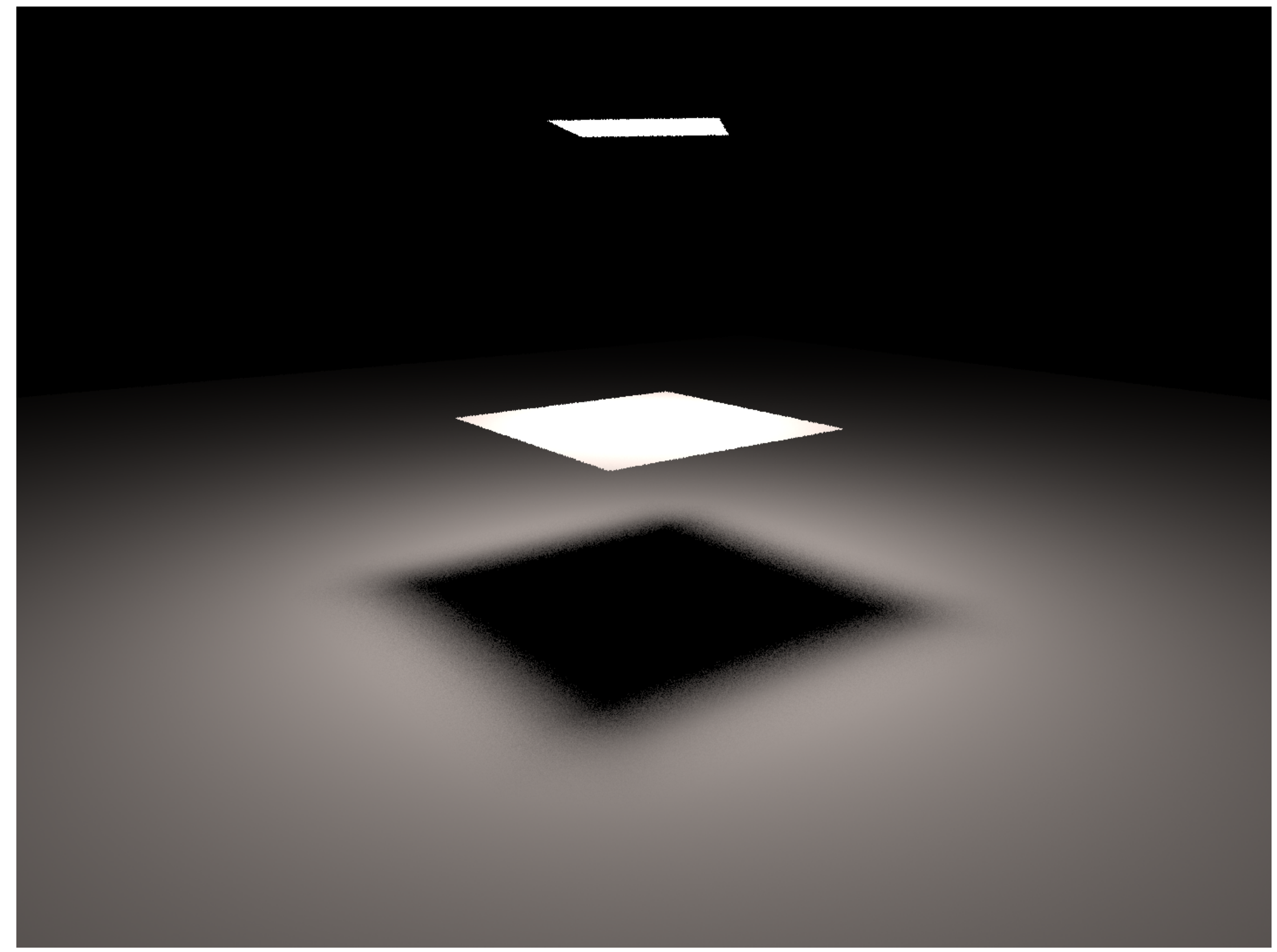


Area

16 shadow rays

Why is Area Better than Hemisphere?





Hemisphere | **Area**

16 shadow rays | **16 shadow rays**

Further Reading: PBRT Book

CHAPTER 13: MONTE CARLO INTEGRATION

- Skip 13.4 Metropolis Sampling